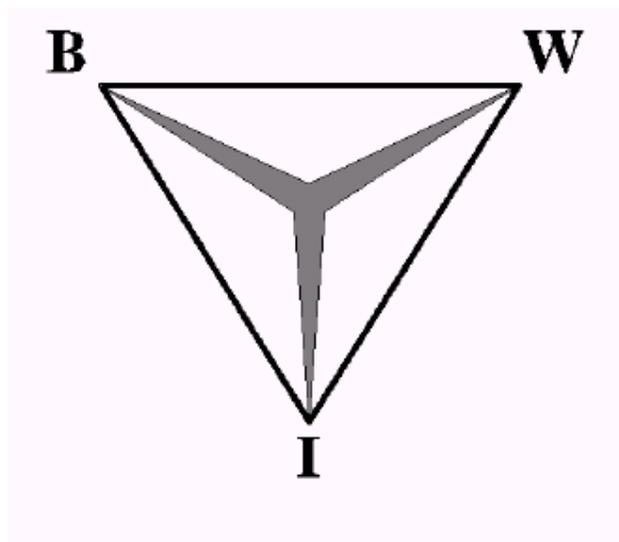


Copulas

Modeling dependencies in Financial Risk Management



BMI Master Thesis

Preface

This paper has been written as part of my study Business Mathematics and Informatics (BMI) at the Vrije Universiteit in Amsterdam. The BMI-paper is one of the final mandatory subjects. The objective is to investigate the available literature in reference to a topic related to at least two out of the three fields integrated in the study.

For the purpose of extending my knowledge on Financial Risk Management and techniques for risk measurement, I decided to dedicate my BMI paper to this area. The subject for this paper was set in consultation with dr. S. Bhulai of the OBP research group at the Faculty of Sciences.

Although several people have supported me during the realization of this paper, I would especially like to express my gratitude to my supervisor, dr. S. Bhulai. Despite his busy schedule he took time to guide me. I am thankful for his advice, comments, and motivational speeches when I needed it.

Purmerend, December 2007

Farzia Habiboellah

Summary

The goal of Financial Risk Management (FRM) is to measure and manage risks across a diverse range of activities used in financial sectors. Risk can be defined as a hazard, a chance of bad consequences, loss or exposure to mischance. There are different types of risks, but we will focus on the three most important: market, credit, and operational risk. There are different ways to measure risk. Two risks measures that are often used, based on loss distributions, are Value at Risk (VaR) and Expected Shortfall.

Within FRM dependencies between random variables play an important role. A popular and often used dependence measure is correlation, also called the correlation coefficient, which indicates the strength and direction of a linear relationship between two random variables. It is a reasonable measure when the random variables are elliptically distributed and a good measure when the random variables are multivariate normally distributed. But research shows that the multivariate normal distribution is inadequate because it underestimates both the thickness of the tails of the marginals of the risks and their dependence structure.

Apart from that correlation has more disadvantages. Correlation is not invariant under strictly increasing transformations of the risks. It is a scalar measure of dependence and therefore cannot tell us everything we would like to know about the dependence structure of risks. Correlation is only defined when the variances of the risks are finite. It is not an appropriate dependence measure for very heavy-tailed risks where variances tend to infinity. This inadequacy of correlation requires an appropriate dependence measure: the copula. The main objective of this thesis is as follows:

“What are the advantages of using a copula model to model dependencies between variables in financial risk models over a more traditional method such as correlation?”

A copula function couples n univariate marginal distributions together to form a multivariate distribution resulting in a joint distribution function of n standard uniform random variables. Assume that you have two random variables (X, Y) . Then the standard formulation is: $H(x, y) = C(F(x), G(y))$, where $C(u, v)$ is the copula, F and G are marginal distribution functions, and H is the joint cumulative distribution function.

Important copula functions are the Fréchet-Hoeffding upper and lower bound given by $M(u, v) = \min(u, v)$ and $W(u, v) = \max(u + v - 1, 0)$, respectively, and the product copula $\prod(u, v) = uv$. Copulas are invariant under strictly increasing transformations of the risks.

In 1959 Abe Sklar was the first who used the term copula in a mathematical sense. The theorem, that was named after him, states that any joint cumulative distribution function \mathbf{F} can be written in terms of a copula and marginal cumulative distribution functions. If the marginals are continuous then the copula is unique for \mathbf{F} .

The copula is in comparison to correlation invariant under transformations of the risks. Correlation is a scalar measure of dependence; it does not tell us everything we would like to know about the dependence structure of risks. A copula determines the dependence relationship by joining the marginal distributions together to form a joint distribution. The scaling and the shape are entirely determined by the marginals. In contrast to correlation the copula function can be applied when risks are heavily tailed.

A copula model that has become a standard market model for valuating collateralized debt obligations (CDOs) is the Gaussian copula model. The risk of a CDO is distributed over several tranches, where each tranche represents a group of investors with different risk degrees. To determine the price of the tranches of a CDO, the following are needed: the default probability, the default severity (or recovery), and the default correlation.

The default correlation is the likelihood that the default of one asset causes the default of another and is much higher between credits within the same industrial sector. Default correlations can be modeled through the use of the one-factor copula model.

Suppose a CDO contains assets from n companies, then the default time of the i th company is denoted by T_i . Its corresponding cumulative probability distribution that company i will default before time t is denoted by $Q_i(t)$. It is assumed that T_i is related to a random variable X_i , so that for any given t , there is a corresponding value x such that: $P(X_i \leq x) = P(T_i \leq t)$. The default of the n companies can then be modeled by the following copula model: $X_i = a_i M + \sqrt{1 - a_i^2} Z_i$. The model maps X_i to t_i on a “percentile to percentile” basis.

When the cumulative distribution of X_i is denoted by F_i and the cumulative distribution of Z_i (assuming that the Z_i are identically distributed) by H , then we can write Z_i as $\frac{X_i - a_i M}{\sqrt{1 - a_i^2}}$. Given that the market variable $M = m$, then its probability can be written as:

$$P(Z_i < x | M = m) = H\left(\frac{x - a_i m}{\sqrt{1 - a_i^2}}\right).$$

Correlation comes in trouble when the random variables are not elliptically distributed. The performance of the copula does not depend on the fact if you are dealing with elliptical distributions or not. Add the fact that copulas possess handy properties and the winner of the two is the copula.

Contents

Preface	3
Summary.....	4
Contents.....	6
Chapter 1 Introduction.....	7
1.1 Background	7
1.2 Object and Scope.....	7
1.3 Thesis structure	8
Chapter 2 Measuring dependencies.....	13
2.1 Dependencies in FRM.....	13
2.2 Correlation and regression, what's the difference?	13
2.3 The classic approach: correlation	14
2.4 Correlation and the normal distribution	14
2.4.1 Pitfalls of correlation	15
Chapter 3 Financial Risk Management	10
3.1 Introduction	10
3.2 Risk	10
3.3 Risk measures.....	11
3.3.1 Value at Risk (VaR).....	11
3.3.2 Expected Shortfall.....	12
Chapter 4 The basic stuff.....	13
4.1 Introduction	16
4.2 What is a copula?	17
4.3 Sklar's Theorem	19
4.4 Copula families	20
4.5 Correlation vs. Copula	21
Chapter 5 Application.....	23
5.1 Introduction	23
5.2 Credit Derivatives & CDO's.....	23
5.3 The one-factor copula model.....	24
Conclusion.....	26
Reference.....	27

Chapter 1 Introduction

1.1 Background

Financial Risk Management is currently a hot topic in the financial world. The goal of Financial Risk Management (FRM) is to measure and manage risks across a diverse range of activities used in, e.g., banking, securities and insurance sectors. Within FRM dependencies between random variables play an important role. Take for example the risk of a portfolio. A portfolio that contains a variety of stock types will be less risky than one holding only a single type of stock.

A popular and often used dependence measure is the correlation coefficient based on a, which is a reasonable measure when the random variables are elliptically distributed and a good measure when the random variables are multivariate normal¹ distributed. But what if the marginals are not normal distributed and a model can make the difference between thousands or even millions of Euros loss or revenue for an institute? Under these circumstances reasonable is not good enough anymore.

The market risk portfolio value distributions, which typically occur in banking, are often approximated by a normal distribution. It is a standard assumption in many risk management applications, because this distribution is easy to implement. However, research shows that the multivariate normal distribution is inadequate because it underestimates both the thickness of the tails of the marginals and their dependence structure [2]. The credit and especially the operational risks are approximated with more skewed distributions because of occasional, extreme losses.

So it seems that the marginals of risks are often not elliptically distributed. However, when you step outside the elliptical world, the marginal distributions and correlations do not suffice to determine the joint multivariate distribution. It no longer tells us everything we need to know about dependence, particularly in the tails. In those cases correlation is no longer a suitable dependence measure. This inadequacy of the use of correlation outside the elliptical world requires an appropriate dependence measure: the copula.

1.2 Object and Scope

The focus in this thesis lies on modeling the dependence structure between risks, an essential activity which is needed in the assessment of risk. In this thesis we will restrict ourselves to the field of banking.

¹ Note that the multivariate normal is a special member in the family of elliptical distributions.

Due to the fact that the copula maneuvers around the pitfalls of correlation, it has become quite popular in financial risk management to model dependencies between risks. Areas of application include credit risk modeling, portfolio Value at Risk calculations, default and credit risk dependence, and tail dependence. In this thesis we will discuss the pitfalls of correlation and how the copula function deals with them. The main objective of this thesis is as follows:

“What are the advantages of using a copula model to model dependencies between variables in financial risk models over a more traditional method such as correlation?”

Based on the objective we have derived the following questions that will be answered in this thesis:

Chapter 3 - When and where does correlation come short in risk modeling?

Chapter 4 - What is a copula function and what are its properties?

Chapter 5 - How can copulas be applied in financial risk management?

1.3 Thesis structure

A graphical representation of the thesis structure can be seen in Figure 1.1. In Chapter 2 we will go deeper into risks and risk measures. It is necessary to cover these subjects to gain a better understanding of Financial Risk Management in order to discuss the main subject: dependence modeling in FRM.

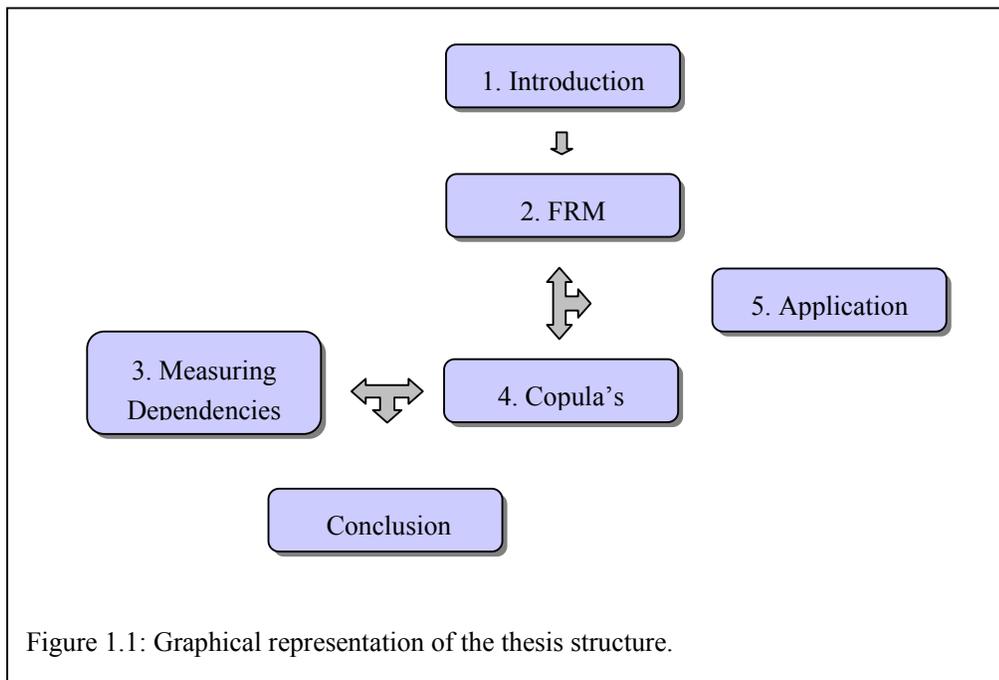


Figure 1.1: Graphical representation of the thesis structure.

As mentioned before, Chapter 3 will explain the concept of correlation and the shortcomings of correlation when it comes to risk modeling.

In Chapter 4 we will explain what a copula is, what its properties are, and how a copula function can be constructed.

Now that the copula methodology is covered, some possible applications of copula functions in financial risk management are presented in Chapter 5.

Finally, the last chapter summarizes the conclusions of our study, and furthermore some points for further research are stated.

Chapter 2 Financial Risk Management

2.1 Introduction

The concept of risk is based on the uncertainty about future outcomes. In the introduction we already mentioned three different types of individual risks: the market, the credit, and the operational risk. In this chapter we will go deeper into these risks, especially the ones last mentioned and we will discuss two generally used risk measures: the variance and the Value at Risk (VaR).

2.2 Risk

According to the Concise Oxford English Dictionary risk can be defined as:

“A hazard, a chance of bad consequences, loss or exposure to mischance.”

In other words, risk indicates any uncertainty that might trigger losses. However, uncertainty is hardly visible in contrast to revenues or costs. Consequently, risks remain intangible until they have materialized into loss. This makes the quantification of risks more difficult.

Financial risks can be split into two parts: the individual risks and the dependence structure between them. This requires an approach for combining different risk types and, hence, risk distributions. There are many types of risks, but for now we will focus on the most important three: the market, the credit, and the operational risk.

Market risk

The market risk is the risk that the value of an investment will decrease due to adverse movements in market factors. These movements cause volatility in Profit & Loss.

Credit risk

The credit risk is the risk that a company or individual will be unable to pay the contractual interest or principal on its debt obligations.

Operational risk

The operational risk is the risk of loss resulting from inadequate or failed internal processes, people and systems, or from external events.

The distributional shapes of each risk type vary considerably; some are better characterized and measured than others. For example, the market risk has portfolio value

distributions that are often approximated by a normal distribution. Credit and operational risk on the other hand are approximated with more skewed distributions because of occasional, extreme losses. These might be due to large lending exposures in the case of credit risk, or large catastrophes such as 9/11, in the case of operational risk.

2.3 Risk measures

In practice, risk measures are used for a variety of purposes. One of the principal functions of financial risk measurement is to determine the amount of capital a financial institution needs to hold. This is important because an institution needs a buffer against unexpected future losses on its portfolio in order to keep the solvency of the institution healthy.

Another purpose is that risk measures are often used by management as a tool for limiting the amount of risk a unit within a firm may take. There are different ways to measure risk. In this section we will discuss two risk measures that are based on loss distributions, namely Value at Risk (VaR) and Expected Shortfall. Losses are the central object of interest in FRM and so it is natural to base a measure of risk on their distribution.

2.3.1 Value at Risk (VaR)

Value at Risk (VaR) is defined as the maximum expected loss (measured in monetary units) of an asset value (or a portfolio) over a given time period and at a given level of confidence (or with a given level of probability), under normal market conditions [Coronado, 2000].

Consider a portfolio of risky assets, with a fixed time horizon Δ , and denote by $F_L(l) = P(L \leq l)$ the distribution function of the corresponding loss distribution. It measures the severity of the risk of holding our portfolio over the time period Δ .

DEFINITION

Given some confidence level $\alpha \in (0,1)$. The VaR of the portfolio at confidence level α is given by the smallest number l such that the probability that the loss L exceeds l is no larger than $(1 - \alpha)$.

$$VaR_\alpha = \inf\{l \in R : P(L > l) \leq 1 - \alpha\} = \inf\{l \in R : F_L(l) \geq \alpha\}. \quad (1)$$

In other words, the VaR of our portfolio is the loss L that is expected to be exceeded with a probability of $(1 - \alpha)$. In probabilistic terms, the VaR is thus simply the quantile of the loss distribution. For example if a bank's 10-day 99% VaR is 3 million Euros, there is considered to be only a 1% probability that losses will exceed 3 million Euro in 10 days.

Typical values for α are 0.95 or 0.99; in market risk management the time horizon Δ is usually 1 or 10 days, in credit risk management and operational risk management Δ is usually one year.

2.3.2 Expected Shortfall

The Expected Shortfall (ES) is closely related to Value at Risk. Mathematically this risk measure can be defined as:

DEFINITION

For a loss L with $E(|L|) < \infty$ and distribution function F_L the Expected Shortfall at confidence level $\alpha \in (0,1)$ is defined as:

$$ES_\alpha = \frac{1}{1-\alpha} \int_\alpha^1 q_u(F_L) du, \quad (2)$$

where $q_u(F_L)$ is the quantile function of F_L . The ES is therefore related to VaR by:

$$ES_\alpha = \frac{1}{1-\alpha} \int_\alpha^1 VaR_u(L) du = E(L | L \geq VaR_\alpha). \quad (3)$$

So, in other words, the ES is defined as the expected size of a loss that exceeds VaR. If you take for example a alpha of 99% over a time horizon Δ of ten days, the ES is the average amount that is lost over a 10-day period, assuming that the loss is greater than the 99th percentile of the loss distribution. In paragraph 2.3.1 we discussed an example of a bank with a 10-day 99% VaR of 3 million. However there is no differentiation between small and very large violations of the 3 million Euro limit. The eventual loss can be 3 million Euro as well as 30 million Euro. That is why ES gives a ‘bigger picture’ of the risk, because you look at the average VaR over all levels $u \geq \alpha$ as can be seen in Equation (3).

Chapter 3 Measuring dependencies

3.1 Introduction

In the previous chapter we generally discussed Financial Risk Management. In this chapter we will focus on the dependencies in FRM and how they can be measured with the correlation coefficient (correlation). In Section 3.5 we will discuss a way to approach the relation between correlation and the Normal distribution. The pitfalls of correlation will be dealt with in the last section of this chapter.

3.2 Dependencies in FRM

Dependencies can be found everywhere, so also in the financial world. Take for example market and credit risk. Both risk types are related to the interest rate. But there can also be relations between risks. Assume an investor has a portfolio with two loans to two different companies. He can reduce the portfolio risk by holding assets that are not related with each other.

For example, if the investor loans money to two Dutch farmers and there is continuing bad weather, which destroys the harvest of all Dutch farmers. Changes are high that both farmers will default. The farmers are related to each other, because they are exposed to many of the same influences. That is why it is smart to hold a risk diversified portfolio. If the investor would loan to a farmer and to an umbrella factory, he balances the risk of bad weather. Because bad weather is a bad influence for the farmer, but it means big business for the umbrella factory. So they can reduce their exposure to asset risk by holding a diversified portfolio of assets.

3.3 Correlation and regression, what's the difference?

Correlation and regression are two concepts that are often confused with each other. Before we will go deeper into correlation, we will shortly discuss the difference between both techniques by looking at their goals. The goal of linear regression is to find the equation of the line that best fits the data. This line is then used to represent the relationship between the variables, or for estimating unknown values of one variable when given the value of the other. The goal of correlation is to see whether two variables co-vary, and to measure the strength of any dependency between the variables. The results of correlation are expressed as a P-value (for the hypothesis test) and an r-value (correlation coefficient) or r^2 value (coefficient of determination).

3.4 The classic approach: correlation

A concept that is often used in risk management, but which is often misunderstood, is correlation. Part of the misunderstanding can be the result of the usage of the word in literature. Correlation is a dependence measure, but is often used for almost every meaning of the word dependence. Even when it is used as a risk measure, there is a tendency to use it as if it were an all-purpose dependence measure. This results in correlation being misused and applied to problems for which it is not suitable.

DEFINITION

The correlation coefficient, indicates the strength and direction of a linear relationship between two random variables. The best known correlation measure is the Pearson product-moment correlation coefficient (r) (www.wikipedia.org).

$$r(X, Y) = \frac{\text{cov}(X, Y)}{\sqrt{\text{var}(X)}\sqrt{\text{var}(Y)}} \quad (4)$$

With $\text{cov}(X, Y) = E(XY) - E(X)E(Y)$, provided that $\text{var}(X)$ and $\text{var}(Y)$ are greater than 0. The correlation coefficient is a scalar measure of dependence with the property that $-1 \leq r(X, Y) \leq 1$.

In Sections 3.2 we discussed that the default of one company can be related to the default of another. This is called the default correlation. Default correlation is the likelihood that the default of one asset causes the default of another and is much higher between credits within the same industrial sector. In Chapter 5 we will go deeper into this matter.

3.5 Correlation and the normal distribution

Although correlation plays a central role in finance, it is important to realize that the concept is only a natural one in the context of multivariate normal or more generally, elliptical models. The visual relation between the normal distribution and correlation is demonstrated in steps one to three in Figure 2.1, which shows the gradual transformation from the bivariate normal distribution to a correlation line (Figure 2.2).

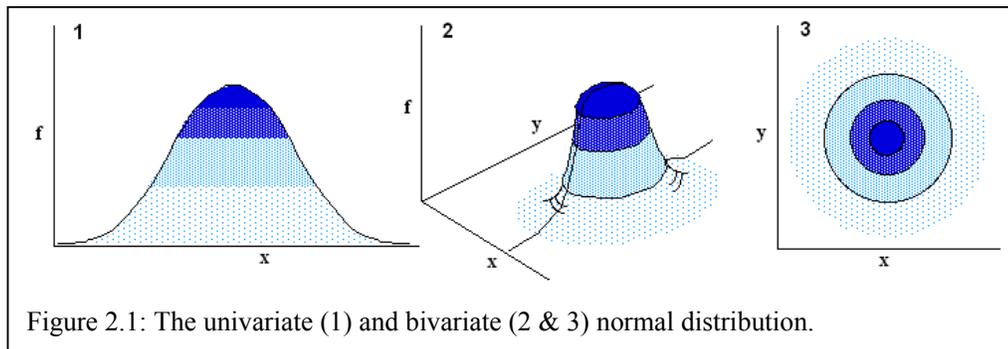


Figure 2.1: The univariate (1) and bivariate (2 & 3) normal distribution.

In the first figure you can see the univariate normal distribution and in the last two figures you can see a bivariate normal distribution from above (2) and directly down on the bivariate normal distribution (3). The last figure (3) is a scatter plot and because of the circularity of the shape it can be said that there is no relationship ($r = 0$). In Figure 2.2, it can be seen that as the correlation between x and y increases, the circles narrow until you see the straight line of the perfect ($r = 1.0$) correlation.

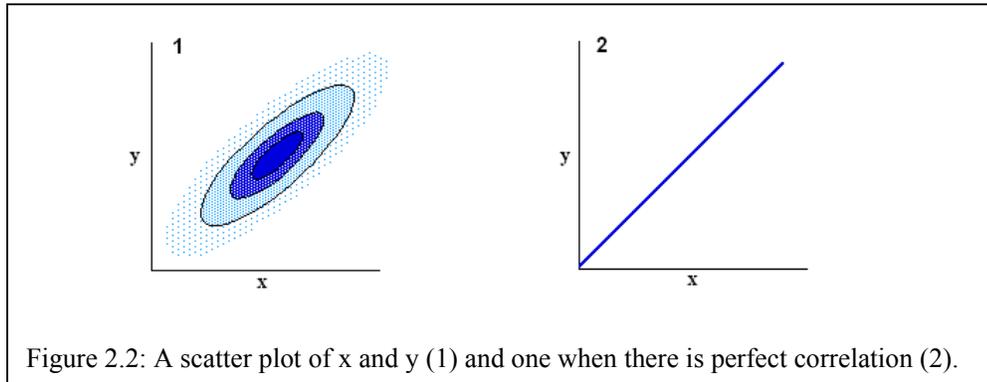


Figure 2.2: A scatter plot of x and y (1) and one when there is perfect correlation (2).

3.5.1 Pitfalls of correlation

Correlation is a reasonable measure of dependence when random variables are distributed as multivariate normal, but when this is not the case correlation gets into trouble. The following pitfalls occur:

- Possible values of correlation depend on the marginal distribution of the risks. Not all values between -1 and 1 are not necessarily attainable.
- Perfect positively dependent risks do not necessarily have a correlation of 1 ; perfect negatively dependent risks do not necessarily have a correlation of -1 .
- A correlation of zero does not indicate independence of risks.

Correlation is not invariant under strictly increasing transformations of the risks. For example, $\log(X)$ and $\log(Y)$ generally do not have the same correlation as X and Y . Hence, transformations of our data can affect our correlation estimates.

Correlation is a scalar measure of dependence; it cannot tell us everything we would like to know about the dependence structure of risks.

Correlation is only defined when the variances of the risks are finite. In Equation (4) you can see that the denominator is determined by the product of the variances, which (see Equation (4)). It is not an appropriate dependence measure for very heavy-tailed risks where variances tend to infinity.

Chapter 4 The basic stuff

4.1 Introduction

In the last chapter we discussed the pitfalls of correlation as a dependence measure in FRM. In this chapter we will discuss a dependence model that deals with most of the shortcomings of correlation, namely the copula. But before we will give a definition of copula functions, one has to be familiar with the following properties and definitions:

- *Grounded*

Suppose a_x is the smallest element of S_x and a_y is the smallest element of S_y . We say a function H from $S_x \times S_y$ into R is *grounded* if $H(x, a_y) = 0 = H(a_x, y)$, for all (x, y) in $S_x \times S_y$.

- *2-increasing*

A function H is 2-increasing if the H-volume of B is greater than or equal to 0, where $B = [x_1, x_2] \times [y_1, y_2]$ is a rectangle whose vertices are in the domain of H for every $(x_1, x_2), (y_1, y_2)$ in $[0, 1]^2$ with $x_1 \leq x_2$ and $y_1 \leq y_2$. The H-volume of B is computed by:

$$V_H(B) = H(x_2, y_2) - H(x_2, y_1) - H(x_1, y_2) + H(x_1, y_1). \quad (5)$$

- *Distribution function*

A distribution function is a function F with domain R such that:

- F is non-decreasing;
- $F(-\infty) = 0$ and $F(+\infty) = 1$.

- *Joint distribution function*

A joint distribution function is a function H with domain R^2 such that:

- H is grounded $\rightarrow H(x, -\infty) = H(-\infty, y) = 0$,
- H is 2-increasing \rightarrow H-volume is ≥ 0 .

H has as marginals the functions F and G given by $F(x) = H(x, \infty)$ and $G(y) = H(\infty, y)$ [4].

- *Domain of $H = S_x \times S_y = I^2$.*

If S_x and S_y are non-empty subsets that contain all possible values of x and y , respectively, then $S_x \times S_y = I^2$ means that all the possible values of x and y lie within the unit square I^2 , where $I = [0,1]$.

- *Marginals of H*

Suppose b_x is the greatest element of S_x and b_y is the greatest element of S_y . If the above mentioned properties hold for $H(x,y)$, then H has as marginals the functions F and G given by:

$$F(x) = H(x, b_y) \text{ for all } x \text{ in } S_x,$$

$$G(y) = H(b_x, y) \text{ for all } y \text{ in } S_y.$$

After explaining how a copula works, we will cover Sklar's Theorem, how a copula function can be constructed and two copula families.

4.2 What is a copula?

The term originates from Latin, and means "a link, tie, bond" and is used to refer to joining together or connecting words. In this thesis we will refer to the mathematical meaning of the term copula, namely the copula function.

DEFINITION

A copula function links n univariate marginal distributions to a full multivariate distribution resulting in a joint distribution function of n standard uniform random variables.

In some sense, the copula still has a connection to the Latin meaning of the word, because the copula actually 'couples' the marginal distributions together to form a joint distribution. Assume that you have two random variables (X, Y) . Then the standard formulation is:

$$H(x,y) = C(F(x), G(y)), \tag{6}$$

where $C(u,v)$ is the copula, F and G are marginal distribution functions, and H is the joint cumulative distribution function. The information of the marginal distributions are retained in $F(x)$ and $G(y)$, and the dependence information is summarized by $C(u,v)$. The dependence relationship is entirely determined by the copula, while the scaling and the shape (e.g., the mean, the standard deviation, the skewness, and the kurtosis) are entirely determined by the marginals.

A bivariate function C is a copula function if it satisfies the following properties [4]:

1. Domain of $C = S_1 \times S_2 = I^2$, where S_1 and S_2 are non-empty subsets of I ;
2. C is a function that is grounded and 2-increasing;
3. for every u in S_1 and every v in S_2 , $C(u,1) = u$ and $C(1,v) = v$.

Note that for every (u,v) in $\text{Dom } C$, $0 \leq C(u,v) \leq 1$, so that $\text{Rang } C$ is a subset of I .

EXAMPLE 3.1 We will show that the bivariate function $H(x,y) = xy$ is in fact a copula on the range $x, y \in [0,1]$.

Solution

- Domain of $H = S_x \times S_y = I^2$.
- $H(0,y) = 0$ and $H(x,0) = 0$ $\rightarrow H$ is grounded.
- For $B = [x_1, x_2] \times [y_1, y_2] \rightarrow V_H(B) \geq 0$ $\rightarrow H$ is 2-increasing.

This has to hold for every $x_1, x_2, y_1, y_2 \in [0,1]$ with $x_1 \leq x_2$ and $y_1 \leq y_2$.

Define $x_2 = x_1 + \Delta_x$ and $y_2 = y_1 + \Delta_y$. Fill this in in $V_H(B)$:

$$V_H(B) = H(x_1 + \Delta_x, y_1 + \Delta_y) - H(x_1 + \Delta_x, y_1) - H(x_1, y_1 + \Delta_y) + H(x_1, y_1) \geq 0$$

$$V_H(B) = (x_1 + \Delta_x)(y_1 + \Delta_y) - (x_1 + \Delta_x)y_1 - x_1(y_1 + \Delta_y) + (x_1y_1) \geq 0$$

$$V_H(B) = x_1y_1 + x_1\Delta_y + y_1\Delta_x + \Delta_x\Delta_y - x_1y_1 - y_1\Delta_x - x_1y_1 - x_1\Delta_y + x_1y_1 \geq 0$$

$$V_H(B) = \Delta_x\Delta_y \geq 0$$

And this holds for all $x_1, x_2, y_1, y_2 \in [0,1]$ with $x_1 \leq x_2$ and $y_1 \leq y_2$.

- $H(x,1) = x$ and $H(1,y) = y$

The function that is used in Example 3.1 is also called the product copula $\prod(u,v) = uv$.

Other important copula functions are the Fréchet-Hoeffding upper and lower bound given by $M(u,v) = \min(u,v)$ and $W(u,v) = \max(u+v-1,0)$, respectively.

THEOREM 3.3

For every copula $C(u_1, \dots, u_n)$ we have the bounds

$$\max\left(\sum_{i=1}^n u_i + 1 - n, 0\right) \leq C(\mathbf{u}) \leq \min(u_1, \dots, u_n).$$

4.3 Sklar's Theorem

In 1959 Abe Sklar was the first who used the term copula in a mathematical sense. In Sklar's Theorem he describes how the copula function works and proves that the copula C is unique for a given distribution F if the marginals are continuous. The theorem states that any joint cumulative distribution function can be written in terms of a copula and marginal cumulative distribution functions.

A copula function shows that it is possible to separately specify the dependence between variables and the marginal densities of each variable. That is why Sklar's Theorem is said to be the most important theorem about copula functions.

THEOREM 3.1 Sklar's Theorem

Let H be a joint distribution function with marginals F and G . Then there exists a copula function C such that for all x, y in \mathbb{R} ,

$$H(x, y) = C(F(x), G(y)). \quad (7)$$

If the marginals F and G are continuous, then C is unique. Otherwise C is uniquely determined on $\text{Range } F \times \text{Range } G$. Conversely, if C is a copula and F and G are distribution functions, then the function H defined by (7) is a joint distribution function with marginals F and G [4].

Theorem 3.1 is presented from the point of view of the joint distribution, but what if we were to turn this around and look at it from the direction of the copula function. The theorem can be inverted to express copulas in terms of a joint distribution function and the inverses of the marginals, but only if the marginals are strictly increasing.

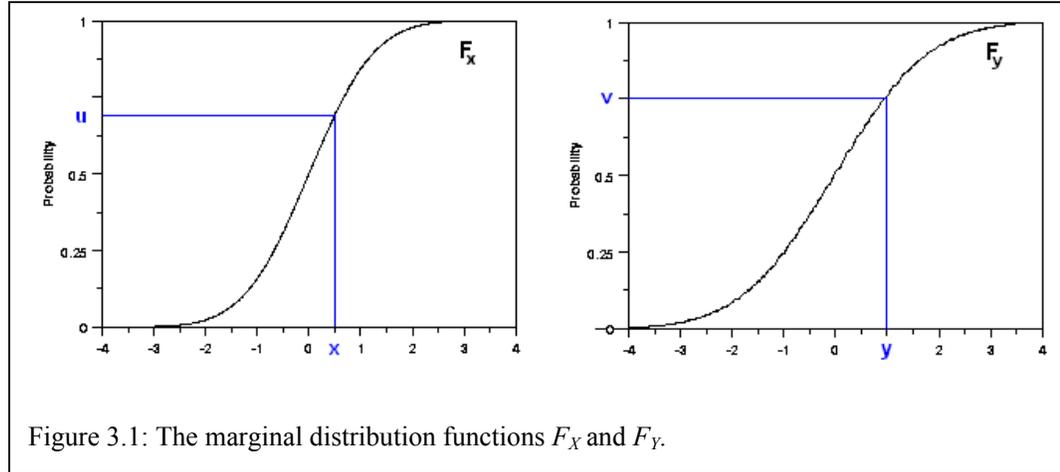
PROPOSITION 3.7

Let $F(x)$ and $F^{-1}(y)$ be two functions. If $F(F^{-1}(x)) = x$ and $F^{-1}(F(y)) = y$ then F^{-1} is the inverse of F and vice versa. The notation for the inverse of F is F^{-1} .

Equation (7) can then be written as

$$C(u, v) = H(F^{-1}(u), G^{-1}(v)), \quad (8)$$

because when $F(x)=u$ and $G(y)=v$, the inverse functions are $F^{-1}(u)=x$ and $G^{-1}(v)=y$. After substitution you obtain Equation (8). A graphical representation can be seen in Figure 3.1.



So a copula is actually a joint cumulative distribution function with uniform marginals. It maps points on the unit square $(u,v \in [0,1] \times [0,1])$ to values between zero and one.

PROPOSITION 3.8

Let (X_1, \dots, X_n) be a random vector with continuous marginals and copula C and let T_1, \dots, T_n be strictly increasing functions. Then $(T_1(X_1), \dots, T_n(X_n))$ also has copula C .

Proposition 3.8 shows the invariance property of the copula under strictly increasing transformations of the marginals.

4.4 Copula families

Copulas can be distinguished in the Elliptical and Archimedean family. Elliptical copulas are the copulas with elliptical distributions, which have an elliptical form and therefore symmetry in the tails. Important copulas in this family are the Gaussian and the student's copula. The Gaussian copula is often used because of its simple form as can be seen in Example 3.2.

EXAMPLE 3.2 Gaussian copula

Assume there are two random variables X and Y where both variables are standard normal distributed, $X \sim N(0,1)$ and $Y \sim N(0,1)$ and let the correlation between X and Y be denoted by $\rho(X, Y) = \rho$.

Then the joint distribution can be written as the following copula:

$$H(x, y) = \Phi_\rho(x, y) = C_\rho(\Phi(x), \Phi(y)), \tag{9}$$

where Φ denotes the standard univariate and Φ_ρ the standard bivariate normal distribution function. (9) can be written as:

$$C_\rho(u, v) = \Phi_\rho(\Phi^{-1}(u), \Phi^{-1}(v)) = \int_{-\infty}^{\Phi^{-1}(u)} \int_{-\infty}^{\Phi^{-1}(v)} \frac{1}{2\pi\sqrt{1-\rho}} e^{-\frac{(x^2+y^2-2\rho xy)}{2(1-\rho)}} dx dy. \quad (10)$$

Archimedean copulas are widely applied, because they are not difficult to construct. In comparison to Elliptical copulas, Archimedean copulas have only one dependency parameter (instead of a dependency matrix) and have many different forms. In Table 3.1 you can see three often used Archimedean copulas.

Copula	$C_\theta(u, v)$	$\varphi_\theta(t)$	Parameter
Gumbel	$\exp\left(-\left((-\ln u)^\theta + (-\ln v)^\theta\right)^{\frac{1}{\theta}}\right)$	$(-\ln t)^\theta$	$\theta \geq 1$
Clayton	$(u^{-\theta} + v^{-\theta} - 1)^{-\frac{1}{\theta}}$	$\frac{1}{\theta}(t^{-\theta} - 1)$	$\theta \geq -1$
Frank	$-\frac{1}{\theta} \ln\left(1 + \frac{(\exp(-\theta u) - 1)(\exp(-\theta v) - 1)}{\exp(-\theta) - 1}\right)$	$-\ln\left(\frac{e^{-\theta t} - 1}{e^{-\theta} - 1}\right)$	$\theta \in \mathbb{R}$

Table 3.1: Summary of three Archimedean copulas.

The copulas in Table 3.1 are of the form: $C(u, v) = \varphi^{-1}(\varphi(u) + \varphi(v))$,

where φ is a decreasing function from $[0,1]$ to $[0, \infty]$, satisfying $\varphi(0) = \infty, \varphi(1) = 0$.

Archimedean copulas enjoy their popularity due to the following reasons:

- Many copula families belong to this class,
- they can be easily constructed,
- they possess nice properties (i.e. $C(u, v) = C(v, u)$).

4.5 Correlation vs. Copula

Now that we have covered the basics of copula we will compare this method with correlation.

As we have seen in the last chapter correlation comes in trouble when the random variables are not Normal distributed. In market risk this is not so much of a problem, but credit and operational risk are approximated with more skewed distributions because of occasional, extreme losses. In those cases copula is preferred over correlation.

Copula is in comparison to correlation invariant under transformations of the risks. For example, $\log(X)$ and $\log(Y)$ generally do not have the same correlation as X and Y . Hence,

transformations of our data can affect our correlation estimates. Proposition 3.1 shows that if you transform a random variable X , which has a copula function C , with a strictly increasing function T , then $T(X)$ also has copula C .

Correlation is a scalar measure of dependence; it cannot tell us everything we would like to know about the dependence structure of risks. Copula on the other hand works by joining together the marginal distributions to form a joint distribution. In this way the dependence relationship is entirely determined by the copula, while the scaling and the shape are entirely determined by the marginals.

Correlation is only defined when the variances of the risks are finite (see Equation (1)). It is not an appropriate dependence measure for very heavy-tailed risks where variances appear infinite.

Chapter 5 Application

5.1 Introduction

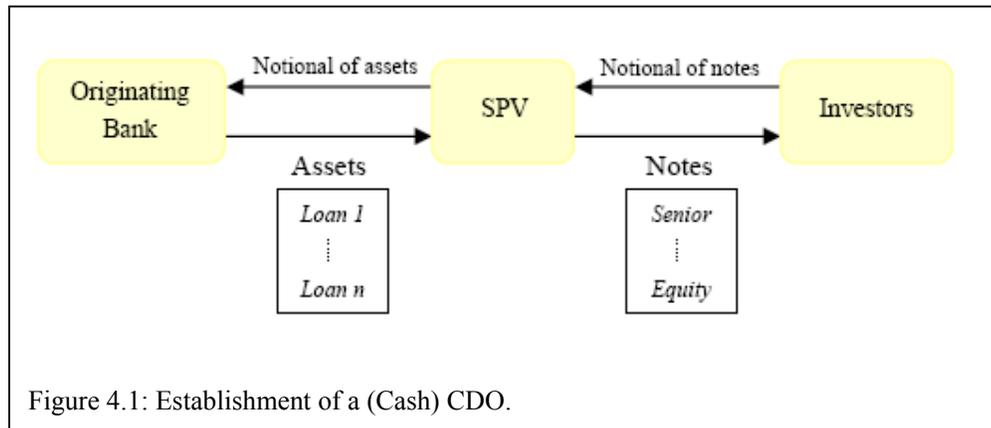
A copula model that has become a standard market model for valuating collateralized debt obligations (CDOs) is the Gaussian copula model. In this chapter we will show how CDO's can be modeled by a copula. In paragraph 4.2 we will go into the basics of Credit Derivatives and CDO's and in paragraph 4.3 we will show how an one-factor copula model can be applied to prize CDO tranches.

5.2 Credit Derivatives & CDO's

Before we discuss the copula model, some knowledge about Credit Derivates and CDO's is required. The purpose of these instruments is to allow market participants to trade the credit risk associated with certain debt instruments.

Credit derivatives can be seen as financial contracts between two parties. One party wants to gain protection against credit risk exposure (the protection buyer) and the other party wants to invest by selling protection (protection seller) in exchange for the cash flows of their investments (premium). In other words, credit derivatives allow one party to transfer credit risk to another in exchange for a fee.

CDO's are categorized as credit derivatives and generally secure a portfolio of bonds and loans by transferring the asset risk to capital market investors in return for the cash flows that are generated from the asset portfolio. This is done by dividing the portfolio into packages named securities, which are sold to investors. Not all securities carry the same risk.



The risk on the portfolio is distributed over several tranches, where each tranche represents a group of investors with different risk degrees. The assets that have the highest risk of default are classified in the lowest tranche and vice versa. So the higher the tranche the lower the risk that the corresponding assets will default. To determine the price of the tranches of a CDO the following are needed: default probability, default severity (or recovery) and default correlation.

The default correlation is the likelihood that the default of one asset causes the default of another and is much higher between credits within the same industrial sector. For example, the default of an US airline will have a high default correlation associated with other US airlines. There could even be a correlation associated with airlines in Europe, because they are exposed to many of the same influences. On the other hand the default correlation of the US airline associated with a telecom company in Europe would be low or even zero.

The structure of a general CDO can be seen in Figure 4.1. Under the CDO a special purpose vehicle (SPV) is created by a sponsoring organization (a bank or other financial institution). The SPV is a legal entity, with its own assets, liabilities and management, whose operations are limited to the acquisition and financing of specific assets. Often, the sponsoring organization acts as the manager of the SPV. The assets, which are generally bought from the sponsoring organization, serve as collateral for the securities that the SPV issued. The SPV funds these assets with the cash proceeds of the securities, which are sold in the capital market to investors.

5.3 The one-factor copula model

In the last paragraph we briefly discussed default correlations. In this section we will show how they can be modeled through the use of the one-factor copula model, see Equation (12).

Suppose a CDO contains assets from n companies, then the default time of the i th company is denoted by T_i . Its corresponding cumulative probability distribution that company i will default before time t is denoted by $Q_i(t)$. It is assumed that the time to default T_i is related to a random variable X_i , so that for any given t , there is a corresponding value x such that:

$$P(X_i < x) = P(T_i < t), \quad i = 1, \dots, n. \quad (11)$$

X_i can be seen as a variable that indicates when an obligator will default: the lower the value of the variable, the earlier a default is likely to occur.

The default of the n companies can then be modeled by the following copula model:

$$X_i = a_i M + \sqrt{1 - a_i^2} Z_i \quad \text{where,} \quad (12)$$

X_i the default indicator variable for the i th company;

a_i correlation of the i th company with the market, where a_i satisfies $-1 \leq a_i \leq 1$;

M a market factor, which is the same for all X_i ;

Z_i individual component affecting only X_i .

The two stochastic components of X_i , M and Z_i 's, have independent probability distributions. In the case of the Gaussian copula model both components are Normal distributed. The correlation of the i th company with the market is denoted by a_i , where a_i satisfies $-1 \leq a_i \leq 1$. Because the correlation is with respect to the common market factor, the correlation between two trigger levels for names i and j is given by $a_i a_j$.

The one factor copula model maps X_i to t_i on a ‘‘percentile to percentile’’ basis, which means that the 5% point on the X_i distribution is mapped to the 5% point on the t_i distribution. In other words: when X_i is small, the time t_i before default is also small.

When the cumulative distribution of X_i is denoted by F_i and the cumulative distribution of Z_i (assuming that the Z_i are identically distributed) by H , then, in general the point $X_i = x$ is transformed to $t_i = t$ where $x = F_i^{-1}(Q_i(t))$ or $t = Q_i^{-1}(F_i(x))$.

We can rewrite (12) into $Z_i = \frac{X_i - a_i M}{\sqrt{1 - a_i^2}}$. Since H is the cumulative distribution of Z_i and given that the market variable $M = m$, it follows from Equation (12) that:

$$P(Z_i < x | M = m) = H\left(\frac{x - a_i m}{\sqrt{1 - a_i^2}}\right), \quad (13)$$

and the conditional default probability is

$$P(t_i < t | M = m) = H\left(\frac{F_i^{-1}(Q_i(t)) - a_i m}{\sqrt{1 - a_i^2}}\right). \quad (14)$$

The idea behind the copula model is that we do not define the correlation structure between the variables of interest directly (t_i), but we map the variables of interest into other more manageable variables (X_i) and define a correlation structure between those variables.

Conclusion

The main object of this thesis was what the advantages of using a copula over correlation were to model dependencies between variables in financial risk models.

Pair wise correlation and the marginal distributions of a random vector is then not enough to determine its joint distribution. The copula works by joining together the marginal distributions to form a joint distribution. In this way the dependence relationship is entirely determined by the copula, while the scaling and the shape are entirely determined by the marginals.

When we look at the distributions of financial risk we see that market risk is often approximated with a Normal distribution and therefore does not cause much of a problem. Credit and operational risk on the other hand are approximated with more skewed distributions because of occasional, extreme losses. Correlation comes in trouble when the random variables are not elliptically distributed. The performance of the copula does not depend on the fact if you are dealing with elliptical distributions or not.

Add the fact that copulas possess handy properties and the winner of the race is copula.

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