Mathematical Methods in Operations Research 2005
Special issue in honor of *Arie Hordijk*

Stochastic Control Problems with Delay

Harald Bauer and Ulrich Rieder

[1] Department of Optimization and Operations Research, University Ulm, Germany e-mail: `hbauer@mathematik.uni-ulm.de`

[2] Department of Optimization and Operations Research, University Ulm, Germany e-mail: `rieder@mathematik.uni-ulm.de`

Received: March 2005 / Revised version: April 2005

Abstract We consider optimal control problems for systems described by stochastic differential equations with delay. We state conditions for certain classes of such systems under which the stochastic control problems become finite-dimensional. These conditions are illustrated with three applications. First, we solve some linear quadratic problems with delay. Then we find the optimal consumption rate in a financial market with delay. Finally, we solve explicitly a deterministic fluid problem with delay which arises from admission control in ATM communication networks.

1 Introduction

In this paper we discuss stochastic control models which are driven by a stochastic differential equation with delay, i.e. the dynamics of the controlled system do not only depend on the current state but also on the states of the system during the last d time units, where $d > 0$ describes a fixed delay. We are looking for a control process that maximizes the total expected reward that is earned by the system during a finite time horizon. Stochastic control problems with delay arise in many contexts. In Elsanosi et al. [2] or Larssen and Risebro [10] harvesting problems with delayed dynamics are discussed. In Øksendal and Sulem [12] consumption and portfolio optimization problems in financial markets with delay are considered, and in Bauer [1] several applications from communication networks are taken into account.

Whereas it is well-known that the dynamic programming principle can be extended to stochastic control problems with delay (see e.g. Kolmanovskiĭ and Shaĭkhet [7], Gihman and Skorokhod [6] or Larssen [9]), most problems remain practically intractable because they have infinite-dimensional state spaces. Therefore, in the applications mentioned, we restrict to a special dependence structure for the stochastic delay differential equation in the following sense. The dynamics of the system may depend on the current state, the state d time units earlier and some (sliding) average of the previous states. For such control problems the value function turns out to be finite-dimensional so that it becomes much easier to solve these problems. In Larssen and Risebro [10] general conditions for a class of control problems with delay are given such that the value function is finite-dimensional.

In this paper we consider more general stochastic control problems with delay. In section 2 the model under consideration is formulated. In section 3 we state conditions which enable us to define a closely related finite-dimensional stochastic control problem (without delay). The solution of this reduced control problem can then be used to derive an optimal control and the value function of the original problem with delay. In the following sections this procedure is illustrated with three different examples. Section 4 covers some stochastic linear quadratic problems with delay. In section 5 we solve an optimal consumption problem in a financial market with delay. Finally, we shortly discuss the special case of deterministic control problems with delay in section 6 and solve explicitly an admission control problem for a fluid network with cross traffic.

2 The Stochastic Control Model with Delay

We consider a filtered probability space $\left(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P}\right)$ that satisfies the usual conditions, i.e. the probability space is complete, $\mathcal{F}_0$ contains all $\mathbb{P}$-null sets in $\mathcal{F}$ and the filtration $\{\mathcal{F}_t\}$ is right continuous. Moreover, let $W = \{W_t, t \geq 0\}$ denote an l-dimensional standard Brownian motion which is adapted to $\{\mathcal{F}_t\}$. Assume that the continuous stochastic process $X = \{X_t, t \geq -d\}$ taking values in a closed set $\mathbb{X} \subset \mathbb{R}^n$ describes the state of a system at time t that started at time $-d < 0$. Here, d describes a (constant) *delay* inherent to the system. Let $C_{\mathbb{X}}[-d, 0]$ denote the space of all continuous functions on $[-d, 0]$ taking values in $\mathbb{X}$. Then, the associated *segment process* $\{\varphi_t, t \geq 0\}$ given by

$$\varphi_t(s) := X_{t+s}, \quad s \in [-d, 0],$$

is a $C_{\mathbb{X}}[-d, 0]$-valued stochastic process. In this paper we consider systems whose dynamics may depend not only on the current state but also on the segment process through the processes

$$Y_t := \int_{-d}^{0} e^{\lambda s} f(X_{t+s}) ds, \quad \zeta_t := f(X_{t-d}),$$

where $f : \mathbb{R}^n \to \mathbb{R}^k$ is continuously differentiable and $\lambda \in \mathbb{R}$ is a constant. The system can be controlled by an $\mathcal{F}_t$-adapted stochastic process $\pi = \{\pi_t, t \geq 0\}$ taking values in a closed subset $\mathbb{U}$ of $\mathbb{R}^m$. Under the control process π the system evolves according to the stochastic delay differential state equation

$$\begin{cases} dX_t = \mu_1(t, X_t, Y_t, \pi_t)\, dt + \mu_2(X_t, Y_t) \cdot \zeta_t \, dt + \sigma(t, X_t, Y_t, \pi_t)\, dW_t, \quad t \geq 0, \\ X_t = \varphi(t) \quad -d \leq t \leq 0, \end{cases}$$

with a given deterministic initial segment value $\varphi \in C_{\mathbb{X}}[-d, 0]$ and drift functions

$$\mu_1 : \mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{R}^k \times \mathbb{U} \to \mathbb{R}^n, \quad \mu_2 : \mathbb{R}^n \times \mathbb{R}^k \to \mathbb{R}^{(n,k)},$$

as well as a volatility function

$$\sigma : \mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{R}^k \times \mathbb{U} \to \mathbb{R}^{(n,l)}.$$

We refer the reader to Mohammed [11] for results concerning the existence and uniqueness of solutions for stochastic differential equations with delay.
At every time instant t, an immediate reward $r(t, X_t, Y_t, \pi_t)$ is accrued and the terminal state of the system earns a reward $h(X_T, Y_T)$. Then we are looking for a control process π that maximizes the overall expected reward over the horizon T. More formally, for $t \in [0, T]$ we consider the family of stochastic control problems

$$(P)\begin{cases} \mathbb{E}_{\varphi,\pi}\Big[\int_t^T r(s,X_s,Y_s,\pi_s)\,ds + h(X_T,Y_T)\Big] \;\longrightarrow\; \max \\ dX_s = \mu_1(s,X_s,Y_s,\pi_s)\,ds + \mu_2(X_s,Y_s)\cdot\zeta_s\,ds + \sigma(s,X_s,Y_s,\pi_s)\,dW_s,\; s\in[t,T], \\ X_s = \varphi(s-t), \quad t-d\le s\le t, \\ \pi_s\in\mathbb{U},\; X_s\in\mathbb{X}, \quad s\in[t,T]. \end{cases}$$

Let $t\in[0,T]$ and $\varphi\in C_{\mathbb{X}}[-d,0]$ be given. Any $\{\mathcal{F}_s\}$-progressively measurable control process $\pi:[t,T]\times\Omega\to\mathbb{U}$ is called *feasible* if under π the stochastic delay differential equation has a unique (strong) solution X taking values in $\mathbb{X}$ such that

$$\mathbb{E}_{\varphi,\pi}\Big[\int_t^T |r(s,X_s,Y_s,\pi_s)|\,ds + |h(X_T,Y_T)|\Big] < \infty.$$

The set of all feasible controls is denoted by $\mathcal{U}[t,T]$. Moreover, the *value function* V of (P) is defined by

$$V(t,\varphi) := \sup_{\pi\in\mathcal{U}[t,T]} \mathbb{E}_{\varphi,\pi}\Big[\int_t^T r(s,X_s,Y_s,\pi_s)\,ds + h(X_T,Y_T)\Big].$$

A feasible control process $\pi^*\in\mathcal{U}[t,T]$ is called *optimal* for (t,φ) if

$$V(t,\varphi) = \mathbb{E}_{\varphi,\pi^*}\Big[\int_t^T r(s,X_s,Y_s,\pi^*_s)\,ds + h(X_T,Y_T)\Big].$$

Note that the value function V is defined on the infinite-dimensional state space $C_{\mathbb{X}}[-d,0]$ so that the stochastic maximum principle or the Hamilton-Jacobi-Bellman theory are not directly applicable. In the next section we will formulate a family of stochastic control problems with finite-dimensional state space such that the corresponding value function gives an upper bound on V and moreover, an optimal control process for (P) can be constructed from an optimal solution of these finite-dimensional stochastic control problems.

3 Reduction to a Finite-Dimensional Model

Considering the dynamics of the controlled system and the reward rates r and h, it is tempting to conjecture that the value function V depends on the initial value $\varphi\in C_{\mathbb{X}}[-d,0]$ only through

$$x(\varphi) := \varphi(0), \quad y(\varphi) := \int_{-d}^0 e^{\lambda s} f\big(\varphi(s)\big)ds, \quad \zeta(\varphi) := f\big(\varphi(-d)\big).$$

But the dynamics of the process ζ_t depend on φ_{t-d} so that this conjecture cannot hold without further assumptions. In order to reduce the stochastic control problem with delay we introduce the following

Assumption (T)**:** (Transformation)

There exists a solution $T:\mathbb{R}^n\times\mathbb{R}^k\to\mathbb{R}^n$ of the system of partial differential equations

$$e^{\lambda d}\cdot T_x(x,y)\cdot\mu_2(x,y) - T_y(x,y) = 0, \quad (x,y)\in\mathbb{R}^n\times\mathbb{R}^k. \tag{1}$$

Here, T_x, T_y denote the Jacobian of T with respect to x, y respectively, whereas T_{xx} is the Hessian of T with respect to x. This transformation yields a new state process

$$Z_t := T(X_t,Y_t).$$

Let $\mathbb{S} := \mathbb{X} \times y\big(C_{\mathbb{X}}[-d,0]\big)$. Then Z takes values in $T(\mathbb{S})$. In order to derive the dynamics of the transformed process Z we need the following *Itô formula*.

Lemma 1 *Let $G \in C^{1,2,1}$ be given and consider a feasible control process $\pi \in \mathcal{U}[0,T]$ with associated state process X. Then the stochastic process $G\big(t, X_t, Y_t\big)$ solves*

$$
\begin{aligned}
dG(t, X_t, Y_t) = \Big\{ & G_t(t, X_t, Y_t) + G_x(t, X_t, Y_t) \cdot \big(\mu_1(t, X_t, Y_t, \pi_t) + \mu_2(X_t, Y_t) \cdot \zeta_t\big) \\
& + \frac{1}{2} tr\Big(G_{xx}(t, X_t, Y_t) \cdot \sigma(t, X_t, Y_t, \pi_t)\, \sigma(t, X_t, Y_t, \pi_t)^T\Big)\Big\}\, dt \\
& + G_x(t, X_t, Y_t) \cdot \sigma(t, X_t, Y_t, \pi_t)\, dW_t \\
& + G_y(t, X_t, Y_t) \cdot \big(f(X_t) - e^{-\lambda d}\zeta_t - \lambda Y_t\big)\, dt.
\end{aligned} \tag{2}
$$

Proof For a given feasible control process π with state process X we introduce

$$
F(t) := \int_0^t f(X_s) ds.
$$

Then the process Y has the representation

$$
\begin{aligned}
Y_t &= \int_{-d}^0 e^{\lambda s} f(X_{t+s}) ds = e^{\lambda s} F(t+s)\Big|_{-d}^{0} - \int_{-d}^0 \lambda e^{\lambda s} F(t+s) ds \\
&= F(t) - e^{-\lambda d} F(t-d) - \int_{-d}^0 \lambda e^{\lambda s} F(t+s) ds.
\end{aligned}
$$

In particular, Y has finite variation so that

$$
dY_t = \big(f(X_t) - e^{-\lambda d} f(X_{t-d}) - \lambda Y_t\big)dt = \big(f(X_t) - e^{-\lambda d}\zeta_t - \lambda Y_t\big)dt. \tag{3}
$$

Applying the multidimensional Itô formula to $G(t, X_t, Y_t)$ the representation (2) follows. □

Now we are able to derive the dynamics for the components of Z by using (1).

$$
\begin{aligned}
dZ_t^i = dT^i(X_t, Y_t) = \Big\{ & T_x^i(X_t, Y_t) \cdot \big(\mu_1(t, X_t, Y_t, \pi_t) + \mu_2(X_t, Y_t) \cdot \zeta_t\big) \\
& + \frac{1}{2} tr\Big(T_{xx}^i(X_t, Y_t) \cdot \sigma(t, X_t, Y_t, \pi_t)\, \sigma(t, X_t, Y_t, \pi_t)^T\Big)\Big\}\, dt \\
& + T_x^i(X_t, Y_t) \cdot \sigma(t, X_t, Y_t, \pi_t)\, dW_t \\
& + T_y^i(X_t, Y_t) \cdot \big(f(X_t) - e^{-\lambda d}\zeta_t - \lambda Y_t\big)\, dt \\
= {} & T_x^i(X_t, Y_t) \cdot \mu_1(t, X_t, Y_t, \pi_t)\, dt + T_y^i(X_t, Y_t) \cdot \big(f(X_t) - \lambda Y_t\big)\, dt \\
& + \frac{1}{2} tr\Big(T_{xx}^i(X_t, Y_t) \cdot \sigma(t, X_t, Y_t, \pi_t)\, \sigma(t, X_t, Y_t, \pi_t)^T\Big)\, dt \\
& + T_x^i(X_t, Y_t) \cdot \sigma(t, X_t, Y_t, \pi_t)\, dW_t, \quad i = 1, \ldots, n.
\end{aligned} \tag{4}
$$

Hence, the drift function $\tilde{\mu} : \mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{R}^k \times \mathbb{U} \to \mathbb{R}^n$ for the process Z is given by

$$
\begin{aligned}
\widetilde{\mu}^i(t, x, y, u) := {} & T_x^i(x, y) \cdot \mu_1(t, x, y, u) + T_y^i(x, y) \cdot \big(f(x) - \lambda y\big) \\
& + \frac{1}{2} tr\Big(T_{xx}^i(x, y) \cdot \sigma(t, x, y, u)\, \sigma(t, x, y, u)^T\Big), \quad i = 1, \ldots, n,
\end{aligned}
$$

whereas the corresponding volatility function $\tilde{\sigma} : \mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{R}^k \times \mathbb{U} \to \mathbb{R}^{(n,l)}$ has the representation

$$
\widetilde{\sigma}^i(t, x, y, u) := T_x^i(x, y) \cdot \sigma(t, x, y, u), \quad i = 1, \ldots, n.
$$

If these functions as well as the rewards would depend on (x,y) through $T(x,y)$ only, then (P) could be reduced to a family of finite-dimensional problems. This yields the

Assumption (R)**:** (Reduction)
There are functions

$$\begin{aligned} &\overline{\mu}: \mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{U} \to \mathbb{R}^n, && \overline{\sigma}: \mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{U} \to \mathbb{R}^{(n,l)}, \\ &\overline{r}: \mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{U} \to \mathbb{R}, && \overline{h}: \mathbb{R}^n \to \mathbb{R}, \end{aligned}$$

such that for all $t \in [0,T], u \in \mathbb{U}, (x,y) \in \mathbb{R}^n \times \mathbb{R}^k$

$$\begin{aligned} \overline{\mu}\big(t, T(x,y), u\big) &= \widetilde{\mu}\big(t,x,y,u\big), & \overline{\sigma}\big(t, T(x,y), u\big) &= \widetilde{\sigma}(t,x,y,u), \\ \overline{r}\big(t, T(x,y), u\big) &= r(t,x,y,u), & \overline{h}\big(T(x,y)\big) &= h(x,y). \end{aligned}$$

Now we can introduce a family of finite-dimensional control problems $(\overline{P})$ associated to (P) via the transformation T. Let $\varphi \in C_{\mathbb{X}}[-d,0]$ be an initial value for problem (P) and define $z := T(x(\varphi), y(\varphi)) \in T(\mathbb{S})$. Then for $t \in [0,T]$ we consider the stochastic control problems

$$(\overline{P}) \begin{cases} \mathbb{E}_{z,\pi}\Big[\displaystyle\int_t^T \overline{r}(s, Z_s, \pi_s)\, ds + \overline{h}(Z_T)\Big] \;\longrightarrow\; \max \\ dZ_s = \overline{\mu}(s, Z_s, \pi_s)\, ds + \overline{\sigma}(s, Z_s, \pi_s)\, dW_s, \quad s \in [t,T], \\ Z_t = z, \\ \pi_s \in \mathbb{U},\ Z_s \in T(\mathbb{S}), \quad s \in [t,T]. \end{cases}$$

The set $\overline{\mathcal{U}}[t,T]$ of feasible controls for $\big(\overline{P}\big)$ is defined analogously to $\mathcal{U}[t,T]$ and the value function $\overline{V}$ of $(\overline{P})$ is given by

$$\overline{V}(t,z) := \sup_{\pi \in \overline{\mathcal{U}}[t,T]} \mathbb{E}_{z,\pi}\Big[\int_t^T \overline{r}(s, Z_s, \pi_s)\, ds + \overline{h}(Z_T)\Big].$$

The stochastic control problem $(\overline{P})$ has a finite-dimensional state space. Problems of this kind are well-studied in the literature (see e.g. Fleming and Soner [5], or Yong and Zhou [13]). In these references verification theorems are provided, which are based on the Hamilton-Jacobi-Bellman equation. To formulate this equation we introduce for any function $v(t,z) \in C^{1,2}$ the so-called *Hamiltonian*

$$\mathcal{H}\big(t,z,u,v_z,v_{zz}\big) := \overline{r}(t,z,u) + v_z(t,z)\cdot\overline{\mu}(t,z,u) + \frac{1}{2} tr\big(v_{zz}(t,z)\cdot\overline{\sigma}(t,z,u)\overline{\sigma}(t,z,u)^T\big).$$

Then, the partial differential equation

$$\begin{cases} v_t(t,z) + \sup_{u\in\mathbb{U}} \mathcal{H}\big(t,z,u,v_z,v_{zz}\big) = 0, \quad (t,z) \in [0,T] \times T(\mathbb{S}), \\ v(T,z) = \overline{h}(z), \quad z \in T(\mathbb{S}), \end{cases}$$

is called the *Hamilton-Jacobi-Bellman* (HJB) equation for $(\overline{P})$. The aforementioned references use slightly varying conditions on the drift and volatility functions $\overline{\mu}, \overline{\sigma}$ as well as on the reward functions $\overline{r}, \overline{h}$ to derive verification results. In order to give a unified treatment we refrain from choosing one specific set of conditions for $\big(\overline{P}\big)$. Instead, we say that $\big(\overline{P}\big)$ satisfies the *verification principle* if for any initial value $(t,z) \in [0,T] \times T(\mathbb{S})$ and any progressively measurable control process the state equation has a unique (strong) solution and if any solution $v \in C^{1,2}$ of the HJB equation has the following properties:

i) $v(t,z) \geq \overline{V}(t,z) \quad \forall\,(t,z) \in [0,T] \times T(\mathbb{S})$.
ii) Let $(t,z) \in \mathbb{R}_+ \times T(\mathbb{S})$. If there exists a control process $\pi^* \in \overline{\mathcal{U}}[t,T]$ with state process Z^* such that

$$\pi^*_s \in \arg\max_{u\in\mathbb{U}} \mathcal{H}\big(s, Z^*_s, u, v_z(s,Z^*_s), v_{zz}(s,Z^*_s)\big)$$

for all $s \in [t,T]$, then π^* is optimal for (t,z) and

$$\overline{V}\big(s, Z^*_s\big) = v\big(s, Z^*_s\big), \quad s \in [t,T].$$

By using the verification principle for $\big(\overline{P}\big)$ we are able to state the main result of this paper.

Theorem 1 *Suppose that Assumptions $(T),(R)$ hold and that $\big(\overline{P}\big)$ satisfies the verification principle. Let $v \in C^{1,2}$ be a solution of the HJB equation. Then*

a) $v(t,z) \geq V(t,\varphi)$ for all $\varphi \in C_{\mathbb{X}}[-d,0]$, $z := T\big(x(\varphi), y(\varphi)\big)$, $t \in [0,T]$.
b) Let $(t,\varphi) \in [0,T] \times C_{\mathbb{X}}[-d,0]$. If there exists a control process $\pi^* \in \mathcal{U}[t,T]$ with state process X^* such that

$$\pi^*_s \in \arg\max_{u\in\mathbb{U}} \mathcal{H}\Big(s, T(X^*_s,Y^*_s), u, v_z\big(s, T(X^*_s,Y^*_s)\big), v_{zz}\big(s, T(X^*_s,Y^*_s)\big)\Big) \tag{5}$$

for all $s \in [t,T]$, then π^* is optimal for (P) with initial value (t,φ) and π^* is also optimal for $(\overline{P})$ with initial value (t,z), $z := T\big(x(\varphi), y(\varphi)\big)$ and moreover,

$$V(s,\varphi^*_s) = \overline{V}\big(s, T(X^*_s,Y^*_s)\big) = v\big(s, T(X^*_s,Y^*_s)\big), \quad s \in [t,T].$$

Proof a) Fix $t \in [0,T]$, $\varphi \in C_{\mathbb{X}}[-d,0]$ and let $\pi \in \mathcal{U}[t,T]$ be a feasible control process with state process X. Due to the Assumptions $(T),(R)$ the process $Z_s = T(X_s,Y_s)$ is the unique (strong) solution of the state equation for problem $(\overline{P})$ with initial value $z = T\big(x(\varphi), y(\varphi)\big)$ at time t, because $(\overline{P})$ satisfies the verification principle. Moreover, Z takes values in $T(\mathbb{S})$ so that $\pi \in \overline{\mathcal{U}}[t,T]$ holds. Hence, we conclude

$$\begin{aligned} v(t,z) \geq \overline{V}(t,z) &\geq \mathbb{E}_{z,\pi}\Big[\int_t^T \overline{r}(s,Z_s,\pi_s)\,ds + \overline{h}(Z_T)\Big] \\ &= \mathbb{E}_{\varphi,\pi}\Big[\int_t^T r(s,X_s,Y_s,\pi_s)\,ds + h(X_T,Y_T)\Big]. \end{aligned}$$

Since $\pi \in \mathcal{U}[t,T]$ was chosen arbitrary, the desired inequality follows.

b) For $(t,\varphi) \in [0,T] \times C_{\mathbb{X}}[-d,0]$ let $\pi^* \in \mathcal{U}[t,T]$ be a control process and let X^* be the associated state process such that

$$\pi^*_s \in \arg\max_{u\in\mathbb{U}} \mathcal{H}\Big(s, T(X^*_s,Y^*_s), u, v_z\big(s, T(X^*_s,Y^*_s)\big), v_{zz}\big(s, T(X^*_s,Y^*_s)\big)\Big)$$

for all $s \in [t,T]$. Under the transformation $Z^*_s = T(X^*_s,Y^*_s)$ this becomes

$$\pi^*_s \in \arg\max_{u\in\mathbb{U}} \mathcal{H}\big(s, Z^*_s, u, v_z(s,Z^*_s), v_{zz}(s,Z^*_s)\big).$$

Since π^* is also feasible for $\big(\overline{P}\big)$ with initial value $z = T\big(x(\varphi), y(\varphi)\big)$ at time t and $\big(\overline{P}\big)$ satisfies the verification principle, π^* is optimal for $(\overline{P})$ with initial value (t,z) and

$$\overline{V}(t,z) = v(t,z).$$

In particular, we conclude from a)

$$\begin{aligned} v(t,z) &= \mathbb{E}_{z,\pi}\Big[\int_t^T \overline{r}(s,Z_s^*,\pi_s^*)\,ds + \overline{h}(Z_T^*)\Big] \\ &= \mathbb{E}_{\varphi,\pi}\Big[\int_t^T r(s,X_s^*,Y_s^*,\pi_s)\,ds + h(X_T^*,Y_T^*)\Big] \\ &\geq V(t,\varphi). \end{aligned}$$

On the other hand, π^* is feasible for (P) so that

$$\mathbb{E}_{\varphi,\pi}\Big[\int_t^T r(s,X_s^*,Y_s^*,\pi_s)\,ds + h(X_T^*,Y_T^*)\Big] \leq V(t,\varphi).$$

Therefore π^* is optimal for (P) with initial value (t,φ). Replacing (t,z) by (s,Z_s^*) in the above argument, we obtain the remaining statements.

Remark From the proof of part a) it follows that

$$V(t,\varphi) \leq \overline{V}(t,z), \quad z := T\big(x(\varphi),y(\varphi)\big),\ t\in[0,T],\ \varphi\in C_{\mathbb{X}}[-d,0],$$

i.e. the value function of $(\overline{P})$ is an upper bound of the value function of (P).

Often, the optimal control process obtained from (5) can be described in *feedback form.* Let $\overline{\pi} : [0,T]\times T(\mathbb{S}) \to \mathbb{U}$ be a function with

$$\overline{\pi}(t,z) \in \arg\max_{u\in\mathbb{U}} \mathcal{H}\big(t,z,u,v_z(t,z),v_{zz}(t,z)\big)$$

and define for $(t,\varphi)\in[0,T]\times C_{\mathbb{X}}[-d,0]$ the function

$$\widetilde{\pi}(t,\varphi) := \overline{\pi}\big(t,T\big(x(\varphi),y(\varphi)\big)\big).$$

As an immediate consequence of Theorem 1 we get the following

Corollary 1 *If the stochastic delay differential equation*

$$\begin{cases} dX_s = \mu_1\big(s,X_s,Y_s,\widetilde{\pi}(s,\varphi_s)\big)\,ds + \mu_2(X_s,Y_s)\cdot\zeta_s\,ds \\ \qquad\quad +\sigma\big(s,X_s,Y_s,\widetilde{\pi}(s,\varphi_s)\big)\,dW_s,\ s\in[t,T], \\ X_s = \varphi(s-t), \quad t-d\leq s\leq t, \end{cases}$$

has a unique solution X^* with segment process φ^* for some given initial value $(t,\varphi)\in[0,T]\times C_{\mathbb{X}}[-d,0]$, then

$$V(s,\varphi_s^*) = \overline{V}\big(s,T(X_s^*,Y_s^*)\big), \quad s\in[t,T].$$

Moreover, the optimal control process for problem (P) with initial value (t,φ) can be given in feedback form

$$\pi_s^* = \widetilde{\pi}(s,\varphi_s^*), \quad s\in[t,T].$$

Proof Let the initial value (t,φ) be given and consider the corresponding solution X^* of the stochastic delay differential equation. Then $\pi^*\in\mathcal{U}[t,T]$ and the statements follow directly from Theorem 1. □

Remarks 1) Let X^* be the optimal state process for some initial value $(0,\varphi_0)$ with segment process φ^*. Then the value function $V(t,\varphi_t^*)$ and the optimal feedback control process $\pi_t^* = \widetilde{\pi}(t,\varphi_t^*)$ depend on φ_t^* only through

$$X_t^* \quad \text{and} \quad Y_t^* = \int_{-d}^0 e^{\lambda s} f(X_{t+s}^*)\,ds.$$

Note that $V(t,\varphi_t^*)$ and π_t^* are independent of $\zeta_t^* := f(X_{t-d}^*)$.

2) In many applications the set of feasible control processes is restricted to some subset of L^p. Then we are able to relax the verification principle to hold for this subclass of control processes and Theorem 1 as well as Corollary 1 are still valid.

4 Stochastic Linear Quadratic Problems with Delay

Stochastic linear quadratic problems (LQ problems) constitute a class of optimization problems which were extensively studied in the literature (see e.g. Yong and Zhou [13]). In this section, we apply the verification technique from the last section to study a delayed version. LQ problems with delay were first investigated by Kolmanovskiĭ et al. (see [7] and references therein). Assume that $f(x) = x$ and let $A_1(t), A_2(t), Q(t) \in \mathbb{R}^{(n,n)}$, $B(t) \in \mathbb{R}^{(n,m)}$, $R(t) \in \mathbb{R}^{(m,m)}$ be (deterministic) matrix valued functions in $L^\infty[0,T]$. Moreover, let $\sigma(t) \in \mathbb{R}^{(n,l)}$ be a (deterministic) matrix valued function in $L^2[0,T]$ and let $A_3 \in \mathbb{R}^{(n,n)}, G \in \mathbb{R}^{(m,m)}$ be given. In addition, we assume that $Q(t), G$ are positive semi-definite and $R(t)$ is positive definite for all $t \in [0,T]$ and continuous on $[0,T]$. Then the stochastic LQ problem with delay can be stated as

$$(LQ) \begin{cases} \mathbb{E}_{\varphi,\pi}\Big[\displaystyle\int_t^T \big(X_s + e^{\lambda d}A_3Y_s\big)^T Q(s)\big(X_s + e^{\lambda d}A_3Y_s\big) + \pi_s^T R(s)\pi_s\, ds \\ \qquad\qquad + \big(X_T + e^{\lambda d}A_3Y_T\big)^T G\big(X_T + e^{\lambda d}A_3Y_T\big)\Big] \longrightarrow \max \\ dX_s = \Big(A_1(s)X_s + A_2(s)Y_s + A_3\zeta_s + B(s)\pi_s\Big)\, ds + \sigma(s)\, dW_s, \quad s \in [t,T], \\ X_s = \varphi(s-t), \quad t-d \le s \le t, \\ \pi_s \in \mathbb{R}^m, \quad s \in [t,T], \end{cases}$$

where $\varphi \in C_{\mathbb{R}^n}[-d,0]$ is a given initial segment. We restrict the set of feasible controls $\mathcal{U}[t,T]$ to all progressively measurable processes π with values in $\mathbb{R}^m$ such that $\pi \in L^2\big(\Omega \times [0,T]\big)$. If we set

$$\begin{aligned} \mu_1(t,x,y,u) &:= A_1(t)x + A_2(t)y + B(t)u, \\ \mu_2(x,y) &:= A_3, \\ r(t,x,y,u) &:= (x + e^{\lambda d}A_3y)^T Q(t)(x + e^{\lambda d}A_3y) + u^T R(t)u, \\ h(x,y) &:= (x + e^{\lambda d}A_3y)^T G(x + e^{\lambda d}A_3y), \end{aligned}$$

then (LQ) is a special case of (P). In order to apply Theorem 1 we have to check whether Assumptions $(T), (R)$ are satisfied. Note that the partial differential equation (1) reads

$$e^{\lambda d} \cdot T_x(x,y) \cdot A_3 - T_y(x,y) = 0.$$

If $A_3 = 0$, then $T(x,y)$ depends on x only and Assumption (R) can only hold if $A_2(t) \equiv 0$ so that the problem reduces to the classical LQ problem without delay. Hence, we assume $A_3 \neq 0$. Obviously, $T(x,y) := x + e^{\lambda d}A_3y$ solves (1). This implies

$$\begin{aligned} \widetilde{\mu}(t,x,y,u) &= A_1(t)x + A_2(t)y + B(t)u + e^{\lambda d}A_3\big(x - \lambda y\big) \\ &= \big(A_1(t) + e^{\lambda d}A_3\big)\big(T(x,y) - e^{\lambda d}A_3y\big) + \big(A_2(t) - \lambda e^{\lambda d}A_3\big)y + B(t)u, \\ r(t,x,y,u) &= T(x,y)^T Q(t)T(x,y) + u^T R(t)u, \\ h(x,y) &= T(x,y)^T G T(x,y). \end{aligned}$$

Therefore, Assumption (R) is satisfied if and only if

$$A_2(t) = e^{\lambda d}\big(\lambda E_n + A_1(t) + e^{\lambda d}A_3\big)A_3. \tag{6}$$

The reduced finite-dimensional stochastic control problem becomes

$$(\overline{LQ})\begin{cases} \mathbb{E}_{z,\pi}\Big[\int_t^T Z_s^T Q(s) Z_s + \pi_s^T R(s)\pi_s\,ds + Z_T^T G Z_T\Big] \longrightarrow \max \\ dZ_s = \big((A_1(s)+e^{\lambda d}A_3)Z_s + B(s)\pi_s\big)\,ds + \sigma(s)\,dW_s, \quad s\in[t,T], \\ Z_t = z, \\ \pi_s \in \mathbb{R}^m, \quad s\in[t,T], \end{cases}$$

where $z := T\big(x(\varphi), y(\varphi)\big) \in \mathbb{R}^n$. In Yong and Zhou [13, Chapter 6] it is shown that $(\overline{LQ})$ satisfies the verification principle and the optimal control is given by the feedback function

$$\overline{\pi}(t,z) := -R^{-1}(t)B(t)^T P(t) z,$$

where $P(t)$ is a solution of the *Riccati equation*

$$\begin{cases} \dot{P}(t) = P(t)B(t)R^{-1}(t)B(t)^T P(t) - \big(A_1(t)+e^{\lambda d}A_3\big)P(t) \\ \qquad\qquad -P(t)\big(A_1(t)+e^{\lambda d}A_3\big)^T - Q(t), \\ P(T) = G. \end{cases} \tag{7}$$

Moreover, the value function $\overline{V}$ has the representation

$$\overline{V}(t,z) = z^T P(t) z + \int_t^T tr\big(\sigma(s)\sigma(s)^T\big)P(s)\,ds.$$

Let X^*, Y^* be the unique solution of

$$\begin{aligned} dX_t &= \Big(\big(A_1(t) - B(t)R^{-1}(t)B(t)^T P(t)\big)X_t + \big(A_2(t) - e^{\lambda d}B(t)R^{-1}(t)B(t)^T P(t)A_3\big)Y_t \\ &\qquad + A_3 X_{t-d}\Big)\,dt + \sigma(t)\,dW_t, \;\; t\in[0,T], \\ X_t &= \varphi(t), \quad -d\le t\le 0, \\ Y_t &= \int_{-d}^0 e^{\lambda s}X_{t+s}\,ds, \quad t\in[0,T]. \end{aligned}$$

From this we can directly derive the solution of the LQ problem with delay.

Theorem 2 *Consider the stochastic control problem* (LQ) *with* $A_3 \neq 0$ *and assume that* (6) *holds. Then the value function* V *of* (LQ) *is given by*

$$V(t,\varphi_t^*) = \big(Z_t^*\big)^T P(t) Z_t^* + \int_t^T tr\big(\sigma(s)\sigma(s)^T\big)P(s)\,ds$$

and the optimal control process is given by

$$\pi_t^* = -R^{-1}(t)B(t)^T P(t) Z_t^*,$$

where $Z_t^* := X_t^* + e^{\lambda d}A_3 Y_t^*$ *and* $P(t)$ *solves the Riccati equation* (7).

5 Optimal Consumption in a Financial Market with Delayed Dynamics

In this section we study a problem of utility maximization from consumption and terminal wealth. We consider an investor whose wealth process follows a stochastic delay differential equation. During the time interval $[0,T]$ the investor consumes from his wealth according to the consumption process $c_t \geq 0$ so that his current wealth X_t is described by

$$\begin{cases} dX_t = \big(\mu(t,X_t,Y_t) + a\zeta_t - c_t\big)\,dt + \sigma(t,X_t,Y_t)\,dW_t, \quad t \geq 0, \\ X_t = \varphi(t), \quad -d \leq t \leq 0, \end{cases}$$

with $a \in [0,\infty)$. Here we assume that $f(x) = x$. Consumption at rate c yields the utility $U_1(c) := \frac{1}{\alpha}c^\alpha$, $\alpha \in (0,1)$ which is discounted at rate $\beta \geq 0$, whereas the terminal utility is defined by $U_2(z) = \frac{1}{\alpha}z^\alpha$. Hence, the investor is faced with the following stochastic consumption problem with delay.

$$(C)\begin{cases} \mathbb{E}_{\varphi,\pi}\Big[\int_t^T e^{-\beta s}U_1(c_s)\,ds + U_2(X_T + ae^{\lambda d}Y_T)\Big] \longrightarrow \max \\ dX_s = \big(\mu(s,X_s,Y_s) + a\zeta_s - c_s\big)\,ds + \sigma(s,X_s,Y_s)\,dW_s, \quad s \in [t,T], \\ X_s = \varphi(s-t), \quad t-d \leq s \leq t, \\ c_s \geq 0,\ X_s \geq 0, \quad s \in [t,T], \end{cases}$$

where $\varphi \in C_{\mathbb{R}_+}[-d,0]$ is a given initial segment. The set of feasible controls $\mathcal{U}[t,T]$ consists of those consumption processes c with $\int_t^T c_s\,ds < \infty$, $P-a.s$ and $X \geq 0$. Obviously, (C) is a special case of (P). The partial differential equation (1) reads

$$ae^{\lambda d}\cdot T_x(x,y) - T_y(x,y) = 0.$$

As in the last section, this problem reduces to the classical consumption problem (without delay) if $a = 0$. Hence, we assume here $a > 0$. Then $T(x,y) := x + ae^{\lambda d}y$ solves (1). This yields

$$\begin{aligned} \widetilde{\mu}(t,x,y,c) &= \mu(t,x,y) - c + ae^{\lambda d}(x - \lambda y), \\ \widetilde{\sigma}(t,x,y,c) &= \sigma(t,x,y), \\ r(t,x,y,c) &= e^{-\beta t}U_1(c), \\ h(x,y) &= U_2\big(T(x,y)\big). \end{aligned}$$

Choosing Cauchy data $\mu(t,x,0) := \mu_t \cdot x$ and $\sigma(t,x,0) = \sigma_t \cdot x$ for some continuous functions μ_t, $\sigma_t > 0$, we obtain by the method of characteristic curves in view of Assumption (R)

$$\mu(t,x,y) = ae^{\lambda d}\big(ae^{\lambda d} + \lambda\big)y + \mu_t T(x,y), \qquad \sigma(t,x,y) = \sigma_t T(x,y). \tag{8}$$

If we define $\overline{\mu}_t := \mu_t + ae^{\lambda d}$, then the reduced finite-dimensional consumption problem becomes

$$(\overline{C})\begin{cases} \mathbb{E}_{z,\pi}\Big[\int_t^T e^{-\beta s}U_1(c_s)\,ds + U_2(Z_T)\Big] \longrightarrow \max \\ dZ_s = \big(\overline{\mu}_s Z_s - c_s\big)\,ds + \sigma_s Z_s\,dW_s, \quad s \in [t,T], \\ Z_t = z, \\ c_s \geq 0,\ Z_s \geq 0, \quad s \in [t,T], \end{cases}$$

where $z := T\big(x(\varphi), y(\varphi)\big) \in \mathbb{R}_+$. In Korn and Korn [8] it is shown that $(\overline{C})$ satisfies the verification principle and the optimal consumption process can be given in feedback form

$$\overline{c}(t,z) := e^{-\frac{\beta t}{1-\alpha}} \cdot \frac{z}{g(t)}$$

where $g(t)$ is a positive solution of the differential equation

$$\dot{g}(t) = a_1(t) \cdot g(t) + a_2(t), \quad g(T) = 1 \tag{9}$$

with

$$a_1(t) := -\frac{1}{2}\frac{\overline{\mu}_t^2}{(1-\alpha)^2}(\sigma_t \sigma_t^T)^{-1} \quad \text{and} \quad a_2(t) := -\frac{1}{\alpha} e^{-\frac{\beta t}{1-\alpha}}.$$

Moreover, the value function $\overline{V}$ has the representation

$$\overline{V}(t,z) = \frac{1}{\alpha} g(t)^{1-\alpha} z^{\alpha}.$$

Let X^*, Y^* be the unique solution of

$$\begin{aligned}
dX_t &= \Big(\mu(t, X_t, Y_t) + aX_{t-d} - e^{-\frac{\beta t}{1-\alpha}} \cdot \frac{1}{g(t)}\big(X_t + ae^{\lambda d} Y_t\big)\Big)\, dt \\
&\quad + \sigma(t, X_t, Y_t)\, dW_t, \quad t \in [0,T], \\
X_t &= \varphi(t), \quad -d \le t \le 0, \\
Y_t &= \int_{-d}^{0} e^{\lambda s} X_{t+s}\, ds, \quad t \in [0,T].
\end{aligned}$$

By using Theorem 1 we can directly state the solution of the consumption problem with delay.

Theorem 3 *Consider the optimal consumption problem* (C) *with* $a > 0$ *and assume that* (8) *holds. Then the value function* V *of* (C) *is given by*

$$V(t, \varphi_t^*) = \frac{1}{\alpha} g(t)^{1-\alpha} Z_t^*$$

and the optimal consumption process is given by

$$c_t^* = e^{-\frac{\beta t}{1-\alpha}} \frac{1}{g(t)} \cdot Z_t^*,$$

where $Z_t^* := X_t^* + ae^{\lambda d} Y_t^*$ *and* $g(t)$ *is a positive solution of* (9).

The special case $a = -\lambda e^{-\lambda d}$ and $\lambda < 0$ was treated by Elsanousi and Larssen [3].

6 Deterministic Control Problems with Delay

One important subclass of problems consists of *deterministic control problems with delay*, i.e. $\sigma \equiv 0$. In this case, problem (P) reduces to the following deterministic control problem

$$\begin{cases}
\displaystyle\int_t^T r\big(s, x(s), y(s), \pi_s\big)\, ds + h\big(x(T), y(T)\big) \;\longrightarrow\; \max \\
\dot{x}(s) = \mu_1\big(s, x(s), y(s), \pi_s\big) + \mu_2\big(x(s), y(s)\big) \cdot \zeta(s), \quad s \in [t,T], \\
x(s) = \varphi(s-t), \quad t-d \le s \le t, \\
\pi_s \in \mathbb{U},\ x(s) \in \mathbb{X}, \quad s \in [t,T],
\end{cases}$$

where $\varphi \in C_{\mathbb{X}}[-d,0]$ is a given initial segment. Under Assumption (T) the dynamics of the transformed state $z(s) = T\big(x(s), y(s)\big)$ are given by

$$\begin{aligned}\dot{z}(s) &= T_x\big(x(s), y(s)\big) \cdot \mu_1\big(s, x(s), y(s), \pi_s\big) + T_y\big(x(s), y(s)\big) \cdot \big(f(x(s)) - \lambda y(s)\big)\\ &= \widetilde{\mu}\big(s, x(s), y(s), \pi_s\big)\end{aligned}$$

Under Assumption (R) we obtain for any initial value $\varphi \in C_{\mathbb{X}}[-d,0]$ and $z := T\big(x(\varphi), y(\varphi)\big)$ the reduced finite-dimensional control problem

$$\begin{cases} \displaystyle\int_t^T \overline{r}\big(s, z(s), \pi_s\big)\, ds + \overline{h}\big(z(T)\big) \;\longrightarrow\; \max \\ \dot{z}(s) = \overline{\mu}\big(s, z(s), \pi_s\big), \quad s \in [t,T], \\ z(t) = z, \\ \pi_s \in \mathbb{U},\; z(s) \in T(\mathbb{S}), \quad s \in [t,T]. \end{cases}$$

In this case the Hamiltonian has the form

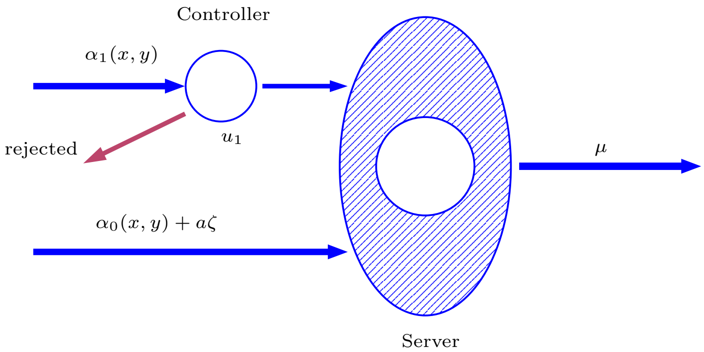
$$\mathcal{H}(t, z, u, v_z) = \overline{r}(t, z, u) + v_z(t, z) \cdot \overline{\mu}(t, z, u).$$

To exemplify the use of Theorem 1 in the deterministic setting, let us consider an *example of congestion control* which is motivated by ATM communication networks where several users can connect to the network over the same User Network Interface. Every user submits ATM cells which are collected in a common buffer and transmitted in FIFO discipline. In order to prevent congestion or excessive transmission delay, some control must be exerted. The most common form of congestion control is given by access control. In Fendick and Rodrigues [4], such a model is studied under cross traffic, i.e. in addition to the controlled arrival stream there is an uncontrolled stream which utilizes some of the available bandwidth. In their model the cross traffic is described by a Brownian motion, whereas the controlled arrival stream is modeled as fluid. We will discuss a model where the cross traffic is described by a fluid stream as well.

Consider a *fluid model* with 2 different arrival streams sending fluid to one infinite buffer from which it is fed into a network. Assume that fluid can be pumped into the network at a maximal rate of μ and that the state $x(t)$ of the system at time t describes the current buffer level. In this example we set $f(x) := \frac{1}{d}e^{-x}$ and $\lambda = 0$. Hence $y(t) = \frac{1}{d}\int_{-d}^{0} e^{-x(t+s)} ds$ represents an exponential mean of the (negative) states over the last d time units taking values in $(0,1]$, whereas $\zeta(t) = f(x(t-d))$ gives the scaled exponential (negative) state d time units earlier.
The rate at which the first traffic source pumps fluid into the buffer follows a self adaptation scheme, i.e. the arrival rate $\alpha_1(x,y)$ varies with the current state as well as with past states.
The second traffic source also follows a self adaptation scheme. Here, the arrival rate is described by $\alpha_0(x,y) + a\zeta$, $a \in \mathbb{R}$ depending explicitly on the buffer level d time units earlier. The use of the special function $f(x) = \frac{1}{d}e^{-x}$ signifies that the rates adapt fast to significant changes in the buffer level.

The buffer level can be controlled by admitting only a fraction $u_1 \in [0,1]$ of the incoming fluid from source one and by utilizing only a fraction $u_2 \in [0,1]$ of the network capacity $\mu > 0$. We impose linear holding costs $c(x + ay)$, $c > 0$. Moreover, a reward at rate $r > 0$ is gained whenever fluid from stream one is admitted to the system. More precisely, we consider the overall reward rate

$$r\big(t, x, y, u_1, u_2\big) = -c\big(x + ay\big) + ru_1.$$



Hence, our deterministic admission control problem with delay is given by

$$(AC)\begin{cases} \displaystyle\int_t^T \Big[-c\big(x(s)+ay(s)\big)+ru_1(s)\Big]ds \longrightarrow \max \\ \dot{x}(s)=\alpha_1\big(x(s),y(s)\big)u_1(s)+\alpha_0\big(x(s),y(s)\big)+a\zeta_s-\mu u_2(s), \quad s\in[t,T], \\ x(s)=\varphi(s-t), \quad t-d\le s\le t, \\ \pi_s=\big(u_1(s),u_2(s)\big)\in[0,1]^2,\ x(s)\ge 0, \quad s\in[t,T] \end{cases}$$

where $\varphi\in C_{\mathbb{R}_+}[-d,0]$ is a given initial segment and $a\in\mathbb{R}$. For this deterministic control problem the partial differential equation (1) has the form

$$a\cdot T_x(x,y)-T_y(x,y)=0.$$

A solution is given by $T(x,y):=x+ay$. This implies

$$\begin{aligned} \widetilde{\mu}(t,x,y,u_1,u_2)&=\alpha_1(x,y)u_1+\alpha_0(x,y)-\mu u_2+af(x),\\ r(t,x,y,u)&=-cT(x,y)+ru_1,\\ h(x,y)&=0. \end{aligned}$$

In view of (R), we assume $\alpha_1(x,y)=\overline{\alpha}_1\big(T(x,y)\big)$ for some function $\overline{\alpha}_1:\mathbb{R}\to\mathbb{R}_+$ and for $\alpha_0(x,0):=\mu(1+e^{-x})$ we obtain

$$\alpha_0(x,y)=-\frac{a}{d}e^{-x}+\mu\Big(1+\big(1+\frac{a}{\mu d}\big)e^{-(x+ay)}\Big).$$

Now we fix $a:=-\mu d$. Then α_0 simplifies to $\alpha_0(x,y)=\mu+\mu e^{-x}$. Hence, Assumption (R) is satisfied if we choose

$$a=-\mu d, \quad \alpha_1(x,y):=\big(-\frac{\alpha}{\mu d}(x-\mu dy)\big)^+, \quad \alpha_0(x,y)=\mu+\mu e^{-x} \tag{10}$$

for some $\alpha>0$. The reduced finite-dimensional admission control problem has now the form

$$(\overline{AC})\begin{cases} \displaystyle\int_t^T \big[-cz(s)+ru_1(s)\big]\,ds \longrightarrow \max \\ \dot{z}(s)=\big(-\frac{\alpha}{\mu d}z(s)\big)^+u_1(s)+\mu(1-u_2(s)), \quad s\in[t,T], \\ z(t)=z, \\ \pi_s=\big(u_1(s),u_2(s)\big)\in[0,1]^2,\ z(s)\ge-\mu d, \quad s\in[t,T], \end{cases}$$

where $z \in [-\mu d, \infty)$. In Bauer [1] it is shown that problem $(\overline{AC})$ satisfies the verification principle and the optimal solution can be described in the following way. Let $z \in [-\mu d, \infty)$ and define the switching time $t^*(z)$ by

$$t^*(z) := \begin{cases} 0 & z \geq -\frac{r}{c}, \\ \left(T + \frac{\mu d}{\alpha} \ln\left(1 + \frac{r}{cz}\right)\right)^+ & z < -\frac{r}{c}. \end{cases}$$

Then the optimal control for $(\overline{AC})$ with initial value $(0, z)$ is given by $u_2^*(t) \equiv 1$ and

$$u_1^*(t) = \begin{cases} 0 & t \leq t^*(z),\ z \leq 0, \\ 1 & \text{else.} \end{cases}$$

The value function for $(\overline{AC})$ has the representation

$$\overline{V}(t, z) = \begin{cases} (r - cz)(T - t) & z > 0, \\ r(T - t) - c\frac{\mu d}{\alpha} z\left(1 - e^{-\frac{\alpha}{\mu d}(T-t)}\right) & t \geq t^*(z),\ z \leq 0, \\ \frac{\mu d r}{\alpha} - cz(T - t) - (r + cz)\frac{\mu d}{\alpha} \ln\left(1 + \frac{r}{cz}\right) & t < t^*(z),\ z \leq 0. \end{cases}$$

Let x^*, y^* be the unique solution of

$$\begin{aligned} \dot{x}(t) &= \left(-\frac{\alpha}{\mu d}\left(x(t) - \mu d y(t)\right)\right)^+ u_1^*(t) + \mu\left(e^{-x(t)} - e^{-x(t-d)}\right), \quad t \in [0, T], \\ x(t) &= \varphi(t), \quad -d \leq t \leq 0, \\ y(t) &= \int_{-d}^{0} \frac{1}{d} e^{-x(t+s)}\, ds, \quad t \in [0, T]. \end{aligned}$$

Theorem 1 yields directly the solution of the admission control problem (AC).

Theorem 4 *Consider the problem (AC) and assume that (10) holds. Then:*

a) The value function of (AC) is given by

$$V(t, \varphi_t^*) = \begin{cases} \left(r - z^*(t)\right)(T - t) & z^*(t) > 0, \\ r(T - t) - c\frac{\mu d}{\alpha} z^*(t)\left(1 - e^{-\frac{\alpha}{\mu d}(T-t)}\right) & t \geq t^*\left(z^*(t)\right),\ z^*(t) \leq 0, \\ \frac{\mu d r}{\alpha} - cz^*(t)(T - t) & \\ \qquad -\left(r + cz^*(t)\right)\frac{\mu d}{\alpha} \ln\left(1 + \frac{r}{cz^*(t)}\right) & t < t^*\left(z^*(t)\right),\ z^*(t) \leq 0, \end{cases}$$

where $z^*(t) = x^*(t) - \mu d y^*(t)$.

b) The optimal control process for (AC) with initial value $(0, \varphi_0)$ has the form

$$u_2^*(t) \equiv 1 \quad \text{and} \quad u_1^*(t) = \begin{cases} 0 & t \leq t^*\left(z(\varphi_0)\right),\ z(\varphi_0) \leq 0, \\ 1 & \text{else,} \end{cases}$$

where $z(\varphi_0) := x(0) - \mu d y(0)$.

References

1. H. Bauer. *Fluid Approximation for Controlled Stochastic Networks with Delayed Dynamics*. PhD thesis, University Ulm, 2004.
2. I. Elsanosi, B. Øksendal, and A. Sulem. Some Solvable Stochastic Control Problems with Delay. *Stochastics and Stochastics Reports*, 71:69–89, 2001.
3. I. Elsanousi and B. Larssen. Optimal Consumption under Partial Observation for a Stochastic System with Delay. Preprint, 2001.

4. K. W. Fendick and M. A. Rodrigues. Asymptotic Analysis of Adaptive Rate Control for Diverse Sources with Delayed Feedback. *IEEE Transactions on Information Theory*, 40(4):2008–2025, 1994.
5. W. H. Fleming and H. M. Soner. *Controlled Markov Processes and Viscosity Solutions*. Springer-Verlag, New York, 1993.
6. I. I. Gihman and A. V. Skorokhod. *Controlled Stochastic Processes*. Springer-Verlag, New York, 1979.
7. V. B. Kolmanovskiĭ and L. E. Shaĭkhet. *Control of Systems with Aftereffect*, volume 157 of *Translation of Mathematical Monographs*. American Mathematical Society, Providence, Rhode Island, 1996.
8. R. Korn and E. Korn. *Optionsbewertung und Portfolio-Optimierung - Moderne Methoden der Finanzmathematik*. Vieweg, 1999.
9. B. Larssen. Dynamic Programming in Stochastic Control of Systems with Delay. *Stochastics and Stochastics Reports*, 74(3-4):651–673, 2002.
10. B. Larssen and N. H. Risebro. When are HJB-Equations for Control Problems with Stochastic Delay Equations Finite Dimensional? *Stochastic Analysis and Applications*, 21(3):643–671, 2003.
11. S. E. Mohammed. *Stochastic Functional Differential Equations*. Pitman, Boston, 1984.
12. B. Øksendal and A. Sulem. A Maximum Principle for Optimal Control of Stochastic Systems with Delay, with Applications to Finance. In J. M. Menaldi, E. Rofman, and A. Sulem, editors, *Optimal Control and Partial Differential Equations*, pages 64–79. IOS Press, Amsterdam, 2001.
13. J. Yong and X. Y. Zhou. *Stochastic Controls*. Springer-Verlag, New York, 1999.