Real Options Valuation in the Airline Industry

BMI Paper
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Preface

This BMI paper is one of the last steps towards the Master of Science degree in Business Mathematics & Informatics at the VU University Amsterdam. The purpose of this paper is to analyze and solve a problem on the crossing of Business, Mathematics & Computer Science.

The goal of this paper is to provide an advanced methodology in airline fleet planning for the decision of buying or leasing aircraft. Research was conducted at KLM for the department Fleet Development & Aircraft Trading and was done in light of replacing 5 Fokker 100s with 5 Embraer 190s. Although methodology is designed for this event, the results are generally applicable for financing decisions and not limited to a specific industry.

I would like to thank Sandjai Bhulai despite his busy schedule for making the necessary time for supervising this paper from the VU University Amsterdam and Marjolein van Reisen for giving me the opportunity to do such an interesting project and providing the necessary support from the Fleet Development & Aircraft Trading department.

Chuck Yuen Liu, April 2011

Executive summary

We will show how to decide between leasing and buying aircraft using real options analysis. The theoretical grounded methodology takes into account different scenarios and different options in a correct and fair way.

The choice for a firm above a lease is a matter of commitment, the more commitment is chosen the less flexibility (implicit and explicit options) is available to management. At the other hand in normal market circumstances, more flexibility costs more on the long term. We provide a general framework in which the financing decision is made transparent and a new decision rule is introduced to make the tradeoff between profitability and flexibility. The financing decision will eventually incorporate all possible scenarios of the future concerning the factors that determine the profit or loss.

Main Goals:

- Provide a sound and objective guide to the financing decision
- Decompose the financing decision
- Provide a framework to do the necessary calculations
- Show how options can be valued
- Determining the optimal financing mix

Furthermore the methodology is implemented in an Excel valuation tool in which calculations are automated. Additionally a manual gives insight in the usage of the tool and validation Excel sheets provide the validation of the calculations mimicking in a step by step approach mimicking the final valuations.

Contents

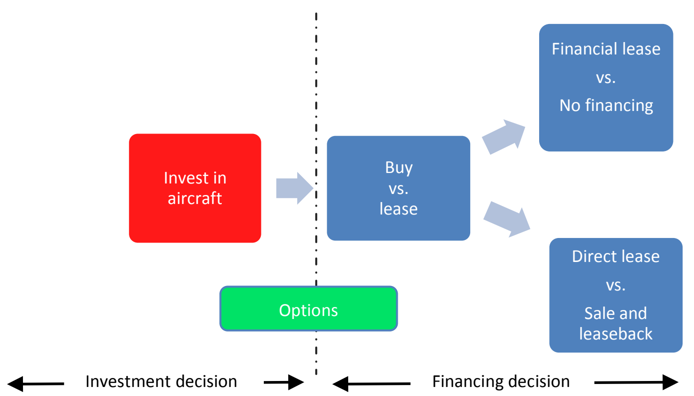
1 Introduction

1.1 Background

The department Fleet Development & Aircraft trading is responsible for the procurement, financing and the sale of aircraft within KLM. Procurement of (new) aircraft is an investment decision, a decision that has been made by management in the light of bringing positive return to the company. This means a profitable investment in terms of euros and/or a better strategic position. When aircraft needs to be replaced due to aging, the situation is the same and the following question arises: How many aircraft is KLM going to invest/replace? KLM could choose for a smaller number (reducing the fleet size), an equal number (keeping the same fleet size) or even a larger number of aircraft (expansion of fleet). The next step in the process is choosing the financing construction, that is the choice between buying (firm) and leasing. In this paper the focus is on unravelling and making the financing decision.

The choice of buying or leasing depends on the advantages and disadvantages accompanying a firm or a lease. For each, one can distinguish two financing constructions. Buying gives us the choice of going for a financial lease or no financing, i.e. paying the aircraft in steps to a bank willing to finance or paying the full amount before delivery to the original equipment manufacturer (OEM) respectively. No financing has a big impact on the liquidities' position (and gearing ratio), as the entire sum is paid before delivery (pre delivery payments) and at delivery. Buying results in ownership over the aircraft, in contrary with leasing where the aircraft returns to the lessor at the end of the leaseterm. In the latter case the risk or uncertainty of the residual value of the aircraft lies with the lessor.

Leasing consists of two choices that are similar. KLM could directly go to the lessor and lease an aircraft which is called an operational or direct lease. The other way is first purchasing an aircraft, immediately selling it back to a lessor with the agreement for a direct lease. This is called a sale and leaseback construction. The advantage is that KLM can customize the aircraft from the OEM, while in the direct lease there might be some restrictions as one is dependent on the fleet of the lessor. Next to that it could be attractive to lease a relative short period when KLM expects in the near future developments will change our choice, for example when a new fuel efficient aircraft type is going to enter the market.



1.1 red – investment decision, green – delaying investment – blue – financing decision

In reality the financing decision is even more complicated. When one chooses to firm, one can acquire option(s) from the OEM to purchase additional aircraft. The decision to invest can then be delayed at some moment in the future, typically 1-2 years later. In 1-2 years time, one can decide to exercise the option and acquire the extra aircraft for a predetermined price. This purchase option is valuable, if the investment decision can be made better in 1-2 years of time.

Options are not solely available in firms, in the direct lease (and sale & lease back) agreement we could obtain an option to extend the direct lease with for example 2 years for a predetermined rate today. We will call such an embedded (explicitly stated) option, an extension option.

As u might have noticed investing in options is neither an investment nor a financing decision, as the decision to invest (and consequently the financing decision) in an aircraft is not made yet. But an option is valuable as it gives us opportunities and no obligations, we must value these options just as any investment for full picture. The choice to invest in options (as options typically costs money) is an investment decision *in the option* but not *in an aircraft* (yet).

We have now made an overview of 5 ways to fill in our financing decision. We give an overview of the available constructions we will take into account in this paper:

1.2 7 Financing constructions

The available financing constructions and the accompanying conditions like lease prices and durations are all dependent on today's market consisting of the Original Equipment Manufacturers (e.g. Boeing, Airbus, Embraer) & the lessors and the negotiations between KLM and these market participants.

The eventual choice will depend on the uncertainty of the return on an investment and the price of a contract with one of the market participants. The more uncertainty there is, the more flexibility is needed for management to act accordingly in the then prevailing economy in the future. The flexibility lies in the duration of the financing construction and options, a shorter lease term and/or options will give more flexibility in readjusting the fleet size over time.

When for example choosing a 6 year direct lease over a 12 year direct lease, we get more flexibility as we can reconsider the investment after 6 years. In other words we only committed to direct lease payments for 6 years instead of an investment of 12 years which includes 12 years of costs. Choosing for 6 years however, will likely result in a higher monthly payment. This will lower downside risk or losses in a bad state of economy but will limit upside potential in a good state of economy.
The last important difference between the buying and leasing is the residual value risk that comes with buying. Predicting the residual value in 20 years time is hard as many factors have to be taken into account.

Clearly, there are pros and cons for each financing choice and it is the goal of this paper to evaluate the pros against the cons in an objective way. In this paper we take on a mathematical approach using real options valuation to solve this problem.

1.2 Problem statement

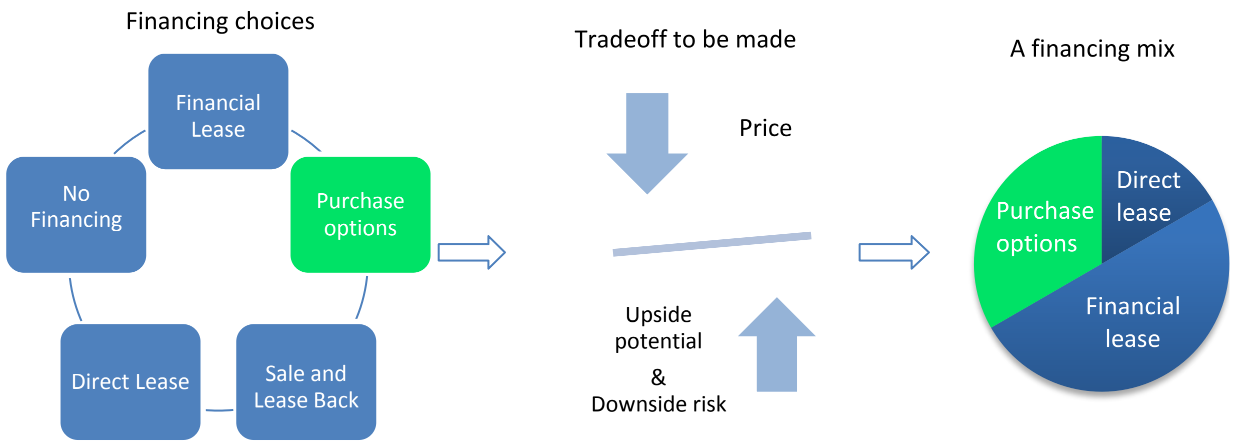
Now that we have a full picture of the available constructions, we are ready to formulate the problem statement. The situation is as follows: The investment decision is made for x airplanes and we are up to the challenge how to allocate the number of aircraft to the five different constructions, where each construction takes up one allocation.

<u>Example</u>

KLM has decided on the investment of 5 Embraer 190's, to replace 5 old Fokker 100's in operation. The question is how to allocate the 5 "slots" over the available constructions.

- One possible allocation or mix of financing is:
  all 5 Direct Lease for 10 years (without extension options).
- Another possible allocation is:
  1x Direct lease for 8 years (without extension options),
  3x financial lease for 20 years and 1 purchase option maturing in 1 year for delivery in 2 years time from option maturity.
- Another possible allocation is:
  3x Sale and lease back for 8 years $0.25 million per month with each an extension option to extend by 2 years and 2x a purchase option.

KLM has these three financing mixes available and has to make a decision which one to choose. The choice will depend on the price, the flexibility a mix provides, upside potential and downside risk for the final choice.



1.3 Find the ideal financing mix, identify the available constructions, measure their pros and cons and make the choice.

We formulate this in a formal research question.

Research question

• *What is the optimal mix of financing for the acquisition of a fixed number of new aircraft given its current fleet and today's financing market?*

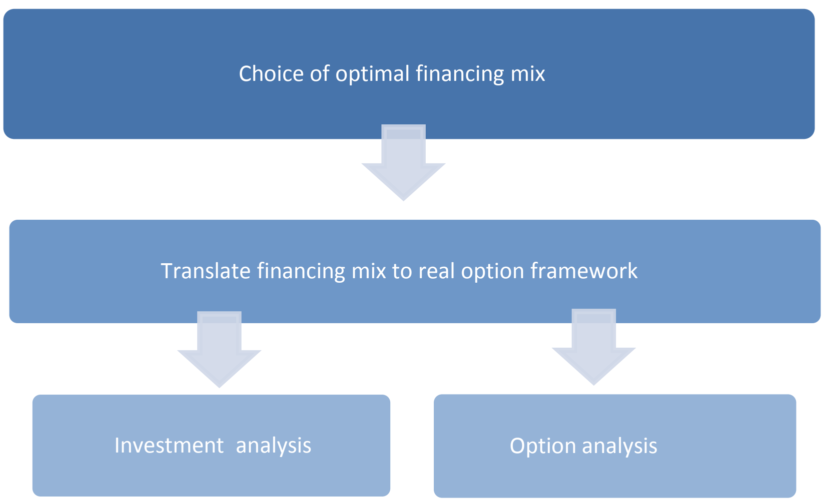
With the following sub questions:

◦ *How to reformulate the financing decision into the real option valuation framework?*
◦ *How do we define optimal?*
◦*How to value options?*

The valuation of the different financing construction is not possible with the well known Net Present Value (NPV) or Discounted Cash Flow (DCF) method. The reason is that options are differently valued than investments and the NPVs of different lengths of contracts cannot be fairly compared to each other. We can translate the financing question into an option framework and use financial options theory to value the options properly, in the next chapter one will also see that it resolves the unequal "investment horizons". The valuation of the options in the option framework will lead us to the optimal financing mix.

1.3 Structure of report

The structure of this paper is as follows:



Translating the financing mix to the real option framework is treated in chapter 2, in chapter 3 investment analysis is treated and the mathematical approach to option analysis is treated in chapter 4, 5 & 6. In chapter 7 we come to the application of the theory to come to the optimal financing mix using an alternative (insightful) way of option valuation. The alternative approach is implemented in Excel. We finish with our conclusion in chapter 8 and in chapter 9 a word on further research directions.

2 Translate financing mix to real options framework

In this chapter we will answer the first sub question:

- *How to reformulate the financing decision into the real option valuation framework?*

2.1 Choices & Assumptions

- Although it is possible to do a sale and leaseback for any aircraft in ownership, we will assume here that this can only be done at the moment of the financing decision, today.
- Escalation of lease payments is assumed to be fixed, thus non-stochastic.
- The analysis assumes aircraft are delivered straight away at initiation of the contract.
- Options are decided on at maturity and delivery of the optional aircraft is instantaneous too.
- When ownership is chosen, we assume the aircraft will be operated on for its natural lifetime typically 20 years after which the aircraft is sold against market value.

2.2 Investment horizon

To compare financing mixes in a fair way, we need to compare financing constructions over an equal time interval. It matters whether we invest for 8 years or for 20 years and a fair comparison is needed. There is a 12 years difference between the two, after the 8 years investment one can make the decision to reinvest or not. These 12 years are an (implicit) optional investment, and by seeing it as an option one can value it accordingly. Thus the total period to consider is 20 years called the investment horizon. If we have an 8 year direct lease contract and there are 2 extension options available for 2 extra years, the investment horizon of this contract is 8+2+2 = 12 years. Comparing it with a 20 year investment, we would value the remaining 8 years of the direct lease as an option again.

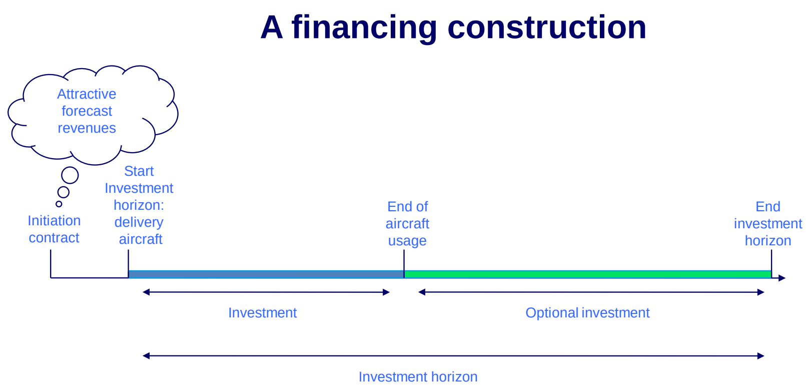
We will formally define the above:

<u>Definition: Investment horizon of a financing construction</u>
The Investment horizon of a financing choice is the operation time plus, if present, the extension time for optional operation (due to options).

The next step would be to compare financing mixes, we take the maximum of the investment horizon of all the available financing constructions and consider that the investment horizon for fair comparison.

2.3 Comparing financing constructions

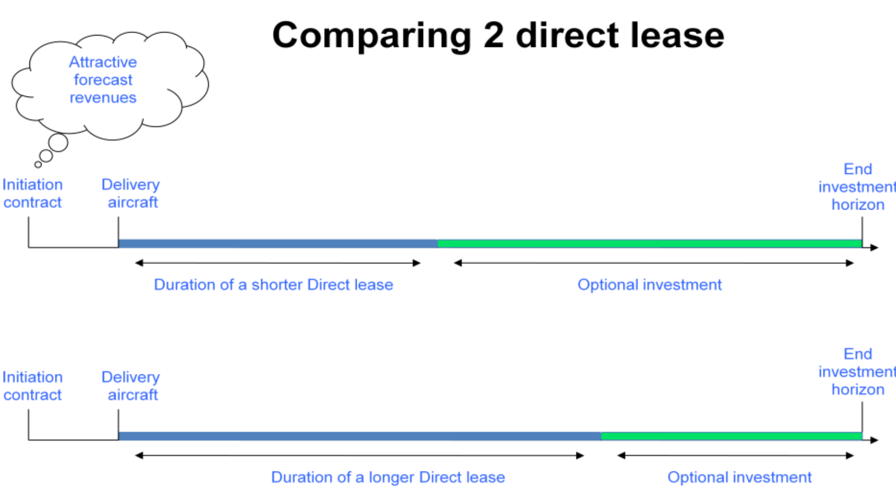
The real options framework is simple. By looking at the investment horizon for all the contracts we can fairly compare them. By definition one financing construction has the length of the investment horizon of the financing mixes. The other contracts are equally compared by realizing we have an opportunity, an implicit option, to reinvest at the end of a contract. This implicit option is a reinvestment option.



Example 1:
Consider the situation where we want to compare two direct lease contracts with lengths of 6 and 12 years in the contract.

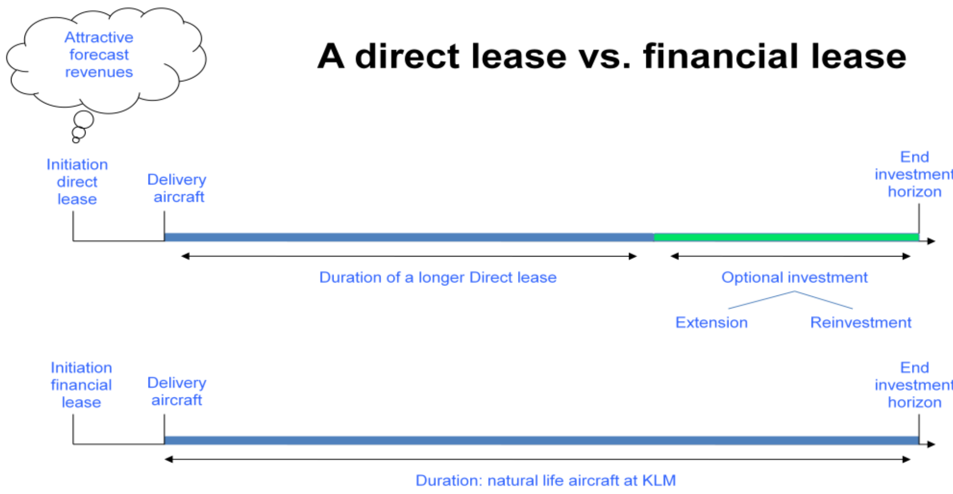
If we would have invested in the 6 years contract, we have an option after 6 years to do a reinvestment for the remaining 6 years. In practice this is a bit before the 6 years, as the contract negotiation and delivery time needs to be taken into account but for convenience we assume this is exactly after 6 years. We can either decide to replace the aircraft or decide not to reinvest. This will depend on the economic outlook and the cost of ownership for a replacement airplane. In case the aircraft is forecasted in year 6 to be loss making then the reinvestment option will not be exercised and the fleet size will decreases by one. At the other hand when market circumstances present themselves favorable we can use our option and reinvest in the remaining 6 years.

If we have chosen the 12 year direct lease, there wouldn't be flexibility after running 6 years in the contract. We would be obliged to continue operating the aircraft and satisfy the lease payment in both favorable and unfavorable market circumstances.



Example 2:
Consider the situation where we want to compare a direct lease contracts with a length of 12 years against a financial lease with a duration of 16 years.

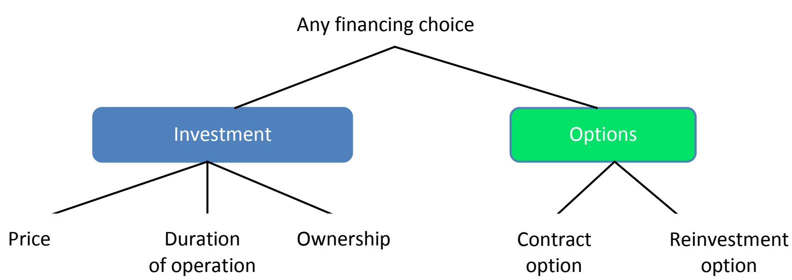
The duration of the financial lease only states the period to pay and the aircraft at the end of the contract is still in ownership of the airliner. Thus the length of the financial lease contract is irrelevant. Assuming the aircraft has a lifetime of 20 years, the investment horizon to consider is 20 years. If we would choose for the 12 year direct lease we would have a reinvestment option at year 12 for a period of 8 years. This is flexibility which the financial lease contract doesn't offer.



2.3.1 Characterizing different financing constructions

All the financing constructions can be decomposed in an investment (indicated blue) and option component (indicated green). The investment component consists of the price or lease rate to be paid, a duration the aircraft can be operated and whether ownership is obtained or not. The duration available for operation is in the case of a direct lease or sale and lease back, the duration of the contract and in case of a firm (i.e. "No financing" or "Financial lease") the natural lifetime of the aircraft (typically 20 years) before KLM would sell the aircraft. The ownership of an aircraft is obtained only in case of a firm. In case of ownership, one receives the sale proceeds of the aircraft at the end of its natural lifetime.

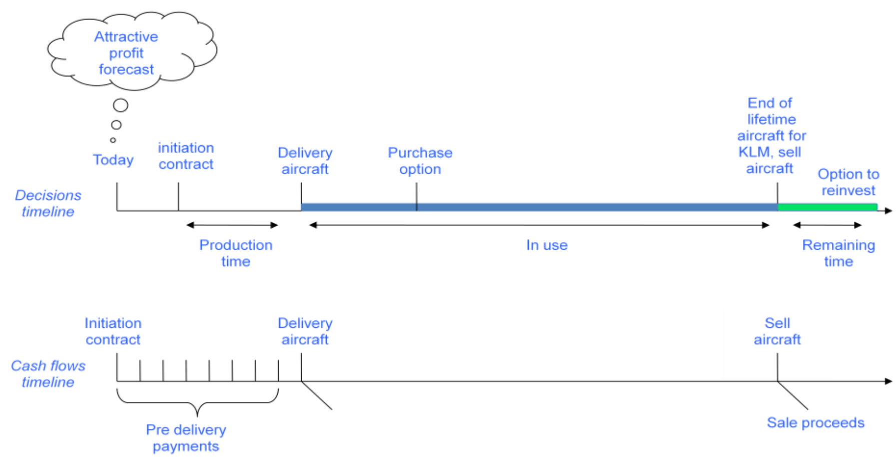
Options can be an extension option in case of a direct lease, a purchase option in case of firm or a reinvestment option in case the operation duration of a specific contract is shorter than the investment horizon. The investment can be valued with the traditional NPV method, while options need to be valued appropriately using discussed in later chapters.



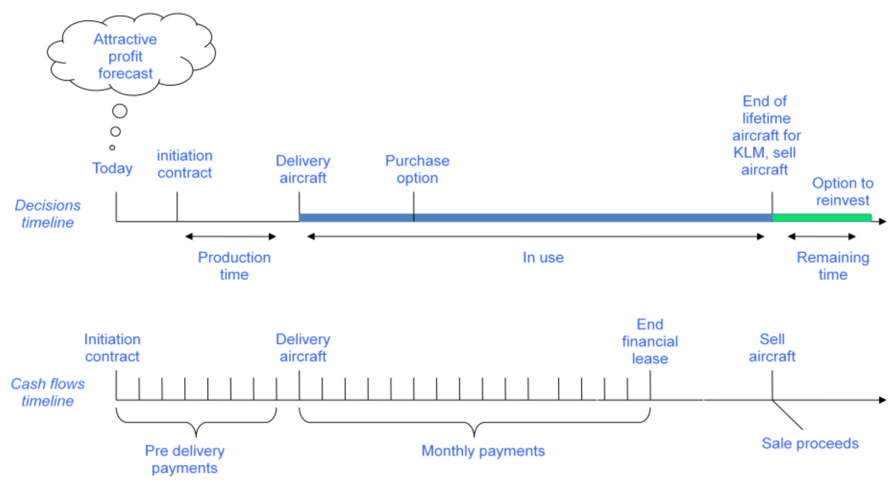
2.4 Overview of the decompositions of any construction

In this section we give a complete overview of all financing constructions. We will visualize the cash flows accompanying each financing construction. That is the payments for the right to use an aircraft (e.g. lease payments) and the sale proceeds when dealing with a firm. Additionally the direct lease and sale and leaseback are similar in our analysis, we assume that the sale and lease back terms from the OEM and lessor to can be determined and is executed today. So that we don't take into account the opportunity to firm and sale and lease back and aircraft in a later stadium. That is a firm stays a firm. As a consequence the difference between a direct lease and sale and lease back lies in the availability of purchase option(s) for the latter one.

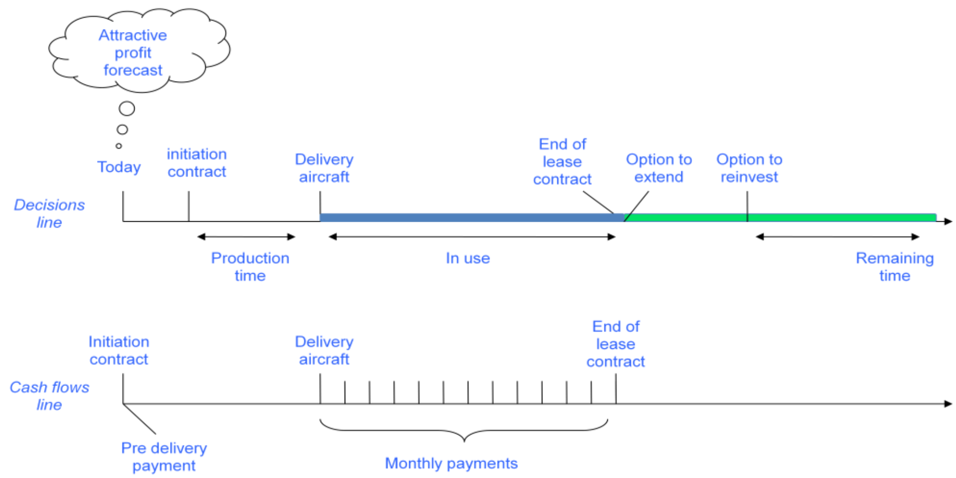
No Financing visualized



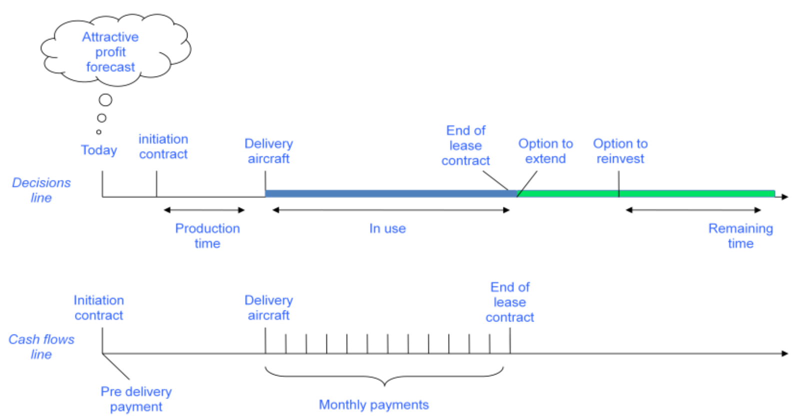
Financial lease visualized

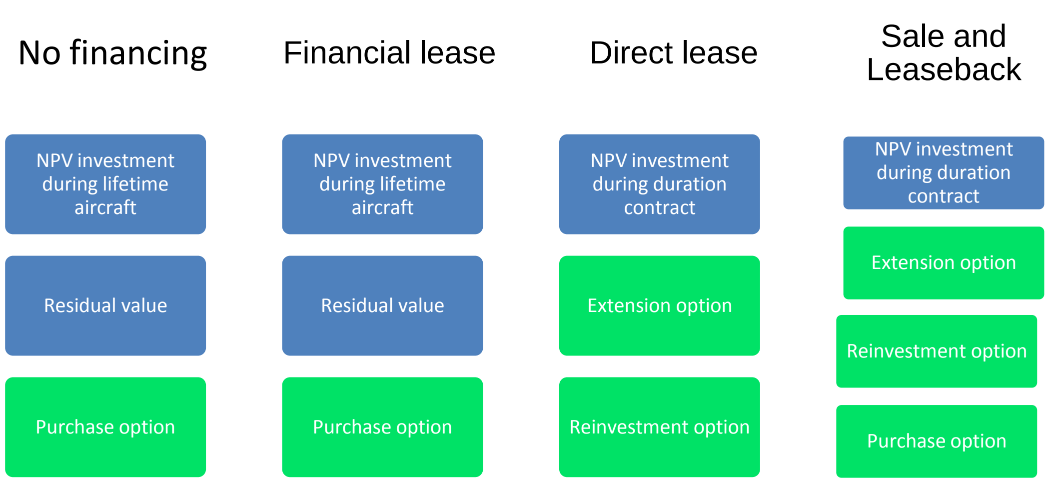


Direct lease visualized

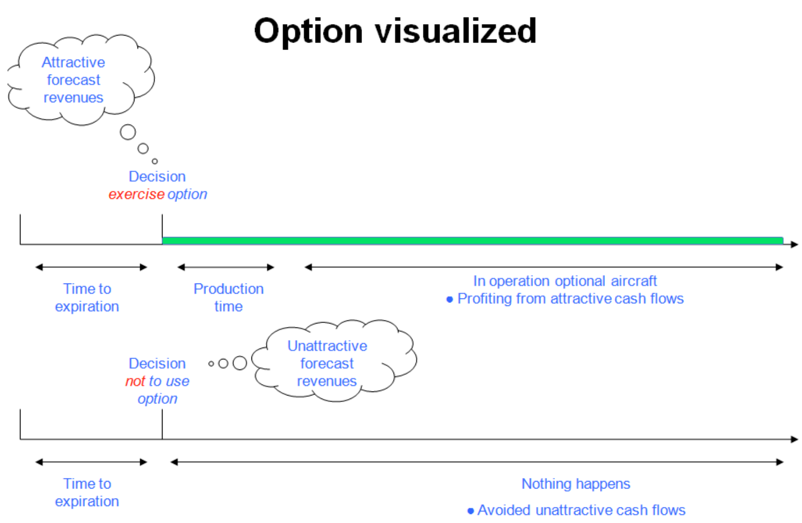


Sale and Lease back visualized





Finally we conclude this paragraph with the visualization of an option. As can be seen we have assumed that whenever the option is exercised at maturity, the optional aircraft is delivered immediately.



2.5 Real options framework

Having gone through the previous sections, we have all the tools to reformulate the allocation or financing question into the real options framework. If the duration of the available contracts were the same and no options were present, then the Discounted Cash Flow method (DCF) or the Net Present Value (NPV) method is the right choice.

In the previous section we saw the difference in time as a reinvestment option. Valuing all other options we can fairly compare and value financing constructions. We can integrate the value of the option with the NPV to come to the financing decision. The fixed component is here the initial commitment to the investment, so in a direct lease this is the duration of the contract while the variable component consists of all the options that come available given that one has chosen for a specific contract. With the explicit options being the purchase and extension options and the implicit options being the reinvestment options.

2.5.1 Expanded NPV

The integration of both the NPV of the project during the contract time with its options leads us to the expanded NPV. The static part or fixed part of a financing construction valued with the NPV method is called the static NPV, this name follows from the fact it fails to incorporate the flexibility in options and only values the fixed component of a contract. The value of the implicit and explicit options in the financing construction is the option premia. Adding the values up, we get the Expanded NPV.

Definition Expanded NPV
Expanded NPV = Static NPV + Option Premiums

The expanded NPV was introduced by Gehr in 1981 and is used by Trigeorgis (2001).

We would calculate the expanded NPV for each financing mix, and just like a normal NPV analysis we choose the project that has the highest NPV, but now the expanded NPV which also incorporates options.

An example:
If today the investment decision is made for 5 aircraft, two different financing mixes could be:

- 2 direct Lease for 12 years, 2 financial lease for 20 years and 1 purchase option (maturity in 1 year).
- 4 direct lease for 10 years and 1 financial lease for 20 years.

Translating this into the real options framework:

- Mix 1: 2 direct lease + 2 options starting in year 12 to reinvest for 8 years + 2 financial lease + 1 purchase option.
- Mix 2: 4 direct lease for 10 years + 4 options starting in year 10 to reinvest for 10 years direct lease + one financial lease for 20 years.

This would yield the following expanded NPV:

Expanded NPV mix 1: 2x NPV direct lease for 12 years + 2x Option value direct lease in 12 years time + 2x NPV financial lease for 20 years + 1x purchase option value
Expanded NPV mix 2: 4x NPV direct lease for 10 years + 4x Option value direct lease in 10 years time + 1x NPV financial lease for 20 years. The obvious decision is choosing the financing mix which holds the highest expanded NPV and can be seen as the expected added value given one chooses for the corresponding financing mix. This is also the answer to the second sub question, how optimality is defined. The choice here is made on the highest expected return, the highest expanded NPV among all possible financing mixes.

In general it holds that the higher the uncertainty, the higher the options are in value. The choice will depend on the amount of uncertainty. The lower the uncertainty the more the expanded NPV will move towards firm (that is going for ownership: No financing or Financial lease), since the options will be worth less. The higher the uncertainty the more the expanded NPV will tend towards flexibility, the options will become more valuable (as having the choice to invest is available) so that shorter contract durations (gives reinvestment options) and purchase options and extension options are preferred.

3 Investment analysis

In this chapter we will view investments from a financial theoretic point of view.

In investment analysis one always has to make the tradeoff between risk and return. In general the more risk one is willing to take, the higher the promised or expected payoff is. Investors get compensated for risk. The starting point of any investment analysis in financial literature is identifying a so called risk free rate, which is the rate of return of an investment without any risk. One could say that putting money into a savings account is safe and risk free. In general banks are reliable and even when a bank defaults, most governments still guarantees the savings. The interest one receives can thus be considered to be the risk free rate. For corporate investments, sovereign bonds are considered risk free and more specifically triple A bonds like German (or Dutch) bonds. We will make the assumption that the risk free rate exists and equals the rates of German bonds. One could still debate between bonds of different lengths, we follow market conventions by taking 1 year bonds.

As a consequence the rate of return of any investment project that has no risk, is the risk free rate. Any risk free cash flows in the future should be discounted by the risk free rate to get the fair value of the investment today.

In the corporate world cash flows of investment projects are far from certain. This has given the rise of different risk adjusted discount factors or risk adjusted rates of return like the weighted average cost of capital (WACC) within corporations. Observe this is always higher than the risk free rate, this rate of return can be seen the required expected rate of return for an investment within the corporation.

The WACC depends on the industry and also varies across firms, as different industries have different degrees of risk and different firms have different risk appetites. One can imagine that the risk of a budget airliner/low-cost carrier is different than from a legacy carrier (which targets different customers). Within KLM two WACCs or discount rates are used for investments in aircraft, 4,5% for the replacement of fleet and 9% for the expansion of fleet. The higher required rate of return for expansion of fleet underwrites the higher risk and thus results in more uncertain cash flows when one expands the fleet.

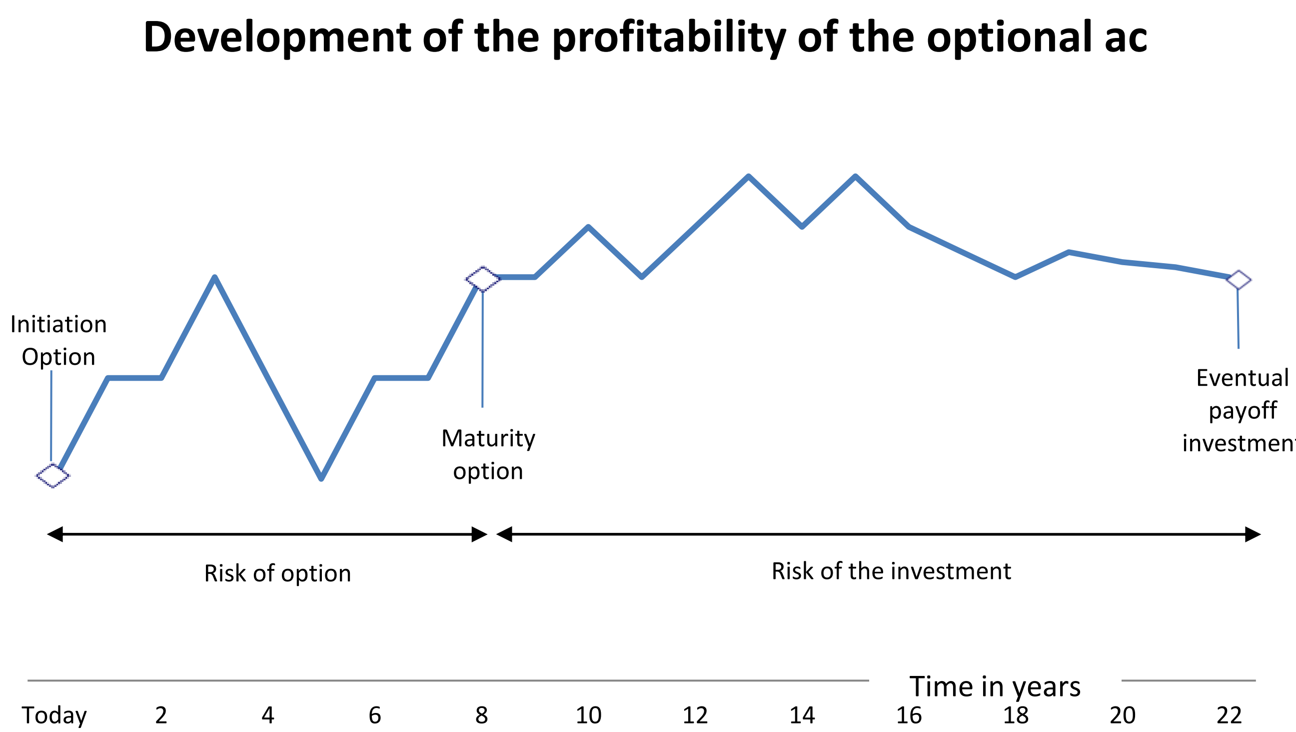
In the next chapter we will show how we are going to deal with the risk of the option or the uncertainty of the cash flows associated with the option. One has to distinguish between the risk or uncertainty of today (the initiation of the option) till the maturity of the option and from this maturity till the end of the (optional) investment. On maturity of the option, we cannot postpone the decision anymore whether to invest or not. Using the risk adjusted WACC one can calculate the added value of the investment at that moment. One can view the calculated added value as the payoff of the option at maturity whenever it is positive. When this amount is negative, the optional aircraft is not exercised. No value is added but also no costs are incurred. The question remains what to do with the uncertain payoff between today and the maturity of the option. It would be incorrect to use the risk free rate as the payoff is uncertain. Essentially there are two ways to resolve this:

1) Use a risk adjusted discount rate, e.g. the WACC at 4,5%. State the risk of the optional aircraft investment equals the risk on the replacement of an aircraft.
2) Use financial option valuation.

In the later chapters we explain the mechanics of financial option valuation. We will prove that the payoff of an option can be replicated with an hedging portfolio. The consequence is that all risk is hedged away. As a result we can value the option by discounting the value of the replicated portfolio at maturity (which is the payoff of the option at maturity or the payoff of the investment at maturity), with the risk free rate. In general the risk free rate is the only discount rate that is used in valuing options using the replication argument.

Although we explain financial options valuation, for the implementation we will choose the other option. That is pricing the risk with another risk adjusted discount rate like the WACC. Financial options theory is highly complicated and hard to fully understand by first encounter, while the WACC methodology can be explained more transparent. Nonetheless, we will be needing a part of the theory of financial options valuation to do the first option.

Below we show we plot of the value of the option over time, once can observe that this changes over time and secondly the volatility decreases the closer to the end. That is when the option option is exercised at the maturity of the option, the revenues and costs will be more and more set. The plot also show there is two different risks or uncertainties having an option. The first risk is the start time of the option till maturity, which is the uncertainty of the value of the option while having the opportunity to exercise or not. And second is the risk or uncertainty once the option is exercised, which is the risk of the investment itself.



3.1 Investment valuation

At the time the decision about the investment has to be decided upon, the project to be invested in needs to be valued. One needs to make an analysis of the expected cash flows, that is the revenues minus the costs for the duration of the investment and discount it back using the appropriate risk adjusted discount factor WACC. This is normally done on a yearly basis and in this paper the analysis will also be. The revenues and costs will be zoomed in and is the basis for the valuation of the investment and the option. Note that in case of a firm, the residual value of the aircraft at the end of its natural lifetime needs to be taken into account in addition to the revenues and costs and is a risk factor as well since there is uncertainty about the residual value of an aircraft at the end of its lifetime.

3.2 Revenues

The income stream of an aircraft is determined by the revenues from ticket sales. In the aviation industry one denotes the revenues in the realized revenue per passenger per kilometre (RPK) and in yield per RPK. In this context RPK is not an amount from revenue but the actual realized occupied seats that travel a kilometre. Yield means the actual revenue or money units that the passenger has paid (by average, think about business and economy class) for this one seat per kilometre. From this it follows that the revenues are given by the following equation (all variables are per year):

$$\boldsymbol{Revenues = RPK * yield}$$

$where\ RPK\ are\ denoted\ in\ passenger\ kilometers\ per\ year$
$yield\ in\ €\ per\ RPK\ per\ year$

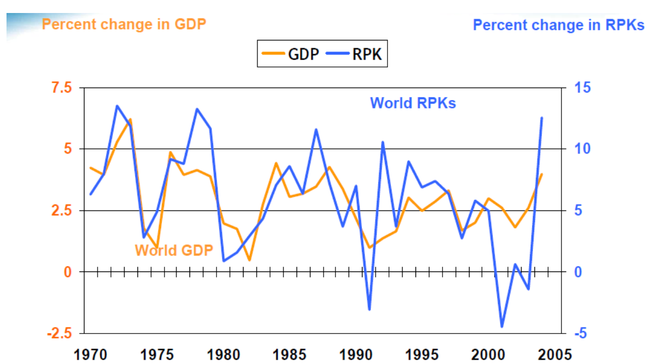
Once the investment must be decided on, the then prevailing yield and RPK in that specific year is assumed to be known.

The yield will depend on the type of aircraft and the specific area of operation of the type of aircraft. For example, the Embraer 190 has a yield that is for 1/3 dependent on the yield of intercontinental flights and 2/3 dependent on the yield of regional flights. Thus the yield for this type of aircraft can be computed to be the weighted average of the intercontinental and the regional flights. The two yields from the regional flights and intercontinental flights are the weighted average of the yields from the economy classes and business classes.

Furthermore we assume a constant network utilisation, that is the planned number of kilometres and number of flights each year for the total fleet of this type remains constant. Also all planned flights (and their corresponding kilometres) are assumed to be carried out. So that in the valuation we assume that aircraft keep flying even for example when the yield fall.

3.2.1 Modeling the relationship between RPK, GDP and Yield

We acknowledge the fact that the RPK generated by the fleet is dependent on the world gross domestic product (GDP). When GDP goes up, the RPK (Revenue per passenger kilometre) generated by KLM's fleet goes up too.



3.1 Source Boeing 2008

Next to GDP, RPK is dependent on the yield. Whenever yield goes up (tickets prices rise) demand expressed in RPK will go down. We model this (deterministic) relationship in the following equation.

$$RPK = GDP^{\gamma} * Yield^{-\varepsilon}$$

$$where\ \varepsilon = elasticity\ of\ yield$$
$$yield\ \gamma = elasticity\ of\ GDP$$

Where the elasticity's are assumed constant.

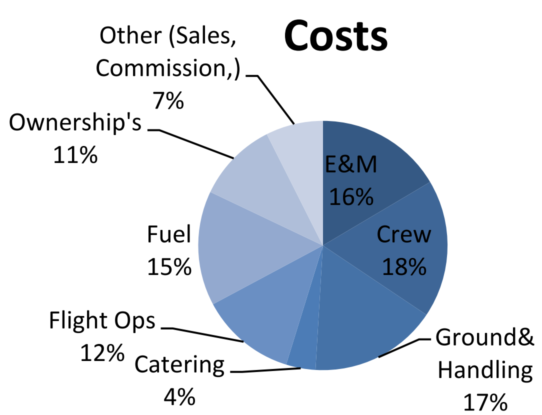
The relationship tells us that when GDP goes up with 1%, the RPK will go up by approximately γ%. And when yield goes up with 1%, the RPK will go down with approximately ε%. Approximately, since the changes are log proportional. Given a yearly change of the GDP and yield and the corresponding elasticities γ and ε, we can determine the RPKs. From this it follows that the revenues can be determined by GDP and yield, which are unknown for the future from today's point of view. Thus these are stochastic.

3.3 Costs

Likewise with the revenues, we take the total costs per year and assume constant network utilization and following that a constant fuel consumption.

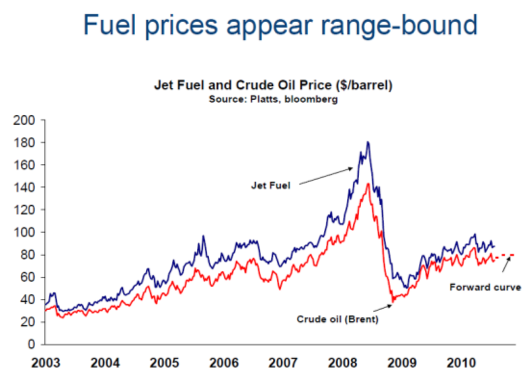
$$Total\ costs\ =\ E\&M\ +\ Crew\ +\ Ground\ \&\ Handling\ +\ Catering\ +\ Flight\ Operations\ +\ Fuel\ +\ Aircraft\ Ownership\ +\ Other$$

Where the costs are divided in:
E&M = engineering and maintenance.
Flight operations = operational costs including take off and landing charges by Schiphol and other airports.
Ground & Handling = costs incurred with baggage handling
Catering = costs occurred with meals by passengers and crew.



3.2 costs pie based on 07/08 Fokker 100, E&M heavily dependent on age of ac

From an investment point of view, the fuel costs are the biggest risk factor. We assume that E&M, Crew, Ground & Handling, Catering & Flight Operations are deterministic costs and in jet fuel lays the uncertainty when considering the costs of the investment over the investment horizon. Thus jet fuel becomes a stochastic factor to consider. The jet fuel price is primarily influenced by the crude oil price quoted in financial markets. The costs excluding fuel and ownship will be put under either fixed or variable costs of which the total variable costs are divided equally over all the aircraft. We could think of the fixed costs as costs due to ground staff which does in essence not depend on the fleet size (under small mutations) while the variable costs like catering is dependent on the fleet size.



3.3 Source IATA

We will assume that the correlation between jet fuel and crude oil prices is perfect and work with a (constant assumed) conversion factor to jump from jet fuel consumption to crude oil consumption.

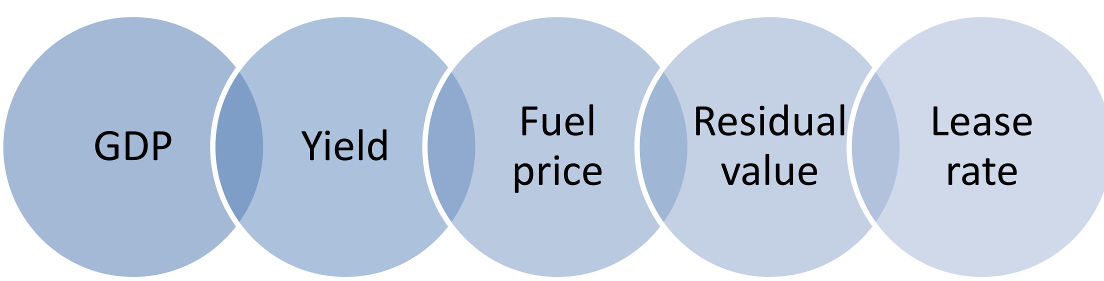
For the ownership costs we need to distinguish between the ownership costs if we would make the investment today and in the future. Today the ownership costs are known. These stems from the lease conditions and the buying terms which are the results of negotiations between KLM and the corresponding lessors, financing banks & OEMs. In contrary the ownership costs in the future are uncertain, the profit from an (implicit) option to reinvest depends on the ownership costs by that time. While the ownership costs for purchase options and extension options are part of today's contract.

$$Total\ costs\ = Fuel\ costs + \ Ownership\ \text{costs}\ + variable\ costs + fixed\ costs$$

For today's investment we would need to give an estimate of all three components for the time of the investment. The biggest uncertainty lies with the fuel and a prediction of the fuel costs in the future needs to be made. This depends on the beliefs of experts and/or in combination with a model to predict crude oil prices. Since from the costs perspective, oil and ownership costs are the biggest risk factors we model these two as risk factors while taking the variable and fixed costs deterministic (constant per plane). The ownership costs in the future are stochastic too and we will simulate a direct lease rate (where we don't distinguish between lease rates of different lengths) to get a proxy for the ownership costs in the future.

3.4 The risk factors

We have determined that the risk for the investment in aircraft are determined by 5 different risk factors which are risks because the development of these factors determine the attractiveness and unattractiveness and are uncertainty (stochastic). Tthey are unknown from the perspective of today. Quantification of risk is done by dispersion and the volatility is one common to quantify risk. The later chapters on option valuation will make use of the volatility and see this a parameter in the stochastic model.



3.4 The volatility of the option is the volatility of all these (correlated) factors

These factors change all the time and the investment decision is primarily based on these factors i.e. the projected revenues and the forecasted costs.

3.4.1 Determining the investment and option value

We look at the investment from a financial perspective, the value of the investment is determined by discounting future cash flows with an appropriate discount rate. The future cash flows are determined by the 5 risk factors and are unknown/stochastic at the moment the investment has to be decided on. At that moment one has to forecast the 5 factors for the investment horizon to determine the cash flows. In this paper we will assume that the 5 stochastic factors follow some stochastic process, elaborated in the option valuation chapters. This can capture both stable future levels for the factors as well as disperse changing behaviors.

Implementation wise, this means we will simulate the factors for different scenarios and from that determine the profit = future revenues – future costs, this is a generalization of Paul Clark's approach on fleet planning using the macro approach to investment valuation (2007). In which the revenues of an additional aircraft are determined by first allocating the RPKs (=market demand) to the "fixed" fleet (the fleet from contracts which is not optional). The new aircraft takes the "leftovers" in terms of RPKs. Capacity in terms of passenger kilometres is denominated by available seat kilometres. From a revenue management perspective it is known that aircraft cannot be filled up 100%, the ASKs (=available seat kilometres, the maximum capacity of an aircraft) without influencing the yield level significantly. That is why we assume there exists a maximum load factor, which we will use to determine the maximum RPK one aircraft can generate.

The value of an option goes in the same fashion, since at maturity of the option the situation is the same: a projection of revenues and costs has to be made on the basis of the current levels of the 5 factors. The following steps are undertaken:

The projected revenues = projected yield * projected RPK.

At maturity of the option, the decision moment, we exercise the option if:
1) Demand RPK cannot be covered by ASK without optional ac
*Projected RPK > ASK without optional ac * maximum load factor*
and
2) "Spill" RPK is profitable:
projected yield(Projected RPK-ASK without optional ac * maximum load factor) > projected cost optional ac*

Else the optional aircraft is not profitable. We do nothing and let option expire to invest.

---

*Projected RPK (which is the total RPK generated by the entire fleet) is also dependent on the current fleet, as the current fixed fleet size changes during this evaluation period. We assume the fleet size, which are not in the financing decision today, stays constant over the investment horizon.

4 Option valuation in discrete time

Option valuation in discrete time means that time is partitioned over a countable number of intervals at which the option is valued. We will start off with option valuation over one period and then generalize to multi period analysis. These are important examples as these are fundamental towards continuous time models which we will eventually use.
In general one needs to define the behavior of the value of an investment or project, this behavior is a process which is uncertain from today's point of view and thus stochastic. The value of the project will be called S and its value at time t is S_t. All options that we consider are call options.

4.1 Assumptions in the binominal tree model

a. There exists a risk free bond discount bond B_0 with continuously compounded interest rate r^f
b. One can borrow money at the risk free rate r^f

4.2 One period binominal tree model

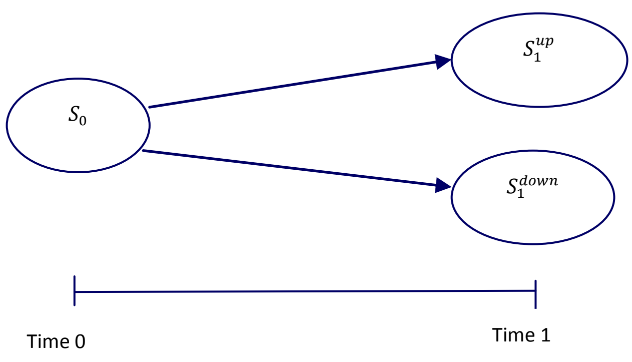
The binominal tree model specifies the process of the value of the project S, assuming that the process either goes up or goes down. The constants $s_0, p, u \; and \; d$ are assumed to be known and satisfy d < u and 0 < p < 1:

$S_0 = s_0$ "The starting value at time 0 of the process is known and is s_0"
$\mathbb{P}(S_1 = us_0) = p$ "At time 1 the process increases in value with factor u with probability p"
$\mathbb{P}(S_1 = ds_0) = 1 - p$ "At time 1 the process decreases in value with factor d with probability p"

The choice of this discrete time model can be motivated, as this process converges to the continuous time Geometric Brownian Motion model which we will consider later on.

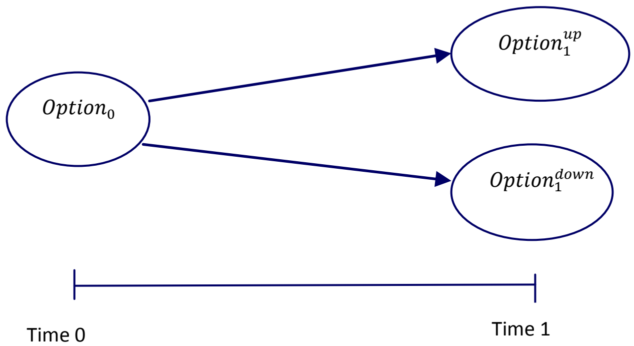
4.3 One period binomial tree model option valuation

The one period binominal tree model can be visualized as follows:



The evolution of the investment

Where the value of the process at time 0 is known $S_0 = s_0$ and the value of the process at time 1, S_1 is unknown. S_1 takes on the value $S_1^{up} = us_0$ or $S_1^{down} = ds_0$ with probability p and 1-p respectively. In same fashion the option value at time 0 is known $Option_0$ and the option value at time 1, $Option_1$ is unknown. $Option_1$ takes on the value $Option_1^{up}$ and $Option_1^{down}$. The functional form or *claim* of a call option at maturity T is $C(S_T) = \max\{S_T - K, 0\}$. We can derive the value of the option where K is a constant. In other words the option becomes valuable when the project value exceeds the threshold K and the option is consequently exercised. Else it is not exercised. Thus for an option to make sense in the one period model, it must hold $0 < S_1^{down} < K < S_1^{up}$.



The evolution of the option's value

The question now, what the fair value at time 0 of an option that expires is at time 1 when the investment decision needs to be made today. As one can notice the value of the option is uncertain and holding it results in risk or uncertainty about the eventual option value at time 1. It turns out that we can compose a portfolio at time 0 and without risk this portfolio will always replicate the option value at time 1.

Construct a portfolio at time 0, V_0. Invest a fraction φ in the investment S_0 and invest a fraction ω in a risk free bond B that matures at time 1. In mathematical notation: $V_0 = \varphi NPV_0 + \omega B_0$. The portfolio at time 1 equals $V_1 = \varphi S_1 + \omega B_1$. The discount bond $B_0 = 1$ will increase in value to $B_1 = e^{r^f}$ and the process S_0 turns into S_1 which is S_1^{up} or $S_1^{down}$with probability p and 1-p respectively. Note we use the continuously compounded interest rate for the bond price process, instead of a periodic compounded in which case $B_1 = (1 + r^f)$. This choice does not change the theory except some notation for discrete time models but will come in handy for continuous time models.

Now we want the portfolio V_1 at time 1 to replicate the option value $Option_1$ at all times, when the project increases in value as well when the project decreases in value.

This results in the following two equations at time 1:

$$\varphi\, us_0 + \omega * e^{r^f} = Option_1^{up}$$
$$\varphi\, ds_0 + \omega * e^{r^f} = Option_1^{down}$$

We point out that the only unknowns are the fractions φ, ω and hence with two equations and two unknowns we can solve for φ, ω.

We get $\varphi = \frac{Option_1^{up} - Option_1^{down}}{us_0 - ds_0}$
and

$$\omega = e^{-r^f}\left[Option_1^{up} - \varphi\, us_0\right] = e^{-r^f}\left[Option_1^{up} - \frac{Option_1^{up} - Option_1^{down}}{us_0 - ds_0} us_0\right] = e^{-r^f}\left[\frac{Option_1^{up}(us_0 - ds_0)}{us_0 - ds_0} - \frac{(Option_1^{up} - Option_1^{down})us_0}{us_0 - ds_0}\right] =$$

$$e^{-r^f}\left[\frac{Option_1^{up}(us_0 - ds_0) - (Option_1^{up} - Option_1^{down})us_0}{us_0 - ds_0}\right] = e^{-r^f}\left[\frac{Option_1^{down}us_0 - Option_1^{up}ds_0}{us_0 - ds_0}\right]$$
$$= e^{-r^f}\left[\frac{Option_1^{down}u - Option_1^{up}d}{u - d}\right]$$

So for these two choices of φ and ω, our portfolio replicates the option's payoff at time 1. Note that the probability p of an upward movement is *irrelevant*!

4.3.1 Revised Expectation: change of measure $\mathbb{P}$ vs. $\mathbb{Q}$

In the previous section we derived an explicit formula for (φ, ω). Since the option value and the replicating portfolio at time 1 are equal (enforced by the two equations) we can find the option value at time 0 by filling in the portfolio weights (φ, ω):

$$
\begin{aligned}
V_0 = \varphi s_0 + \omega B_0 &= \frac{Option_1^{up} - Option_1^{down}}{us_0 - ds_0} s_0 + e^{-rf}\left[\frac{Option_1^{down}u - Option_1^{up}d}{u-d}\right] * 1 \\
&= \frac{Option_1^{up} - Option_1^{down}}{u-d} + \frac{e^{-rf}Option_1^{down}u - e^{-rf}Option_1^{up}d}{u-d} \\
&= Option_1^{up}\frac{1-e^{-rf}d}{u-d} + Option_1^{down}\frac{e^{-rf}u-1}{u-d} \\
&= e^{-rf}[Option_1^{up}\frac{e^{rf}-d}{u-d} + Option_1^{down}\frac{u-e^{rf}}{u-d}] \\
&= e^{-rf}\ [\,qOption_1^{up} + (1-q)Option_1^{down}] = \mathbb{E}_q[e^{-rf}Option_1]\,, where\ q := \frac{e^{rf}-d}{u-d}
\end{aligned}
$$

The subscript in the expectation operator means we have to replace the real probability p with the new defined q to evaluate the expectation.

Thus the option value can be evaluated as the discounted expectation under a new probability measure $\mathbb{Q}$, defined above. The correct price of the option is **not** the discounted expectation under the real physical measure $\mathbb{P}$, that is the real probability of going up and down in value. Calculating the price of the option using this new measure $\mathbb{Q}$ is just an artificial way of changing the probabilities in such a way that the fair price of the option is found. If one would use the real probabilities one would be evaluating the value of the option through its expected payoff. Doing so one does not take into account the risk of the option. This is only valid when investors are indifferent to risk or that investors are risk neutral. Taking the expectation under the $\mathbb{Q}$ measure to calculate the fair price of the option seems like as if risk is not taken into account, this is the reason why this is called risk neutral valuation. In fact the original expectation under the real probability p is adjusted with q and exactly this operation takes into account risk. As a consequence the new artificial probabilities (which are only used for computation for the option price) are called risk neutral probabilities.

The third name to these artificial probabilities or the $\mathbb{Q}$ measure is the martingale measure. One can see that:

$$
\begin{aligned}
\mathbb{E}_q\left[e^{-rf}S_1\right] &= e^{-rf}\left[q\,S_1^{up} + (1-q)\,S_1^{down}\right] = e^{-rf}[qus_0 + (1-q)ds_0] \\
&= e^{-rf}\left[\frac{e^{rf}-d}{u-d}us_0 + \left(1-\frac{e^{rf}-d}{u-d}\right)ds_0\right] = e^{-rf}\left[\frac{e^{rf}-d}{u-d}us_0 + \left(\frac{u-d}{u-d} - \frac{e^{rf}-d}{u-d}\right)ds_0\right] \\
&= e^{-rf}\left[\frac{e^{rf}us_0 - uds_0}{u-d} + \left(\frac{uds_0 - ds_0d}{u-d} - \frac{e^{rf}ds_0 - dds_0}{u-d}\right)\right] \\
&= e^{-rf}\left[\frac{e^{rf}us_0 - uds_0 + uds_0 - ds_0d - e^{rf}ds_0 + dds_0}{u-d}\right] = e^{-rf}\left[\frac{e^{rf}us_0 - e^{rf}ds_0}{u-d}\right] \\
&= e^{-rf}\left[\frac{u-d}{u-d}e^{rf}s_0\right] = s_0
\end{aligned}
$$

A martingale is a variable that doesn't change in expectation, that is the variable does not tend to grow or decrease over time but has as expectation the current (known) value. In general, a process Z is a martingale if it satisfies the following (with the subscript in Z denoting the time index):

$$
\mathbb{E}[Z_n|Z_m] = Z_m \\
for\ m\ \leq n
$$

We have left out the subscript on the expectation operation which denotes measure to use, typically one would say it is a martingale with respect to a certain measure if it is needed to explicit specify it. In this paper processes are typically a martingale with respect to $\mathbb{Q}$ but not with respect to the measure $\mathbb{P}$.

The variable S_1 is unknown from the perspective of time 0, we have shown that its discounted value, the expectation is s_0 (given s_0 is known). We can make the known information explicit by writing the conditional expectation: $\mathbb{E}_q\left[e^{-r^f} S_1 | s_0\right] = s_0$. The measure $\mathbb{Q}$ is such that the variable S becomes a martingale, hence the name martingale measure.

4.3.2 No arbitrage

In the previous section we saw that the option and the replicating portfolio have the same cash flows and thus they are essential the some security. Since they are identical there should not be a reason why they would be priced different, this is in economics called the *law of one price*. At the moment there is an imbalance in pricing, there will be a higher demand for the cheaper security. This higher demand will results in a rise in the price of the cheaper security until the two prices coincides again.

The Financial market allows short selling of securities. Whenever there are two securities that are identical or have the same cash flows and they differ in price, one can buy the cheaper one and short sell the expensive one. At the maturity of the security, one has covered its obligations and has a neutral position. The difference in price results in a risk free profit. This is called an arbitrage opportunity. In financial markets *no arbitrage* is assumed as these opportunities will disappear quickly whenever they become available.

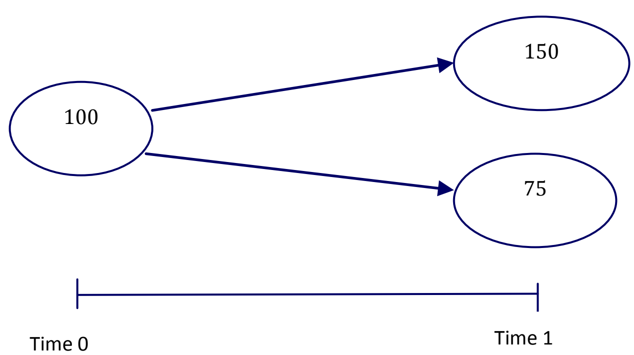
As a consequence the price of the replicating portfolio equals the option price.

4.3.3 An example

Now we turn to a numerical example, we assume:

$$
\begin{aligned}
&S_0 = 100 \\
&u = 1.5 \ \rightarrow S_1^{up} = 150 \\
&d = 0.75 \rightarrow S_1^{down} = 75 \\
&K = 110 \\
&r^f = 5\% \ \rightarrow B_0 = \ e^{-r^f} = e^{-0.05} \approx 0.95
\end{aligned}
$$

This results in the following visualization of the project value.

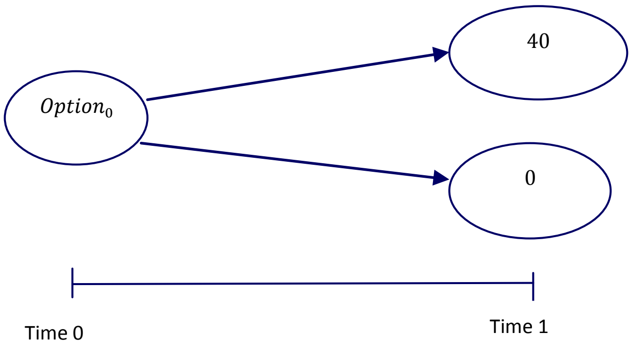


The evolution of the investment

From this we can derive the call option on such a project:

$$
\begin{aligned}
Option_1^{up} &= \mathrm{C}\left(S_1^{up}\right) = \max\left\{S_1^{up} - 110{,}0\right\} = 40 \\
Option_1^{down} &= \mathrm{C}(S_1^{down}) = \max\{S_1^{down} - 110{,}0\} = 0
\end{aligned}
$$

The question is what the value is of the option at time 0, $Option_0$.



The evolution of the option's value

We answer this by valuing the replicating portfolio:

$$\varphi = \frac{Option_1^{up} - Option_1^{down}}{NPV_1^{up} - NPV_1^{down}} = \frac{40-0}{150-75} = 8/15$$

$$\omega = \left[\frac{Option_1^{down} NPV_1^{up} - Option_1^{up} NPV_1^{down}}{NPV_1^{up} - NPV_1^{down}}\right] = \left[\frac{(0*150)-(40*75\,)}{150-75}\right] = -40$$

The second weight ω is negative, it would mean we wouldn't buy the bond but short sell the bond against the interest free rate.

Now we construct our portfolio at time 0: $V_0 = \varphi S_0 + \omega * B_0 = \frac{8}{15} * 100 - 40 * e^{-0.05} \approx 15.28$

Now one period later we have:

$$\varphi S_1^{up} + \omega B_1 = \frac{8}{15} * 150 - 40 * 1 = 40$$

$$\varphi S_1^{down} + \omega B_1 = \frac{8}{15} * 75 - 40 * 1 = 0$$

We just exactly replicated the option buying 8/15 of the project and short selling 40 riskless discount bonds. As we can see the payoff of the portfolio exactly replicates the portfolio of the option. Thus the option's value is 15.238.

Alternatively we can derive the option value using the discounted expectation under the $\mathbb{Q}$ measure. This gives the identical result as the portfolio value at time 0. We find the new probability measure $\mathbb{Q}$ and then evaluate the discounted expectation under this artificial measure.

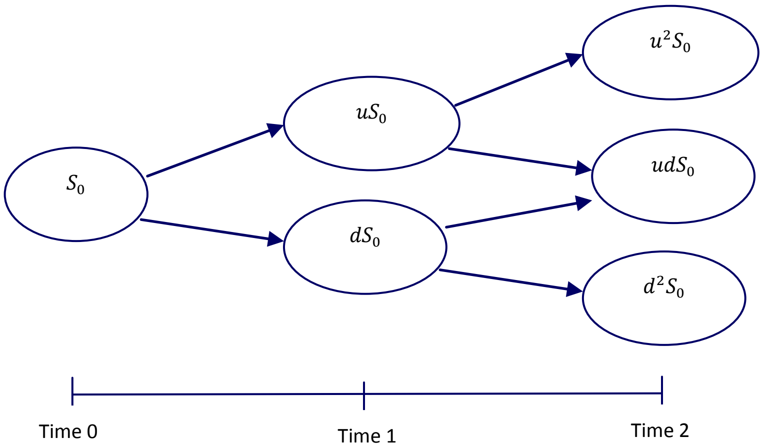
$$q = \frac{e^{rf} S_0 - S_1^{down}}{S_1^{up} - S_1^{down}} = \frac{e^{0.05} * 100 - 75}{150-75} \approx 0.27$$

$$Option_0 = \mathbb{E}_q\left[e^{-rf} Option_1\right] = e^{-rf}\mathbb{E}_q[C(S_1)] = e^{-rf}\left(q Option_1^{up} + (1-q) Option_1^{down}\right)$$
$$= e^{-0.05}(0.27.. * 40 + (1 - 0.27..) * 0) = 15.238$$

As one can see option valuation using an expectation under a new measure $\mathbb{Q}$ is quick, it is the way option pricing is done in practice.

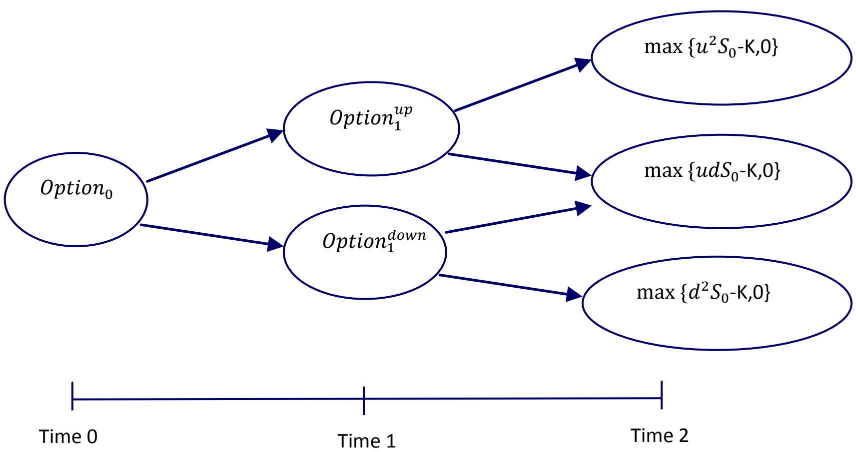
4.4 Option valuation over two periods

We extend the one period binomial tree to two periods, using the same setting



The evolution of the investment

And this results in a tree for the call option:



The evolution of the option's value

From the final values of the project at maturity, we can derive the option price at time 2. To find the price of the option today at time 0, we work backwards from time 2 to time 0. We can use the developed theory of the one period binomial tree to do this. First look at the upper three nodes, where we have a known payoff at time 2 and an unknown option price $Option_1^{up}$ at time 1. This is exactly the same situation as in the one binomial tree and can be evaluated in the same fashion:

$$Option_1^{up} = \mathbb{E}_q\left[e^{-rf}Option_2\right] = e^{-rf}[qC(u^2s_0) + (1-q)C(dus_0)]$$

Now we look at the lower three nodes and calculate the option price in similar fashion:

$$Option_1^{down} = e^{-rf}\mathbb{E}_q[Option_2] = e^{-rf}[qC(uds_0) + (1-q)C(d^2s_0)]$$

Now that we have found the two option price at time 1, we can value the option at time 0.

$$Option_0 = e^{-r^f}\mathbb{E}_q[Option_1] = e^{-r^f}\{qOption_1^{up} + (1-q)Option_1^{down}\}$$
$$= e^{-r^f}\left\{q(1+r^f)^{-1}[qC(u^2s_0) + (1-q)C(dus_0)]\right.$$
$$\left.+ (1-q)e^{-r^f}[qC(uds_0) + (1-q)C(d^2s_0)]\right\}$$
$$= e^{-2r^f}\{q^2C(u^2s_0) + 2q(1-q)C(dus_0) + (1-q)^2C(d^2s_0)\}$$

Observe that we have a binomial tree again using the probability q for an upward movement. The expression equals $e^{-2r^f}\mathbb{E}_q[Option_2]$, that is the discounted expected value of the option under the $\mathbb{Q}$ measure.

We have seen that in the one period binomial tree the discounted project value is a martingale the $\mathbb{Q}$ measure:

$$\mathbb{E}_q\left[e^{-r^f}S_1|s_0\right] = s_0$$

And in the same fashion:

$$\mathbb{E}_q\left[e^{-r^f}S_2|s_1, s_0\right] = s_1$$

The same holds for the project value over two periods (given we are in time 0):

$$\mathbb{E}_q\left[e^{-2r^f}S_2|s_0\right] = e^{-2r^f}\mathbb{E}_q[quS_1 + (1-q)dS_1] = que^{-r^f}\mathbb{E}_q\left[e^{-r^f}S_1\right] + (1-q)de^{-r^f}\mathbb{E}_q\left[e^{-r^f}S_1\right]$$
$$= que^{-r^f}s_0 + s_0e^{-r^f}(1-q)d = s_0\left(e^{-r^f}qu + e^{-r^f}d - e^{-r^f}qd\right)$$
$$= s_0\left[\frac{e^{r^f}-d}{u-d}e^{-r^f}u + e^{-r^f}d - \frac{e^{r^f}-d}{u-d}e^{-r^f}d\right]$$
$$= s_0\left[\frac{u - e^{-r^f}ud + e^{-r^f}ud - e^{-r^f}dd - d + e^{-r^f}dd}{u-d}\right] = s_0\left[\frac{u-d}{u-d}\right] = s_0$$

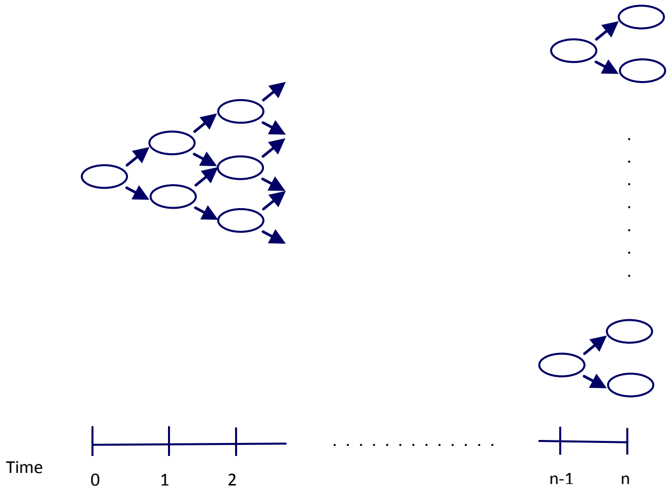
Once again we see that the project value over two period under the $\mathbb{Q}$ measure.

4.5 Option valuation over n periods

We can extend the result to a general n period binomial tree. Starting from the last three nodes where given the evolution of the project value we can price the option through the claim. Working from time n, we can move back to today, time 0. Just like in the one period and two period binomial tree, the price of the option at time 0 is the discounted expected payoff under the measure $\mathbb{Q}$, the formula is given by:

$$Option_0 = e^{-nr^f}\mathbb{E}_q[Option_n]$$
$$where\ q \coloneqq \frac{e^{r^f}S_0 - S_1^{down}}{S_1^{up} - S_1^{down}}$$

The quantity q can be seen as the new probability of an upward movement. Again the real probabilities of an upward movement p are not relevant!



The evolution of the option's value

Just as in the previous sections the project value is a martingale too:

$$\mathbb{E}_q\left[e^{-nr^f}S_n|s_m, s_{m-1}, \dots, s_0\right] = s_m$$

Or that $\tilde{S}_i := e^{-ir^f}S_i \; for \; i = 0,1,2, \dots$ (the discounted project value) is a martingale under the $\mathbb{Q}$ measure.

4.6 Binomial representation theorem

In the previous section we saw that the discounted project value in the binomial tree $\widetilde{S}_i$ is a martingale under the $\mathbb{Q}$ measure. The binomial representation theorem says that any other martingale under the same measure $\mathbb{Q}$ can be constructed from this martingale using a predictable process. It means that without knowing what a martingale $\widetilde{S}$ will be tomorrow, we can use another martingale M under the same measure to construct the value of S tomorrow using a pre-determinable fraction of M. This theorem is not needed for option valuation in the binomial tree but is useful for intuition in the continuous time setting.

5 Option valuation in continuous time

In the following we present continuous time models. The price of the option evolves continuously over time. This makes valuation convenient. We could then model each factor, like fuel and yield as a stochastic process.

5.1 Probability space

Probability Triple: $(\Omega, \mathcal{F}, \mathcal{P})$
Ω: Sample space, the set of all possible outcomes.
$\mathcal{F}$: Sigma algebra or "set of events", where each event is a set containing zero or more outcomes.
$\mathcal{P}$: Probability measure, assigns probabilities to the events.

Probability quadruple $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathcal{P})$
In words one can say $\mathcal{F}_t$ is the historic information up to time t, it is an increasing collection of sub sigma algebras and is called a filtration.

When we write down $\sigma(X_1, X_2, \dots, X_n)$, we mean it is the smallest possible sigma algebra generated by the process $X_1, X_2, \dots, X_n$. A sigma algebra is mathematical concept of a set satisfying certain properties, so that $\sigma(X_1, X_2, \dots, X_n)$ means that we are given the set of information containing the values of $X_1, X_2, \dots, X_n$. A filtration $\mathcal{F}_n$ of a process $X_1, X_2, \dots, X_n$is adapted if $\sigma(X_n) \subset \mathcal{F}_n$. The smallest adapted filtration is called the natural filtration, it is the filtration that only contains the information of the process itself up to time n.

5.2 Stochastic processes

There exist several types of stochastic processes, in the previous chapter the project value described by the binomial tree was a discrete stochastic process. Discrete in time and values or states it can attain. In the following we present continuous stochastic processes in time and values to model the project value.

An important class of continuous time stochastic processes are Itō processes:

$$X_t = X_0 + \int_0^t \mu_s \, ds + \int_0^t \sigma_s \, dW_s$$
$$\rightarrow X_t - X_0 = \int_0^t \mu_s \, ds + \int_0^t \sigma_s \, dW_s$$

For convenience most literature represent the above equation in differential notation, we say X_t satisfies the stochastic differential equation (SDE):

$$dX_t = \mu_t dt + \sigma_t \, dW_t$$

Where $\mu_t = \mu(t, X), \sigma_t = \sigma(t, X)$. That is the parameters can be deterministic functions of the time and current state of the process.

5.2.1 Brownian motion

In the year 1827 botanist Robert Brown studied pollen in water under the microscope, he discovered a rugged unpredictable movement of the pollen. Later this (stochastic/unpredictable) process, Brownian motion, was named after Brown.

In the literature it is also known as the Wiener process, it is the basic stochastic process and functions as the building block of more sophisticated processes. It can be seen as the limit process of a discrete time random walk. A discrete time random walk is a stochastic process in which the next period's value jumps with +1 or -1 with probability 0.5. Thus if today's value is 5, next period's value will be 6 with probability 0.5 and 4 with probability 0.5. The following period the value will again jump with +1 or -1 with probability 0.5, and so on. If we take the step size and the period's interval smaller and smaller, one eventually will take an infinitesimal (infinite small) time step and an infinitesimal jump size. Then the the jump size and time step becomes continuous.

Practically one can see Brownian motion as a random behaviour, in which the value of the process stays unbiased due to the equal probability and the equal step size up and down. Brownian motion is used as the building block that models the randomness of processes including Itō processes.

We can denote a Brownian motion as W, after Wiener. Formally a stochastic process W is a Brownian motion if:

i. The increment $W_t - W_s$ is normally distributed with mean 0 and variance t-s, for any $0 \leq s \leq t$.
ii. The increment $W_t - W_s$ is stochastically independent of W_u, for any $0 \leq u \leq s \leq t$.
iii. W_0= 0.
iv. W_t is continuous in $t \geq 0$.

5.2.2 Variation

The variation of a function f is a measure for its roughness. We will start off with the definition.

Let π be a sequence of partitions $0 = t_0 < t_1 < .. < t_{N(\pi)} = t$ on the interval [0,t], $N(\pi)$ be the number of intervals and the mesh $\delta(\pi) := \max_i (t_i - t_{i-1})$ of the partition and as mesh goes to zero the variation of f is defined to be:

$$\sup_{\pi}\{ \sum_{i=1}^{N(\pi)} |f(t_i) - f(t_{i-1})| \}$$

Then the p-variation is defined to be:

$$\sup_{\pi}\{ \sum_{i=1}^{N(\pi)} |f(t_i) - f(t_{i-1})|^p \}$$

The quadratic variation is the 2-variation and the quadratic variation of any continuously differentiable function is zero. To see this we give the following inequality.

$$\sup_{\pi}\{ \sum_{i=1}^{N(\pi)} |f(t_i) - f(t_{i-1})|^2 \} \leq \max_i |f(t_i) - f(t_{i-1})| * \sum_{i=1}^{N(\pi)} |f(t_i) - f(t_{i-1})|$$

This follows from the fact $\max_i |f(t_i) - f(t_{i-1})|$ goes to zero as $\max_i (t_i - t_{i-1})$ goes to zero.
Brownian motion doesn't hold this property, this implies that Brownian motion is not differentiable. In fact the quadratic variation of Brownian motion converges to a non trivial limit. This following lemma is needed when we start manipulating Brownian motion and in general Itō processes.

Lemma: Quadratic variation of an Itō process
Let A_t be an Itō process and π be a partition on [0,t], then

$$\sum_{i=1}^{N(\pi)} |A_{t_i} - A_{t_{i-1}}|^2 \xrightarrow{P} \int_0^t \sigma_s^2 \, ds$$

Where the arrow with the p stands for convergence in probability, this means the convergence is almost sure.

The previous quadratic variation was for a single process, next to that there also exists a cross quadratic variation or quadratic covariation of a pair of two processes A and B. This is defined as:

$$[A,B]_t = \sum_{i=1}^{N(\pi)} (A_{t_i} - A_{t_{i-1}})(B_{t_i} - B_{t_{i-1}})$$

The quadratic covariation maybe written in terms of quadratic variation by the polarization identity:

$$[A,B]_t = \frac{1}{4}([A+B]_t - [A-B]_t)$$

5.2.3 Stochastic integrals

In the previous section we have shown that the quadratic variation of any differentiable function is zero and that the quadratic variation of a stochastic process is non-zero. Hence stochastic process is not differentiable. Now we would like to evaluate a stochastic integral of the form: $\int_0^t g(s)\, dW_s$. In ordinary Newtonian calculus, we would try to evaluate a function by:

$$\int g(t)\, df(t) \; = \int g(t) f'(t)\, dt$$

Clearly we cannot we cannot do this, due to the fact we cannot compute the derivative of Brownian motion. Instead we can redefine the stochastic integral. We can see it as the limit of a sum, or as a so called Riemann-Stieltjes integral:

$$\int_0^t g(s)\, dW_s = \lim_{\delta(\pi)\to 0} \sum_{j=1}^{N(\pi)} g(t_j)(W_{t_j} - W_{t_{j-1}})$$

Let X be adapted and satisfy $\mathrm{E}\int_0^t X_s^2\, ds < \infty$ then the following properties hold:

i. $\mathrm{E}\int_0^t X_s\, dW_s = 0$
ii. $\mathrm{E}(\int_0^t X_s\, dW_s)^2 = \mathrm{E}\int_0^t X_s^2\, ds$

The first property is needed if one wants to calculate the expectation of some process integrated with respect to Brownian motion, the second property is called Itō isometry and can be used to calculate the variance.

5.2.4 Itō's Lemma

Itō's lemma or also known as Itō's formula is the stochastic version of Taylor's formula for deterministic functions, it gives an formula to approximate the difference in the value of an function f from a given starting point.

Theorem: Itō's formula for one stochastic processes

$$df(X_t) \; = \; f'(X_t)\, dX_t \; + \frac{1}{2} f''(X_t)\, d[X]_t$$

Itō's formula for a function of two stochastic processes R_t and S_t

Where f is deterministic function and the derivatives $f_r(R_t, S_t), f_s(R_t, S_t)$, $f_{rr}(R_t, S_t), f_{ss}(R_t, S_t)$ and $f_{rs}(R_t, S_t$ exist.

$$df(R_t, S_t) \; = \; f_r(R_t, S_t)\, dR_t \; + f_s(R_t, S_t)\, dS_t + \frac{1}{2} f_{rr}(R_t, S_t)\, d[R]_t + \frac{1}{2} f_{ss}(R_t, S_t)\; d[S]_t + f_{rs}(R_t, S_t)\; d[R,S]_t$$

Example:
The Itō's product rule is an application of Itō's lemma for two stochastic processes.
Consider the following stochastic process $f(R_t, S_t) = R_t S_t$.

Then $df(R_tS_t) = S_t dR_t + R_t dS_t + \frac{1}{2} * 0\ d[R]_t + \frac{1}{2} * 0\ d[S]_t + 1\ d[R,S]_t$

So that the **Itō product rule** is: $d(R_tS_t) = S_t dR_t + R_t dS_t + d[R,S]_t$

The above formula's can be extended to a system of differential equations.
Theorem: Multifactor Itō's formula for a system of stochastic differential equations

Let $dX_t^i = \mu_i(t)dt + \sum_{j=1}^{n} \sigma_{ij}(t) dW_t^j, for\ i = 1, \ldots, n$ be the system of stochastic differential equations, of n processes. Then $df(X_t) = \sum_{i=1}^{n} \frac{\partial f}{\partial X_i}(X_t) dX_t^i + \frac{1}{2}\sum_{i,j=1}^{n} \frac{\partial^2 f}{\partial X_i X_j}(t, X_i) C_{ij}(t) dt$, where $C_{ij}(t) = \sum_{k=1}^{n} \sigma_{ik}(t)\sigma_{jk}(t)$

5.3 Geometric Brownian motion

This is the most common continuous time stochastic process and is used in the Black Scholes model for financial options. It is in fact the limit process of the binomial tree when one takes the time steps smaller and smaller.

The geometric Brownian motion is a stochastic process that satisfies the following SDE:

$$dX_t = (\mu + \frac{1}{2}\sigma^2) X_t dt + \sigma X_t dW_t$$
$$\frac{dX_t}{X_t} = (\mu + \frac{1}{2}\sigma^2) dt + \sigma dW_t$$
$$where\ \sigma > 0\ and\ \mu \in (-\infty, \infty)$$

We can find the explicit form of the stochastic process by using Itō's Lemma, for $X_t = f(Z) = exp(Z)$ and $Z_t = \mu t + \sigma W_t$.

$dX_t = df(Z) = dexp(Z) = X_t\, dZ_t + \frac{1}{2} X_t d[Z]_t = X_t d(\mu t + \sigma W_t) + \frac{1}{2} X_t d[\mu t + \sigma W_t] = \mu X_t dt + \sigma X_t dW_t + \frac{1}{2} X_t \sigma^2 dt = (\mu + \frac{1}{2}\sigma^2) X_t dt + \sigma X_t dW_t$

Thus we can conclude **the explicit solution of the geometric Brownian motion process** is:

$$X_t = f(Z_t) = e^{Z_t} = X_0 e^{\mu t + \sigma W_t}$$

Thus $X_t \sim Lognormal(ln(X_0) + \mu t, \sigma^2 t)$, this is equivalent to stating $log(\frac{X_t}{X_0}) \sim Normal(\mu t, \sigma^2 t)$

And it follows that the expectation $E[X_t] = X_0 e^{\mu t + \frac{1}{2}\sigma^2 t}$ and the variance $Var[X_t] = X_0^2 e^{2\mu + \sigma^2 t}(e^{\sigma^2 t} - 1)$.

The quadratic variation $[X]_t = \int_0^t \sigma_s^2\, ds = \int_0^t \sigma^2\, ds = \sigma^2 t$

There are two notes to make: taking the exponential of the process Z_t ensures that the process stays positive at all times or i.e. its outcome space is mapped on the positive real line $\in (0, \infty)$, this is why GBM is a popular choice for any process that needs to stay positive at all times. Black and Scholes chose exactly this process for modelling stock behaviour. One critic is that variance grows linearly with time and so that the variance grows to infinite with an infinite horizon.

5.4 Understanding Geometric Brownian motion

In the previous section we have seen that the log growth of the process X is normally distributed:

$$log(\frac{X_t}{X_0}) \sim Normal(\mu t, \sigma^2 t)$$

This is not an unusual way for representing growths or returns in Finance, for intuition log growths are approximately the same as normal growths for small values.

Consider the following situation an asset grows 6% (=r_{year}) per year, then this is not equal to a monthly growth of 6%/12 = 0.5%. This is because after the first month the asset value is higher than the original value. In other words there is growth on growth. This is called compounding and one has to distinguish between different frequencies of compounding. In Finance continuous compounding is preferred, this is when the compounding is continuous.

We can formalize this, consider the following continuous growth factor $(1 + r)$, then we want to equate this to a periodic growth factor $(1 + r_n)^n = (1 + \frac{nr_n}{n})^n$, as we continue to make the interval smaller and eventually take the limit n to infinity we get:

$$\lim_{n\to +\infty} (1 + \frac{nr_n}{n})^n = e^r$$

Where nr_n tends to r in the limit.

The growth r is known as the continuously compounded growth rate. Using this, we can easily find the growth over any period n since growth factors which are multiplied turns out to be the sum of the growth rates. For example:

$$(e^r)^2 = e^{2r}$$

One can now find the growth rate of a growth factor by taking the inverse operation of an e power, which is the natural logarithm. This is exactly the growth of the Geometric Brownian Motion, so that growth can be seen as the continuously compounded growth rate which is normally distributed.

Maclaurin series are Taylor series in case the approximating area is around 0, it follows from Taylor's theorem that:

$$e^x = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$$

For example when we want to approximate $e^{0.02}$, we fill in the formula for the first two terms: $1 + x = 1.02$, the third and higher order terms are of little influence, so that we can see that putting a "growth rate" of 0.02 in the e power results approximately in a growth factor for small growth rates. When growth rates become bigger, third order and higher order terms become of high significance and such an approximation is less suitable.

5.5 Ornstein–Uhlenbeck process (Mean reversion)

An alternative stochastic process is the (arithmetic) Ornstein–Uhlenbeck process X_t is given by the following SDE:

$$dX_t = \theta(\mu - X_t)dt + \sigma dW_t$$
$$where\ \theta, \mu, \sigma > 0$$

Here μ represents the long term mean and θ is the rate of convergence.

By Itō's Lemma, using $f(X_t, t) = X_t e^{\theta t}$. We get:

$$df(X_t, t) = e^{\theta t} dX_t + \theta X_t e^{\theta t} dt = e^{\theta t}\theta(\mu - X_t)dt + e^{\theta t}\sigma dW_t + \theta X_t e^{\theta t} dt = e^{\theta t}\theta\mu dt + e^{\theta t}\sigma dW_t$$

So that we obtain: $X_t e^{\theta t} = e^{\theta t}\theta\mu\, dt + e^{\theta t}\sigma dW_t$

Writing this in integral notation we get: $X_t e^{\theta t} - X_0 = \int_0^t e^{\theta s}\theta\mu\, ds + \int_0^t e^{\theta s}\sigma dW_s$

$$X_t = e^{-\theta t}X_0 + e^{-\theta t}\int_0^t e^{\theta s}\theta\mu\, ds + e^{-\theta t}\int_0^t e^{\theta s}\sigma dW_s$$

Solve the integral $\int_0^t e^{\theta s}\theta\mu\, ds = \mu e^{\theta s}|_0^t = \mu e^{\theta t} - \mu$

Now filling this integral in:

$$X_t = e^{-\theta t}X_0 + e^{-\theta t}(\mu e^{\theta t} - \mu) + e^{-\theta t}\int_0^t e^{\theta s}\sigma dW_s = e^{-\theta t}X_0 + \mu(1 - e^{-\theta t}) + e^{-\theta t}\int_0^t e^{\theta s}\sigma dW_s$$

Thus we can conclude **the explicit solution of the Ornstein–Uhlenbeck process** is:

$$\longrightarrow X_t = e^{-\theta t}X_0 + \mu(1-e^{-\theta t}) + e^{-\theta t}\int_0^t e^{\theta s}\sigma dW_s$$

The expectation is $E[X_t] = E[e^{-\theta t}X_0 + \mu(1-e^{-\theta t}) + e^{-\theta t}\int_0^t e^{\theta s}\sigma dW_s] = E[e^{-\theta t}X_0 + \mu(1-e^{-\theta t})] = e^{-\theta t}X_0 + \mu(1-e^{-\theta t})$ and the variance is $Var[X_t] = \frac{\sigma^2}{2\theta}(1-e^{-2\theta t})$

This follows from:

$$Var[X_t] = E[(X_t - E[X_t])(X_t - E[X_t])] = E[(e^{-\theta t}\int_0^t e^{\theta s}\sigma dW_s)(e^{-\theta t}\int_0^t e^{\theta s}\sigma dW_s)] = E(e^{-2\theta t}(\int_0^t e^{\theta s}\sigma dW_s)^2]$$
$$= e^{-2\theta t}E[(\int_0^t e^{\theta s}\sigma dW_s)^2] = e^{-2\theta t}\int_0^t e^{2\theta s}\sigma^2 ds = e^{-2\theta t}\sigma^2\int_0^t e^{2\theta s}ds = e^{-2\theta t}\sigma^2(\frac{1}{2\theta}e^{2\theta t} - \frac{1}{2\theta})$$
$$= \frac{\sigma^2}{2\theta}(1-e^{-2\theta t})$$

The quadratic variation $[X]_t = \int_0^t \sigma_s^2\, ds = \int_0^t \sigma^2\, ds = \sigma^2 t$. Thus the quadratic variation of the geometric Brownian motion and the Ornstein–Uhlenbeck process are the same.

We see that the variance is bounded, as the time horizon t goes to infinity the variance becomes $\frac{\sigma^2}{2\theta}$, this is one of the advantage on top of the geometric Brownian motion process. The process is also known as a mean reverting process. When $\mu > X_t$ or when the value of the process is lower than μ, the process will have a positive drift. When $\mu < X_t$ or when the value of the process is higher than μ, the process will have a negative drift. Thus the process reverts to the long term meanμ. The futher away the process is in value, the bigger the drift is towards this mean.

Furthermore the Ornstein–Uhlenbeck process X_t is normally distributed since the stochastic integral with respect to Brownian motion itself normally distributed. This follows from the definition of the stochastic integral, which is the limit of a sum of normal random variables and the sum of normal random variables is again normally distributed. The parameters follow from the previously computed expectation and variance. So that $X_t \sim Normal(e^{-\theta t}X_0 + \mu(1-e^{-\theta t}), \frac{\sigma^2}{2\theta}(1-e^{-2\theta t}))$.
Observe this process can attain negative values and one parameter more has to be estimated in contrast with the geometric Brownian process.

5.6 Doob martingale

Here we state that the conditional expectation under some conditions is martingale. To do so we need to the tower law which states: $E[E[X|Y,Z]|Z] = E[X|Z]$. Now we can turn to the Doob martingale.

If y is a random variable with $E[Y] < \infty$ and $\mathcal{F}_n$ an arbitrary filtration then $X_n = E[Y|\mathcal{F}_n]$ is a martingale. To check this the tower law and the definition of a martingale can be used:

$$\mathbb{E}[E[Y|\mathcal{F}_n]|\mathcal{F}_m] = \mathbb{E}[Y|\mathcal{F}_m]\ for\ m < n$$

5.7 Girsanov theorem

This theorem states the existence of a measure Q, such that any stochastic process becomes a martingale under this new measure Q.

Theorem:

Let W_t be a Brownian motion under the measure $\mathbb{P}$ with the natural filtration $\mathcal{F}_t$ and let μ be an adapted process satisfying $E\left[e^{\frac{1}{2}\int_0^T \mu_s\, dt}\right] < \infty$. Then there exists a probability measure $\mathbb{Q}$ such that the process $W_t + \int_0^t \mu_s\, ds$ is a Brownian motion under $\mathbb{Q}$ for $0 \leq t \leq T$.

5.8 Brownian Martingale Representation Theorem

The following theorem is the equivalent of the binomial representation theorem of the binomial tree model. It states that one can replicate a martingale from another martingale if they are under the same measure.

Theorem:
Let $\mathcal{F}_t$be the natural filtration of a Brownian motion process W under a probability measure $\mathbb{P}$. If M is another martingale relative to $\mathcal{F}_t$ under $\mathbb{P}$ and $E[|M_t|^2] < \infty\ for\ each\ t > 0$, then there exists a predictable process X such that $M_t = M_0 + \int_0^t X_s\, dW_s$.

5.9 Discounted investment under Q

Changing the measure from a real world probability measure $\mathbb{P}$ to an artificial theoretical probability measure $\mathbb{Q}$ is not straightforward. However this is possible for the two continuous time processes Geometric Brownian motion and mean reversion, for the geometric Brownian motion we will need Ito's lemma.

The SDE of the geometric Brownian Motion is:

$$dX_t = (\mu + \frac{1}{2}\sigma^2)X_t dt + \sigma X_t dW_t$$

With explicit solution: $X_t = X_0 e^{\mu t + \sigma W_t}$
Now define$\widetilde{W}_t = W_t + \frac{u-r}{\sigma} t$, then by Ito's lemma we get $d\widetilde{W}_t = dW_t + \frac{u-r}{\sigma} dt$ and by Girsanov theorem $\widetilde{W}_t$ is a martingale under a new measure $\mathbb{Q}$, under which $\widetilde{W}_t$ is a Brownian motion. Filling this in the original SDE we get:

$$dX_t = (r + \frac{1}{2}\sigma^2)X_t dt + \sigma X_t d\widetilde{W}_t$$

This is again a Geometric Brownian motion, with the only difference that $\mu = r$ under the new measure $\mathbb{Q}$.
Thus we can simulate a Geometric Brownian motion under the Q measure such that $\widetilde{W}_t$ is a Brownian motion using the explicit solution $X_t = X_0 e^{rt + \sigma W_t}$.

5.10 Valuing the option in the general univariate case

Now that we have gone through the necessary theory, we turn back to our goal that is perform continuous time valuation of an option. There are several steps to be taken. First we find a measure $\mathbb{Q}$ such that the discounted value of the investment or project becomes a martingale, this is guaranteed by Girsanov theorem. We form a portfolio of the investment and the risk free bond (like in the discrete time model) to replicate the option price. This process is $V_t = E_{\mathbb{Q}}[e^{-rT} C_T | \mathcal{F}_t]$ is martingale under the same measure $\mathbb{Q}$ where C is the value of the call option at time T. This is a Doob martingale. Now we have two processes that are both martingales under $\mathbb{Q}$ and by the Brownian Martingale Representation theorem, there exists a predictable process or a hedging strategy such that we can replicate the option. Thus by no arbitrage both have the same price and we find the option value by evaluating $E_{\mathbb{Q}}[e^{-rT} C_T | \mathcal{F}_t]$. We will do so by simulating under the $\mathbb{Q}$ measure and find the option price which is by the law of large numbers the expectation under many simulated payoffs.

6 Multivariate option valuation in continuous time

The previous chapter was option valuation using only one stochastic process, now we will extend to the multivariate setting. This will be needed since we have multiple stochastic processes on which the option value depends on.

6.1 General n-factor model

The "market" consists of a bond $\{B_t^1\}_{0\leq t\leq T}$ and n different securities with price processes $\{X_t^1, X_t^2, \dots., X_t^n\}_{0\leq t\leq T}$, governed by the system of stochastic differential equations:

$$dB_t = rB_t dt$$

$$dX_t^i = X_t^i(\mu_i(t)dt + \sum_{j=1}^{n} \sigma_{ij}(t) dW_t^j), i = 1,2, \dots, n$$

Where $\{W_t^j\}_{t\geq 0}, j = 1, \dots, n$ are n independent Brownian motions. We assume that the matrix $\sigma = (\sigma_{ij})$ is invertible.

Solution:

The (explicit) solution of the n factor model is $\{X_t^1, X_t^2, \dots., X_t^n\}_{0\leq t\leq T}$ where each component is defined as follows:

$$X_t^i = X_0^i e^{(\int_0^t (\mu_i(s) - \frac{1}{2}\sum_{k=1}^{n} \sigma_{ik}^2(s))ds + \int_0^t \sum_{j=1}^{n} \sigma_{ij}(s) dW_s^j)}$$

As can be seen the solution of each component is parallel with the solution of the Geometric Brownian motion. The solution can be varied in the same as in the univariate model but here using the multifactor Itō's formula.

6.2 Theorems in multivariate setting

In the same fashion as in the univariate case, the mechanism for the pricing of an option is through Girsanov theorem and the Brownian martingale representation theorem, now in a multivariate setting. We state the theorem, the proofs can be found in Protter (1990).

Multivariate Girsanov Theorem:

Let $\{W_t^i\}_{t\geq 0}$, i=1,…,n, be independent Brownian motions under the measure $\mathbb{P}$ generating the filtration $\{\mathcal{F}_t\}_{t\geq 0}$ and let $\{\theta_i\}_{t\geq 0}$, i=1,…,n, be $\{\mathcal{F}_t\}_{t\geq 0}$-previsible processes (that is predictable given the filtration) such that $\mathbb{E}^{\mathbb{P}}\left[e^{\frac{1}{2}\int_0^T \sum_{i=1}^{n} \theta_i^2(s) ds)}\right] < \infty$ holds. Then there exists a new measure $\mathbb{Q}$ such that the processes $\{X_t^i\}_{t\geq 0}$, i=1,…n, defined by $X_t^i = W_t^i + \int_0^t \theta_i(s)\, ds$ are all martingales.

Multivariate Martingale Representation Theorem:

Let $\{W_t^i\}_{t\geq 0}$, i=1,…,n, be independent Brownian motions under the measure $\mathbb{P}$ generating the filtration $\{\mathcal{F}_t\}_{t\geq 0}$. Let $\{M_t^1, M_t^2, \dots., M_t^n\}_{t\geq 0}$ be given by: $dM_t^i = \sum_{j=1}^{n} \sigma_{ij}(t) dW_t^j$ and let $\mathbb{E}\left[e^{\frac{1}{2}\int_0^T \sum_{j=1}^{n} \sigma_{ij}(t)^2 dt}\right] < \infty$ and the volatility matrix (σ_{ij}) be non-singular. Then if $\{N_t\}_{t\geq 0}$ is any one-dimensional $(\mathbb{P}, \{\mathcal{F}_t\}_{t\geq 0})$ martingale, there exists an n-dimensional $\{\mathcal{F}_t\}_{t\geq 0}$-previsible process $\{\phi_t^1, \phi_t^2, \dots., \phi_t^n\}_{t\geq 0}$ such that $N_t = N_0 + \sum_{j=1}^{n} \int_0^t \phi_s^j\, dM_s^j$.

6.3 Valuing the option in the general multivariate case

The theorems state the same in the univariate case, the result is that one prices the options in a similar fashion. Assuming that the processes are Geometric Brownian motions we can price the option by using the solution of the n-factor model. The solution corresponds to the univariate case, when no correlations are assumed. Thus the option is priced by simulating sample paths of all risk factors under $\mathbb{Q}$. This means the discounted processes are martingales, and thus the drift is set by the risk free rate. Constructing all the processes from beginning to maturity of the option, we can then price the option by using the processes to calculate the payoff of the option $E_{\mathbb{Q}}[e^{-rT}C_T|\mathcal{F}_t]$ and discounting back to time 0.

7 Applying real option valuation

As we have stated in chapter three, for the implementation we have chosen for the more transparent method of option valuation. Another advantage is that we are able to choose for the mean reverting process for the stochastic factors which can be more realistic for certain processes.

7.1 Three different options

As we have seen there are three different options to be priced, and they are all priced slightly differently.

Extension option	$\mathbb{E}[Yield * RPK - (Oil + variable\ costs + Contract\ extension\ costs)]$
Purchase option	$\mathbb{E}[Yield * RPK - (Oil + variable\ costs) + Residual\ value] + Exercise\ price$
Reinvestment option	$\mathbb{E}[Yield * RPK - (Oil + variable\ costs\ + future\ lease\ costs)]$

Where RPK follows GDP and Yield, variable costs are deterministic and the future lease cost is the simulated direct lease rate. The fixed costs are assumed to be independent of the options.

All stochastic factors are simulated and at maturity of the options, the corresponding expression is evaluated. At the moment the expression is positive the option will be exercised else not. At the moment the option is exercised, this equals the option value at that moment.

7.2 Justification of option valuation using simulation

One reason why we use simulation is the fact that valuation not easy to understand. Unlike the Black Scholes formula, which is an analytic formula for the calculation of the option we have a multivariate setting. Due to the structure of the option, in which the revenues are subtracted from the costs, an analytic solution is not possible. For example, when all factors like yield, GDP, oil are Geometric Brownian motion processes, then the revenues is still a Geometric Brownian motion. But subtracting the oil costs, results in two lognormally distributed that are subtracted from each other. And there exists no analytic expression for the subtraction of two lognormally distributed variables.

Another reason is that the replication argument, that is using the alternative measure, for valuing real options might not hold as it would for financial options. Since in financial options no arbitrage is enforced, due to the possibility of going long and short. However real options on aircraft are not traded (directly) on open markets. Thus arbitrage opportunities are more likely to occur. Nonetheless from a financial perspective the cash flows from a real option can be replicated by identifying the right securities in financial markets. A recent development is the development of economic derivatives. In 2002 these were introduced and includes derivatives on the American GDP, with this the first steps toward the replication of cash flows as we is done with the pricing of financial options are set.

7.3 Implementation in Excel

The tool calculates the expanded NPV **given** one financing mix and a set of parameters.

	A	B	C	D	E	F	G	H	I
1	Contract A	Contract B	Contract C	Contract D	Contract E	Contract F	Contract G		Total allocated
2	2	3	0	0	0	0	0		5
3									
4									
5	**Expanded NPV of mix**	Static NPV	Option premia		**Average profit mix**				
6	555,826,969	277,236,438	278,590,531		125,297,788		Show parameter screen		
7	-350,915,638	-350,915,638	0						
8	-215,994,232	-215,994,232	0						
9	186,908,452	113,038,004	73,870,447		Average static NPV				
10	221,699,989	96,854,394	124,845,595		52,046,116				
11	134,239,795	83,438,503	50,801,292						
12	-80,861,027	-86,960,729	6,099,702		Average option Premia				
13	150,087,496	60,358,644	89,728,851		83,251,672				
14	144,109,018	-74,862,111	218,971,129						
15	106,893,081	62,578,444	44,314,636		Fixed costs				
16	60,108,434	39,726,483	20,381,952		10,000,000				
17	351,215,245	192,078,238	159,137,007						
18	319,574,492	227,009,228	92,565,265						
19	153,994,535	83,302,449	70,692,086						
20	201,591,286	121,118,561	80,472,725						
21	122,057,911	52,277,258	69,780,653						
22	-21,481,708	-23,492,990	2,011,282						
23	287,347,123	158,814,612	128,532,511						
24	169,598,625	27,270,956	142,327,669						
25	-39,394,720	-39,394,720	0						
26	169,296,592	81,532,477	87,764,115						
27	418,766,512	218,916,070	199,850,442						
28	159,130,704	46,191,344	112,939,360						
29	128,959,374	101,344,822	27,614,552						
30	49,686,405	49,686,405	0						

In this example we consider the financing of 5 aircraft of the type Embraer 190 which replaces the Fokker 100, we have considered 2 contracts in our financing mix. Two aircraft from Contract A which is a direct lease contract for 8 years and three times Contract B which is a financial lease contract for 20 years, given this mix the average expanded NPV from 25 scenarios is €125,297,788. More scenarios can be taken and the more one takes, the more precise the average profit which is the pay off of the financing mix will become.

Next we change the given mix into 3 of contract A and 2 of contract B. Here the average expanded NPV is €118,919,358. Which means that trading in one of the financial lease contract into a direct lease contract is more attractive. One can continue this comparison with different mixes until the highest expanded NPV is found, which is the optimal financing mix.

	A	B	C	D	E	F	G	H	I
1	Contract A	Contract B	Contract C	Contract D	Contract E	Contract F	Contract G		Total allocated
2	3	2	0	0	0	0	0		5
3									
4									
5	**Expanded NPV of mix**	Static NPV	Option premia		**Average profit mix**				
6	517,705,750	205,250,862	312,454,888		118,919,358		Show parameter screen		
7	-315,325,800	-315,325,800	0						
8	-227,526,169	-227,526,169	0						
9	171,774,148	80,280,640	91,493,508		Average static NPV				
10	261,535,957	74,267,565	187,268,392		31,241,441				
11	139,907,109	66,474,974	73,432,135						
12	-81,557,265	-87,945,849	6,388,584		Average option Premia				
13	162,351,914	39,393,322	122,958,592		97,677,918				
14	53,007,785	-92,972,968	145,980,753						
15	113,177,881	46,705,927	66,471,954		Fixed costs				
16	51,630,929	22,702,499	28,928,430		10,000,000				
17	355,180,135	147,255,786	207,924,348						
18	306,806,681	203,235,220	103,571,460						
19	143,942,043	56,477,520	87,464,523						
20	210,871,146	102,916,543	107,954,603						
21	128,383,620	35,418,503	92,965,118						
22	-13,584,547	-16,601,470	3,016,923						
23	269,049,335	116,450,991	152,598,344						
24	148,925,233	-4,672,013	153,597,247						
25	-34,554,727	-34,554,727	0						
26	149,076,336	58,481,764	90,594,572						
27	387,952,800	165,989,312	221,963,488						
28	153,506,027	10,007,777	143,498,250						
29	120,813,278	79,391,449	41,421,828						
30	49,934,357	49,934,357	0						

We have taken the highest expanded NPV, that is the highest average profit mix, as the 'optimal' choice. In fact looking at the scenarios once can spot different ranges of profits and possible losses. The optimal choice can be different than the highest expanded NPV, this could be the case when an airliner as a specific risk appetite. That is there are financing mixes with lower expanded NPV but also lower risk (that is less scenarios) with (high) losses.

Thus the choice is different for different companies and management, depending on the strategy of the company, the tool is a decision support tool.

7.4 Determining the parameters of the stochastic factors

The parameters concern the behavior of the stochastic factors in the future. This can depend on the historic behavior of the relevant factors but is more importantly based on expectation and vision on the future. Historic evolution is not a guarantee for future evolution. From March 1988 to March 2010, the yield decreased by an average of 1,5%. This is due to increasing competition and regulation and support of national governments on their airline market. Because of this fact airliners have sunk in operational losses and it cannot be expected that the trend of 1,5% decrease in yield will and can continue. Because of these facts, the parameters will have to be chosen carefully and estimation of parameters must be taken with care. The observed volatility in the same period was 0,085, whether this holds for the future is a question. The same difficulties arise with estimating the cost of oil.

In general one is advised to increase the volatility parameter, whenever one is uncertain about the parameters. This will enable the financing decision to be proof to large uncertainties concerning the future.

For the implementation the set of parameters are chosen and is influenced primarily by corporating both historic performance and future expectations. For more information one is advised to read the manual of the tool where all the parameter are explained.

The above implementation example works with the following parameters:

Process	Type of SP	Today's value	vol	change per year	rate of mean reversion	long term mean
GDP	Geometric Brownian Motion	100.00	0.10	0.030	0,1	85.00
Yield	Geometric Brownian Motion	0.15	0.10	0.000	0,1	0,14
Crude oil	Geometric Brownian Motion	100.00	0.10	0.000	0,1	120.00
Direct lease rate	Geometric Brownian Motion	225000.00	0.01	0.000	2.000	280000.00
Residual value ac	Geometric Brownian Motion	30000000.00	0.00	-0.100	0,2	8000000.00

7.5 Calculation steps

The program follows the following steps:

1) Calculate fleet size from contracts over time
2) Extract all options out of contract
3) Calculate ownership costs from contract
4) Calculate RPK for contract and options and profit of all contract (revenues & costs)
5) **Now iterate over time, for each year:**
   Look if contract option and/or reinvestment option available.
   For each that is available:
   forecast its profitability.
   by looking at the revenues, costs (type option dependent)
   Now if forecasted profit > 0, then exercise option and reduce available RPK for options.

8 Conclusion

We have proposed a methodology describing how the financing decision for fleet mutations can be handled. This methodology, real options analysis, incorporates the valuation of options both implicit and explicit as opposed to the discounted cash flow method. This has the benefit that purchase and extension options are valued correctly, which are available in different financing constructions in the airline industry. Additionally the methodology identifies reinvestment options which makes the comparison of different possible financing constructions in a fair way possible.

Using real options analysis we have also modeled the determinants of profitability of an investment in a stochastic way. That is the behavior of both the revenues and costs fluctuate over time as a result the influence of determinants such as oil prices and yields are evaluated in a realistic. This poses another important advantage over existing traditional methodologies in the airline industry where oil prices and yields are assumed constant or increasing constantly over time.

One potential disadvantage of such an analysis is the difficulty of comprehending the theory on stochastic processes, although the processes are essentially composed of normal distributions. The theory on martingales and mathematical measures posses another potential difficulty. For the implementation we have chosen to avoid the necessary understanding of martingales and mathematical measures by using the WACC discount factor which is an alternative method used in existing financing decisions.

In combination of the WACC discount factor, the valuation of the different options and thus the fair comparison between financing constructions becomes transparent. Given the parameters of the stochastic behavior of the different stochastic factors, the value of flexibility in the form of options is visible. In general the more uncertainty is present, the more valuable options are and the more attractive a financing constructions with options becomes. This is translated in an overall measure called the expanded NPV, which enables us to make a choice. In general prices ownership is cheaper but results in less flexibility, while flexible leases are more expensive. The expanded NPV incorporates these effects and makes the tradeoff and the calculations show exactly what the added value of options and fixed ownership is. This can be used to show why one alternative is more attractive than the other in a comprehensible transparent way.

9 Further research

Further research can be conducted on the calibration of the parameters on the different stochastic processes. The choice of the type of stochastic processes can be broadened outside the Geometric Brownian Motion and Mean version. The current assumed constant parameters can be replaced by stochastic parameters, for example the volatility parameter. These models are known as stochastic volatility models. Next to that the stochastic factors are assumed to evolve independently except the GDP and Yield. Specifying the correlations between the current stochastic factors is available in the implementation.

Besides the five identified stochastic factors as the driving force influencing the financing decision, other factors like the Euro/Dollar exchange rate and maintenance costs can be included as well. When there is enough confidence and understanding of martingale theory along with changing measures, the current implementation can easily be adapted to risk neutral pricing.

In the current analysis the ASK, that is the market, is assumed independent of other type of markets. In general a distinction is made in short, medium and long haul and within these markets distinctions can be made as well. These markets not operating independently as one of airplane can operate on different distances and markets.

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