Markov decision processes with unbounded transition rates: structural properties of the relative value function

Master's thesis in Stochastics and Financial Mathematics

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April 20, 2011

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Abstract

In this thesis we have studied Markov decision processes with unbounded transition rates. In order to facilitate the search for optimal policies, we are interested in structural properties of the relative value function of these systems. Properties of interest are for example monotonicity or convexity as a function of the input parameters. This can not be done by the standard mathematical tools, since the systems are not uniformizable. In this study we have examined whether a newly developed method called Smoothed Rate Truncation can overcome this problem.

We introduce how this method is used for a processor sharing queue. We have shown that it can be applied to a system with service control. We have also obtained nice results in the framework of event-based dynamic programming. Due to Smoothed Rate Truncation new operators arise. We have shown that for these operators propagation results, similar to results for existing operators, can be derived. We can conclude that Smoothed Rate Truncation can be used to analyse other processes that have unbounded transition rates.

Preface

This thesis finishes the master Stochastics and Financial Mathematics at the Utrecht University. I enjoyed doing research in this field.

First of all I would like to thank my supervisors Sandjai Bhulai and Floske Spieksma. They have provided me with a nice subject and gave me the opportunity to build on their work. It has been a pleasure to work with them, their enthusiasm often motivated me to continue the research. I look forward to the further cooperation as a PhD student. I am thankful for Karma Dajani who has agreed to read this thesis and has been my advisor in the last years of my study.

This thesis would not have been possible without the help of my family. Special thanks to my mother, Hester, Christien and Geertine for giving me time to study while they were looking after Benjamin. I would like to thank my parents for their continuous support and providing me with a place to study in peace. Finally I would like to thank my Aryéle for her love, for her understanding and encouragement.

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Chapter 1

Introduction

If we look at Markov control problems, it is desirable that optimal strategies for these problems are well-behaved as a function of the input parameters. For practical applications, this makes computation or approximation of optimal policies possible. To check if a model has such a structure on its policies, we can study the so-called relative value function. Important properties of the value function, necessary to show the desirable structure, are monotonicity, convexity or supermodularity. A standard work is Puterman [10], and monotonicity results are given in Stidham and Weber [12], [13]. A common method for deriving such properties is the successive approximations method for discrete-time Markov decision processes.
If the original problem is modelled in continuous-time, it is possible to use the successive approximations scheme, but that requires that the Markov decision process is uniformizable (see [9]). A continuous-time Markov decision process is uniformizable if the transition rates are bounded. For continuous-time MDPs that have bounded rates, there is a nice framework developed, called event-based dynamic programming, which unifies results obtained for different models. The founders of this framework were Topkis [14] and later Glasserman and Yao [4] and [5]. Koole [8] has further developed this area, in this thesis we will use his work as a main reference.

In modern applications of Markov decision processes the transition rates are often unbounded as a function of actions or states. This can be due to impatient customers. The successive approximations method is not applicable then. There have been attempts to overcome this problem. In a paper of Down et al. [3], where a system with service control on a single server with abandonments is treated, a finite state space truncation is used. To keep the structural properties of the model intact, a special choice of the transition rates on the boundary is necessary. Unfortunately, this method is very model-dependent and seems not to be applicable in general.
Bhulai et al. [2] have studied a different system where jump rates are unbounded. A processor sharing queue with impatient customers who can retry if they abandon the queue. Two solutions are presented, a coupling method and a method called Smoothed Rate Truncation principle (abbreviated SRT). This last method uses linearly truncated rates, such that the adapted system has a recurrent closed class which is finite. As a consequence the transition rates become bounded.

We have investigated if the SRT principle is more widely applicable. We have successfully applied the SRT method on the model that is treated in [3]. With this method the same results are obtained as in the paper. Further, we have studied to what extent SRT fits in the framework of event-based DP. The theory of event-based dynamic programming can be extended with theory on MDPs with unbounded transition rates.

The structure of the thesis is as follows. In the next chapter we give an introduction of the existing methods for showing structural properties and we specify the problem with unbounded rates. In Chapter 3 we introduce the smoothed rate truncation and demonstrate its application on the processor sharing queue and the system with service control. In Chapter 4 we derive some results in event-based DP that can be obtained for events in systems with unbounded transition rates. We finish by drawing some conclusions and recommendations for further research in Chapter 5.

Chapter 2

Basic theory

2.1 Markov decision processes

A Markov decision process is a stochastic process, where the probabilities can be influenced by a decision maker. Markov decision processes provide a mathematical framework for modelling decision making in many applications.
We are interested in giving a structure on optimal policies of Markov control problems. For example, in many systems it would be desirable that the optimal policy is a threshold policy. I.e. the optimal action would depend on a threshold value as a function of the input parameters. This can be done by using properties of the relative value function of these models. Without this structure the search for optimal policies is very difficult, if not impossible. For discrete-time MDPs there is a method for showing such structural properties of the value function: the successive approximations scheme.
In real life, most problems are in continuous time. The method for showing structural properties for these problems, is to translate these processes into discrete-time processes. In order to do so a uniformization step is needed. In this chapter we will first explain how to compute the relative value function using the successive approximations scheme. Then we will explain uniformization of continuous-time MDPs.

A discrete-time Markov decision process consists of the following quadruple

$$\{\mathcal{X},\ \mathcal{A},\ p(x,a,y),\ r(x,a)\}.$$

1. A denumerable set $\mathcal{X}$, called the *state space*. It is the set of all states of the system under observation;
2. The compact *action space* $\mathcal{A}$, the set of all actions that the decision maker can take;
3. The *transition probabilities* $p(x,a,y)$, the probability that under action a the system moves from state x to y;
4. A function $r : \mathcal{X} \times \mathcal{A} \to \mathbb{R}$, the *reward function*. We can replace $r(x,a)$ by $c(x,a)$ and interpret it as a cost function instead of a reward function.

A deterministic stationary policy $R : \mathcal{X} \to \mathcal{A}$ is a function which assigns an action to each state. Let $\mathcal{R}$ be the set of all policies. We assume that the conditions for existence of a limiting distribution are fulfilled. For a theoretical background on MDPs see Hernandez-Lerma and Lassere [6] and [7]. Let π_*^R be the limiting distribution under policy R. The goal of our optimization is to maximize the expected average reward g^R, where

$$g^R = \sum_{x \in \mathcal{X}} \pi_*^R(x) r(x, R(x)).$$

Calculation and maximization of the expected average reward directly is not trivial. In order to evaluate the expected average rewards, for each policy, it is necessary to compute the limiting distribution π_*^R. To avoid this, we use optimality criteria for the maximum expected average reward g. This is stated in the optimality equation or Bellman equation:

$$V(x) + g = \max_{a \in \mathcal{A}} \{r(x, a) + \sum_{y \in \mathcal{X}} p(x, a, y) V(y)\},$$

where V is the relative value function of the optimal policy. This is a system with a countable collection of variables and equations and is not easy to solve. A general method to overcome this problem is to approximate the value function via the successive approximations scheme. Define a sequence V_n as follows. Let $V_0 \equiv 0$ and iteratively

$$V_{n+1}(x) = \max_{a \in \mathcal{A}} \{r(x, a) + \sum_{y \in \mathcal{X}} p(x, a, y) V_n(y)\}.$$

Then we have that $V_n(x) - V_n(0) \to V(x) - V(0)$ as $n \to \infty$, for some reference state 0. This algorithm plays a crucial role when we want to show properties of the relative value function. We will see that we can only prove anything if the properties propagate through this recursion. In practice, for most problems the time is continuous. Under some conditions the continuous-time processes have a discrete-time equivalent. Let a continuous-time Markov decision process be given. Let $\lambda(x, a, y)$ be the rate from x to y under action a. Then the process is uniformizable if there exists a number γ such that for all x and all a we have $\sum_{y \neq x} \lambda(x, a, y) \leq \gamma$. The *uniformization* works as follows. In each state x with $\sum_y \lambda(x, a, y) < \gamma$ we add fictitious transitions from x to x, such that the rates out of a state sum up to γ. This new process will have expected transition times equal to $1/\gamma$ for all states. Then we can define the related discrete-time process with transition probabilities $p(x, a, y) = \lambda(x, a, y)/\gamma$ if $y \neq x$ and $p(x, a, x) = 1 - \sum_{y \neq x} \lambda(x, a, y)/\gamma$. For this discrete time process we are able to approximate the relative value function using the successive approximations scheme.

Through an example, we will illustrate how this uniformization works. Let us look at the $M|M|s$-queue. This is a multi-server queue model with the following specifications. Arrivals occur according to a Poisson process with rate λ. The service time is exponentially distributed with mean $1/\mu$, there is a fixed number of s servers. All servers are mutually independent and independent of arrivals. The queue has an infinite buffer, so $\mathcal{X} = \mathbb{Z}_+$. There are no actions in this example. If there are s or more customers in the system then the servers are working at maximum speed with rate $s\mu$. The service rate is $x\mu$ if there are less than s customers in the system. So we get the transition rates

$$\lambda(x, y) = \begin{cases} \lambda & \text{if } y = x + 1, \\ x\mu & \text{if } y = x - 1, 0 \leq x \leq s, \\ s\mu & \text{if } y = x - 1, \ x > s. \end{cases}$$

All rates are zero elsewhere. We see that the maximum jump rate out of a state is $\lambda + s\mu$, so we take $\gamma = \lambda + s\mu$. Then artificial rates are introduced such that we have rate out equal to γ for all x. For $x < s$ we add dummy transitions; $\lambda(x,x) = (s-x)\mu$. The discrete-time equivalent of this process has the following transition probabilities

$$p(x,y) = \begin{cases} \lambda/\gamma & \text{if } y = x+1, \\ x\mu/\gamma & \text{if } y = x-1,\ 0 \le x \le s, \\ s\mu/\gamma & \text{if } y = x-1,\ x > s, \\ (s-x)\mu/\gamma & \text{if } y = x,\ 0 \le x \le s. \end{cases}$$

Now that the uniformization is finished, we are ready to use the successive approximations scheme. In the next section we specify which properties of the relative value function we are interested in.

2.2 Monotonicity

Let $f : \mathcal{X} \to \mathbb{R}$. Let $\mathcal{X} = \mathbb{Z}_+^m$. For $1 \le i,j \le m$, define the following properties, let f is

$$\begin{aligned} I(i) \text{ if } & \quad f(x+e_i) \ge f(x) & \forall x \in \mathcal{X}; \\ C(i) \text{ if } & \quad f(x+2e_i) - f(x+e_i) \ge f(x+e_i) - f(x) & \forall x \in \mathcal{X}; \\ \text{Super}(i,j) \text{ if } & \quad f(x) + f(x+e_i+e_j) \ge f(x+e_i) + f(x+e_j) & \forall x \in \mathcal{X}; \\ DI(i,j) \text{ if } & \quad f(x+e_i) \ge f(x+e_j) & \forall x \in \mathcal{X}, \end{aligned}$$

where e_i is the i-th unit vector. In words, f is $I(i)$ if it is non-decreasing in variable i; $C(i)$ means convex in variable i. A function is Super(i,j) if it is supermodular in the variables i and j, this is a sort of 2-dimensional generalization of convexity. $DI(i,j)$ means that f is increasing in the direction $(e_i, -e_j)$.

We apply successive approximations to the $M|M|s$-queue with holding cost $r(x) = x$. Without loss of generality we can assume that $\gamma = 1$. Then define V_n by the successive approximations scheme. $V_0 \equiv 0$ and

$$V_{n+1}(x) = \begin{cases} x + \lambda V_n(x+1) + \mu x V_n(x-1) + \mu(s-x)V_n(x) & 0 \le x \le s, \\ x + \lambda V_n(x+1) + \mu s V_n(x-1) & x > s. \end{cases}$$

As an example, we will prove that V_n is non-decreasing, for all n.

Proof. The proof is done with induction. Clearly $V_0 \equiv 0$ is non-decreasing. Now suppose that

V_n is non-decreasing, then V_{n+1} is non-decreasing, since for $0 \leq x < s$ we have

$$\begin{aligned} V_{n+1}(x+1) - V_{n+1}(x) &= (x+1) - x \\ &+ \lambda V_n(x+2) - \lambda V_n(x+1) \\ &+ \mu(x+1)V_n(x) - \mu x V_n(x-1) \\ &+ \mu(s-x-1)V_n(x+1) - \mu(s-x)V_n(x) \\ &= 1 + \lambda[V_n(x+2) - V_n(x+1)] \\ &+ \mu x[V_n(x) - V_n(x-1)] + \mu V_n(x) \\ &+ \mu(s-x-1)[V_n(x+1) - V_n(x)] - \mu V_n(x) \\ &\geq 0. \end{aligned}$$

All terms between the square brackets are greater than or equal to zero, because of the induction hypothesis. This gives the desired inequality. For $x \geq s$ we have

$$\begin{aligned} V_{n+1}(x+1) - V_{n+1}(x) &= (x+1) - x \\ &+ \lambda[V_n(x+2) - V_n(x+1)] \\ &+ \mu s[V_n(x) - V_n(x-1)] \\ &\geq 0. \end{aligned}$$

With induction we conclude that V_n is non-decreasing, for all n. □

Hence, by taking the limit $n \to \infty$, we conclude that the relative value function V has the same property, i.o.w. V is non-decreasing. If we would add actions to this model, this structure could help in finding more efficient algorithms to determine an optimal policy.

2.3 Formal specification of problem

There are models where the Markov decision process does not have all properties necessary for uniformization. For example, in many telecommunication systems the total rate out of a state grows infinitely large as a function of the state variable. If the transition rates are unbounded, then uniformization is not possible. Hence it is not possible (or very difficult) to prove structural properties for the relative value function. This in spite of the fact that there may be strong evidence for the process to have these properties.

In the previous example we have seen how uniformization works in the $M|M|s$-queue. Now we look at the $M|M|\infty$-queue. This queue is the same as the $M|M|s$-queue, except that there is an infinite number of servers. We have transition rates

$$\lambda(x, y) = \begin{cases} \lambda & \text{if } y = x + 1, \\ x\mu & \text{if } y = x - 1. \end{cases}$$

The jump rates out of a state sum up to $\lambda + x\mu$,which tends to infinity when x grows to infinity. The transition rates are unbounded, so uniformization is not possible.

It would be desirable to have a method that allows us to solve such problems. That is the main question we consider in this thesis. How can we prove structural properties of the relative value

function of Markov decision processes with unbounded rates?
We will focus especially on the smoothed rate truncation principle. Is it possible to use the SRT-principle to prove properties of the value function for different systems with unbounded rates? Further, we want to know if the smoothed rate truncation principle can be embedded in the event-based dynamic programming framework? How should this be done?

Chapter 3

Smoothed Rate Truncation principle

3.1 Definition and intuition

In order to deal with processes with unbounded rates, we will use an adaptation of these processes such that the rates become bounded. We will do this by truncating part of the transition rates, such that the resulting process has a finite recurrent closed class. Therefore, as the transition rates increase as a function of the state, the rates remain bounded inside the recurrent class. We approximate the model on a countable state space $\mathcal{X}$ by a series of models with finite state spaces $\mathcal{X}^N$, such that $\mathcal{X}^N \subset \mathcal{X}^{N+1}$ and $\lim_{N\to\infty} \mathcal{X}^N = \mathcal{X}$. Let the N^{th} model have transition rates $\lambda^N(x,a,y)$. These should be a perturbation of $\lambda(x,a,y)$ such that there are no transitions from $\mathcal{X}^N$ to $\mathcal{X} \setminus \mathcal{X}^N$.
One naïve way to do this is to leave all transition rates unchanged, except for the transitions that move out of $\mathcal{X}^N$, i.e.

$$\begin{aligned}
\lambda^N(x,a,y) &= \lambda(x,a,y) \text{ for } x \neq y \in \mathcal{X}^N, \\
\lambda^N(x,a,y) &= 0 \text{ if } x \in \mathcal{X}^N,\ y \notin \mathcal{X}^N, \\
\lambda^N(x,a,x) &= \lambda(x,a,x) + \sum_{y \notin \mathcal{X}^N} \lambda(x,a,y) \text{ if } x \in \mathcal{X}^N, \\
\lambda^N(x,a,y) &= 0 \text{ if } x \notin \mathcal{X}^N.
\end{aligned}$$

Unfortunately, close to the boundary the value function will lose its inherent properties as a consequence of the sharp cut in the rates.

An example where the value function loses its properties is the following. It is possible to solve the Poisson equations for the $M|M|1$-queue (see Bhulai [1]), and we obtain an exact formula of the value function

$$V(x) = \frac{x(x+1)}{2\mu(1-\rho)},$$

with $\rho = \lambda/\mu < 1$. The value function is clearly convex. On the other hand, if we truncate the system as described above, then the resulting process obtains the following value function:

$$V^N(x) = \frac{x(x+1)}{2\mu(1-\rho)} - c\left(x + \frac{\left(\frac{1}{\rho}\right)^x - 1}{1-\rho}\right),$$

where the constant c is given by

$$c = \frac{(N+1)\rho}{\mu\left(\left(\frac{1}{\rho}\right)^N - \rho\right)(1-\rho)}.$$

The value function of the truncated system consists of a positive part and a dominant negative part which increases exponentially, hence V^N is not convex anymore.

We wish to avoid this problem, therefore it is necessary to truncate the transition rates in a more smoothed way. This is the idea of the Smoothed Rate Truncation principle: the rates of all transitions leading in a direction outside $\mathcal{X}^N$ are linearly decreased until they equal zero. They are kept zero outside $\mathcal{X}^N$. All transitions that make the state variables smaller stay unchanged. In this way stability and ergodicity are preserved.
The smoothed rate truncation, which we shall abbreviate as SRT, is linear in the state variable. The transition rates are linearly truncated as the state increases until they equal zero. Suppose that a transition moves in the direction x_i with rate λ, then in the perturbed process this rate is replaced by $\lambda^N := \lambda(1 - x_i/N)^+$.

3.2 Limit theorems

The SRT method can only work, if the relative value function of the smoothed MDP converges to the relative value function of the original Markov decision process. We will need a limit theorem to get the validity of this approach.

Suppose we have a collection of parametrized countable state Markov processes, $X(a) = \{X_t(a)\}_t$, where a is a parameter from a compact parameter set A. Let $Q(a) = \left(q_{xy}\right)$ be the associated rate matrix. We assume each process $X(a)$ has at most one closed class, plus possibly inessential states.

A function $f : E \to \mathbb{R}_+$ is a *moment function* if there exists an increasing sequence of finite sets $E_n \uparrow E,\ n \to \infty$, such that $\inf\{f(x) | x \notin E_n\} \to \infty$ as $n \to \infty$. We can use the following theorem, that is proven by Spieksma [11].

Theorem 1. *Let the collection $\{X(a)\}_{a\in A}$ as above. Suppose the following conditions hold.*

i) $\{X(a)\}_{a\in A}$ is f-exponentially recurrent for some moment function f. I.e. there exists a moment function $f : E \to \mathbb{R}_+$, constants $c,\ d > 0$ and a finite set K such that

$$\sum_y q_{xy}(a) f(y) \le -cf(x) + d\boldsymbol{1}_{\{K\}}(x), \quad x \in E.$$

ii) $a \mapsto q_{xy}(a),\ a \mapsto \sum_y q_{xy}(a)f(y)$ *are continuous functions, for each* $x, y \in E$.

Then we have the following properties.

1) $\{X(a)\}_{a\in A}$ *is* f*-exponentially ergodic, in other words, there exist constants* $\alpha, \kappa > 0$ *with*

$$\sum_y |P_{t,xy}(a) - \pi(a)|f(y) \le \kappa e^{-\alpha t} f(x), \quad t \ge 0, a \in A,$$

where $\pi(a) = (\pi_x(a))_{x\in\mathcal{X}}$ *is the unique stationary distribution of* $X(a)$.

2) Let $c(a) = (c(x,a))_{x\in E},\ a \in A$*, be a cost function with* $a \mapsto c(x,a)$ *continuous for each* $x \in E$*. If* $\sup_x |\sup_a c(x,a)|/f(x) < \infty$*, then* $a \mapsto g(a) = \pi(a)c(a) = \sum_x \pi_x(a)c(x,a)$ *is continuous and* $a \mapsto V(a) = \int_0^\infty P_t(a)(c(a) - g(a))dt$ *is component-wise continuous.*

Now we wish to apply this theorem to a parametrized collection of Markov processes, associated with the smoothed rate truncation. Let $A = 0 \cup \{1/N\}_{N\in\mathbb{N}}$. For an optimal policy R^*, on A we define the rate matrix $Q(a)$ as follows.
For $a = 0$ we take $Q(0) = \big(\lambda(x, R^*(x), y)\big)_{xy}$, for $a = 1/N$ let $Q(1/N) = \big(\lambda^N(x, R^*(x), y)\big)_{xy}$.
The following corollary gives the desired convergence of the value function.

Corollary 1. *Let* V^N *be the associated value function of the process* $\{X(1/N)\}_{N\in\mathbb{N}}$*, let* V *be the value function associated to* $\{X(0)\}$*.*
If $\{X(a)\}$ *satisfies conditions i) and ii) from Theorem 1, then for all* $x \in \mathcal{X}$

$$V^N(x) \to V(x).$$

The models that we have studied satisfy the conditions of the theorem easily.

3.3 Processor-sharing retrial queue

The processor-sharing queue has been examined by Bhulai et al. [2]. This is the model for which they have developed the SRT principle. We will first introduce the model and then look at how the SRT principle does its work. We will pay special attention to the difficulties that arose and the solutions to these difficulties.

The system is as follows. There is a service facility Q_1 where customers arrive according to a Poisson process with rate λ. The customers are served in a processor-sharing discipline, where the potential inter-departure times are exponential with mean $1/\mu$. Customers are impatient in the following way. They will leave Q_1 after an exponentially distributed period with mean $1/\beta$, independent of all other customers. After a customer has left Q_1 it abandons the system with probability $1-\psi$ and it will go to a retrial queue Q_2 with probability ψ. Each customer in the retrial queue independently rejoins the tail of Q_1 after an exponentially waiting time with mean $1/\gamma$. Further, the system is controlled by the joining rule $[s]$, for $s \in [0,1]$ upon arrival a customer joins facility Q_1 with probability s and gets blocked with probability $1-s$, independent of other customers. In the paper properties of the value function as a function of the control parameter s are obtained. This requires a difficult approximation technique and we will not discuss it here. We will regard s as if it is constant.

The system can be modelled as follows. Let $(X(s), Y(s)) = \{X_t(s), Y_t(s)\}_{t\geq 0}$ be a Markov process representing the number of customers in facilities Q_1 and Q_2, under joining rule $[s]$ at time t. The state space $\mathcal{X} = \mathbb{Z}_+ \times \mathbb{Z}_+$, so $(X_t(s), Y_t(s)) = (x, y) \in \mathcal{X}$ if at time t there are x customers in Q_1 and y customers in Q_2. Denote the transition rate from (x, y) to (x', y') by $q_{xy,x'y'}(s)$. Then for $(x, y), (x', y') \neq (x, y) \in \mathbb{Z}_+ \times \mathbb{Z}_+$ we have

$$q_{xy,x'y'}(s) = \begin{cases} \lambda s & \text{if } (x', y') = (x+1, y), \\ \mu + \beta x(1-\psi) & \text{if } (x', y') = (x-1, y), x > 0, \\ \beta x \psi & \text{if } (x', y') = (x-1, y+1), x > 0, \\ y\gamma & \text{if } (x', y') = (x+1, y-1), y > 0, \\ 0 & \text{else.} \end{cases}$$

We have a cost function $d(x, y) = dx + y$, with $d = 1 + \beta(1-\psi)R$, representing the holding costs of both facilities plus the future potential loss due to abandonments. Further it is important to note that this process satisfies the stability conditions stated in Section 3.2 if either $\psi < 1$, or $\psi = 1$ and $\lambda < \mu$.

In order to show structural properties of the relative value function, smoothed rate truncation, as described in Section 3.1 was applied. Hence, the jump rates that increase one of the components of the state variable are adapted in the following way

$$\begin{aligned} \lambda^N_{x,y} &= \lambda(1 - \frac{x}{N})^+, \\ \gamma^N_{x,y} &= \gamma(1 - \frac{x}{N})^+, \\ (\beta\psi)^N_{x,y} &= \beta\psi(1 - \frac{y}{b(N)})^+, \end{aligned}$$

where N and $b(N)$ are natural numbers. The smoothing ensures that there is one finite closed class, $\mathcal{X}^N = \{(x, y) \in \mathcal{X} |\ x \leq N, y \leq b(N)\}$. On $\mathcal{X}^N$ the transition rates are bounded, w.l.o.g. we can assume that the total jump rate is less than 1. So on $\mathcal{X}^N$ uniformization is possible and we are able to compute the relative value function by successive approximations. We suppress N in the notation of this scheme, so V^N_n will be just V_n. We will always do this if write down the successive approximations scheme. Define $V_0 \equiv 0$, and on $\mathcal{X}^N$

$$\begin{aligned} V_{n+1}(x, y) = {} & \lambda^N_{x,y} s V_n(x+1, y) + \mu V_n((x-1)^+, y) + \gamma^N_{x,y} y V_n(x+1, y-1) \\ & + \beta x(1-\psi)(V_n(x-1, y) + R) + (\beta\psi)^N_{x,y} x V_n(x-1, y+1) \\ & + (1 - \lambda^N_{x,y} s - \mu - \gamma^N_{x,y} y - \beta x(1-\psi) - (\beta\psi)^N_{x,y} x) V_n(x, y). \end{aligned}$$

Set $V_n(x, y) = 0$ if $(x, y) \notin \mathcal{X}^N$. Using the successive approximations scheme, the following properties of the relative value function have been shown.

Theorem (properties for the processor-sharing retrial queue). *On $\mathcal{X}^N$ the following holds. The value function V^N is non-decreasing in x and y. If $\gamma/N = \beta\psi/b(N)$, then V^N is also $C(x)$, $C(y)$ and Super(x, y).*

This relationship between N and $b(N)$ emerges in the proof of supermodularity of V^N. This means that $b(N) = N\psi\beta/\gamma$. In general, $\psi\beta/\gamma$ is not rational, hence $b(N)$ is not integer. To solve this problem γ is slightly perturbed, such that $b(N)$ is integer. This perturbation goes to 0, when N tends to infinity.

We will only give the proof that V^N is supermodular, as this is the hardest proof and it is typical for the other proofs. First make some remarks on the other proofs. The proof of non-decreasingness in both variables is pretty straightforward. To show convexity in the variables x and y, it is necessary that the value function is also Super(x, y). To prove that V^N is Super(x, y), we need to show that for all n, V_n is Super(x, y).

Proof. We will prove this with induction. It is clear that $V_0 \equiv 0$ is $C(x)$, $C(y)$ and Super(x, y). Now suppose that V_n is $C(x)$, $C(y)$ and Super(x, y), then for $0 \leq x \leq N - 1$, $0 \leq y \leq b(N) - 1$ we obtain

$$
\begin{aligned}
&V_{n+1}(x,y) + V_{n+1}(x+1,y+1) - V_{n+1}(x+1,y) - V_{n+1}(x,y+1) \\
&= \lambda\left(1-\frac{x}{N}\right)sV_n(x+1,y) + \lambda\left(1-\frac{x+1}{N}\right)sV_n(x+2,y+1) \\
&\quad -\lambda\left(1-\frac{x+1}{N}\right)sV_n(x+2,y) - \lambda\left(1-\frac{x}{N}\right)sV_n(x+1,y+1) \\
&\quad +\mu\Big[V_n((x-1)^+,y) + V_n(x,y+1) - V_n(x,y) - V_n((x-1)^+,y+1)\Big] \\
&\quad +\gamma\left(1-\frac{x}{N}\right)yV_n(x+1,y-1) + \gamma\left(1-\frac{x+1}{N}\right)(y+1)V_n(x+2,y) \\
&\quad -\gamma\left(1-\frac{x+1}{N}\right)yV_n(x+2,y-1) - \gamma\left(1-\frac{x}{N}\right)(y+1)V_n(x+1,y) \\
&\quad +\beta x(1-\psi)V_n(x-1,y) + \beta(x+1)(1-\psi)V_n(x,y+1) \\
&\quad -\beta(x+1)(1-\psi)V_n(x,y) - \beta x(1-\psi)V_n(x-1,y+1) \\
&\quad +\beta\psi\left(1-\frac{y}{b(N)}\right)xV_n(x-1,y+1) + \beta\psi\left(1-\frac{y+1}{b(N)}\right)(x+1)V_n(x,y+2) \\
&\quad -\beta\psi\left(1-\frac{y}{b(N)}\right)(x+1)V_n(x,y+1) - \beta\psi\left(1-\frac{y+1}{b(N)}\right)xV_n(x-1,y+2) \\
&\quad +\Big(1-\lambda\left(1-\frac{x}{N}\right)s-\mu-\gamma\left(1-\frac{x}{N}\right)y-\beta x(1-\psi)-\beta\psi\left(1-\frac{y}{b(N)}\right)x\Big)V_n(x,y) \\
&\quad +\Big(1-\lambda\left(1-\frac{x+1}{N}\right)s-\mu-\gamma\left(1-\frac{x+1}{N}\right)(y+1)-\beta(x+1)(1-\psi) \\
&\qquad\qquad -\beta\psi\left(1-\frac{y+1}{b(N)}\right)(x+1)\Big)V_n(x+1,y+1) \\
&\quad -\Big(1-\lambda\left(1-\frac{x+1}{N}\right)s-\mu-\gamma\left(1-\frac{x+1}{N}\right)y-\beta(x+1)(1-\psi) \\
&\qquad\qquad -\beta\psi\left(1-\frac{y}{b(N)}\right)(x+1)\Big)V_n(x+1,y) \\
&\quad -\Big(1-\lambda\left(1-\frac{x}{N}\right)s-\mu-\gamma\left(1-\frac{x}{N}\right)(y+1)-\beta x(1-\psi) \\
&\qquad\qquad -\beta\psi\left(1-\frac{y+1}{b(N)}\right)x\Big)V_n(x,y+1) \\
&\geq \lambda\left(1-\frac{x+1}{N}\right)s\Big[V_n(x+1,y) + V_n(x+2,y+1) - V_n(x+2,y) - V_n(x+1,y+1)\Big] \\
&\qquad +\frac{\lambda s}{N}(V_n(x+1,y) - V_n(x+1,y+1)) \\
&\quad +\gamma\left(1-\frac{x+1}{N}\right)y\Big[V_n(x+1,y-1) + V_n(x+2,y) - V_n(x+2,y-1) - V_n(x+1,y)\Big] \\
&\qquad +\frac{\gamma y}{N}V_n(x+1,y-1) + \gamma\left(1-\frac{x+1}{N}\right)V_n(x+2,y) \\
&\qquad -\frac{\gamma(y+1)}{N}V_n(x+1,y) - \gamma\left(1-\frac{x+1}{N}\right)V_n(x+1,y)
\end{aligned}
$$

$$
\begin{aligned}
&+\beta x(1-\psi)\Big[V_n(x-1,y)+V_n(x,y+1)-V_n(x,y)-V_n(x-1,y+1)\Big]\\
&\quad+\beta(1-\psi)(V_n(x,y+1)-V_n(x,y))\\
&+\beta\psi\left(1-\frac{y+1}{b(N)}\right)x\Big[V_n(x-1,y+1)+V_n(x,y+2)-V_n(x,y+1)-V_n(x-1,y+2)\Big]\\
&\quad+\frac{\beta\psi x}{b(N)}V_n(x-1,y+1)+\beta\psi\left(1-\frac{y+1}{b(N)}\right)V_n(x,y+2)\\
&\quad-\frac{\beta\psi(x+1)}{b(N)}V_n(x,y+1)-\beta\psi\left(1-\frac{y+1}{b(N)}\right)V_n(x,y+1)\\
&+\Big(1-\lambda\left(1-\frac{x}{N}\right)s-\mu-\gamma\left(1-\frac{x}{N}\right)(y+1)-\beta(x+1)(1-\psi)\\
&\quad-\beta\psi\left(1-\frac{y+1}{b(N)}\right)(x+1)\Big)\Big[V_n(x,y)+V_n(x+1,y+1)-V_n(x+1,y)-V_n(x,y+1)\Big]\\
&\quad+\frac{\lambda s}{N}(V_n(x+1,y+1)-V_n(x+1,y))\\
&\quad+\gamma\left(1-\frac{x}{N}\right)V_n(x,y)+\frac{\gamma(y+1)}{N}V_n(x+1,y+1)\\
&\quad-\gamma\left(1-\frac{x+1}{N}\right)V_n(x+1,y)-\frac{\gamma(y+1)}{N}V_n(x+1,y)\\
&\quad+\beta(1-\psi)(V_n(x,y)-V_n(x,y+1))\\
&\quad+\beta\psi\left(1-\frac{y}{b(N)}\right)V_n(x,y)+\frac{\beta\psi(x+1)}{b(N)}V_n(x+1,y+1)\\
&\quad-\beta\psi\left(1-\frac{y}{b(N)}\right)V_n(x,y+1)-\frac{\beta\psi x}{b(N)}V_n(x,y+1)\\
\geq\;&\frac{\gamma y}{N}\Big[V_n(x+1,y-1)-V_n(x+1,y)+V_n(x+1,y+1)-V_n(x+1,y)\Big]\\
&+\gamma\left(1-\frac{x+1}{N}\right)\Big[V_n(x+2,y)-V_n(x+1,y)+V_n(x,y)-V_n(x+1,y)\Big]\\
&+\frac{\beta\psi x}{b(N)}\Big[V_n(x-1,y+1)-V_n(x,y+1)+V_n(x+1,y+1)-V_n(x,y+1)\Big]\\
&+\beta\psi\left(1-\frac{y+1}{b(N)}\right)\Big[V_n(x,y+2)-V_n(x,y+1)+V_n(x,y)-V_n(x,y+1)\Big]\\
&\quad+\frac{\gamma}{N}(V_n(x+1,y+1)-2V_n(x+1,y)+V_n(x,y))\\
&\quad+\frac{\beta\psi}{b(N)}(V_n(x+1,y+1)-2V_n(x,y+1)+V_n(x,y))\\
\geq\;&0.
\end{aligned}
$$

The first inequality is a result of supermodularity on the terms after μ, the second inequality follows from supermodularity of all terms between the square brackets. Further, the terms with $\frac{\lambda s}{N}$ and $\beta(1-\psi)$ cancel each other out. For the last inequality we need the induction hypothesis that V_n is convex in x and y. The last two lines are greater than 0 if $\gamma/N = \beta\psi/b(N)$.
Some extra attention to the boundary states, if $x = 0$ then the square brackets with μ in front of it are equal to 0. all other boundaries cause no trouble since the jump rates outside $\mathcal{X}^N$ are 0. This finishes the prove that V_n is supermodular for all n. From this we can conclude that V^N is supermodular on $\mathcal{X}^N$. □

Corollary 2. *The value function V is $I(x)$, $I(y)$, $C(x)$, $C(y)$ and Super(x, y).*

Proof. The properties of the value function V follow directly by applying Corollary 1 on V^N as $N \to \infty$. □

As we have seen, proving properties for the value function V consists of multiple steps. Using the successive approximation scheme we prove that a desired property holds for V_n. Then we take the limit $n \to \infty$ to get this results for V^N. By the theorems of Section 3.2 we can transfer these properties to V.

3.4 Service control on a single server with abandonments

The smoothed rate truncation principle has been designed to determine structural properties of the relative value function of the processor-sharing retrial queue. To test whether this method also works for other problems, we consider a system of two queues and one server attending both queues. This system has unbounded rates as a consequence of impatient customers. Two variations of this system are discussed in Down et al. [3]. They show sufficient conditions for a priority-rule for the server to hold. In that paper both variations of the problem require a different approach. The first problem is solved by a coupling method. The second problem is solved by a finite state space approximation. This solution is quite awkward, since it first requires a change of the state variables and then it needs an ad hoc specification of how to choose the transitions leading out of the closed set. With the SRT principle we are able to solve both problems directly. The results presented are not new, they are exactly the same as in Down et al., but it gives an idea of the strength of the SRT principle.
The difference between the two models that we will study is the goal of the optimization in both cases. In the two systems the reward / costs are obtained as follows:

1. In the first model, for each completed customer, a class-dependent *reward* is received.

2. In the second model, each queue has holding *costs* and there is a class-dependent penalty for each customer that abandons due to impatience.

We shall refer to the problems as the *reward model* and the *cost model*. Let us describe the dynamics of the problems. Suppose two stations are served by a single server. Let $\mathcal{X} = \mathbb{Z}_+ \times \mathbb{Z}_+$ be the state space, and let $(x, y) \in \mathcal{X}$ be the number of customers in stations 1 and 2. Customers of types 1 and 2 arrive according to independent Poisson processes with rates λ_1 and λ_2, respectively. The service times of both types of customers are exponential with mean $1/\mu$. Both classes have limited patience, customers are only willing to wait an exponentially distributed amount of time with means $1/\beta_1$ and $1/\beta_2$, respectively. Hence the abandonment rate of station 1 is $x\beta_1$, if the number of customers in station 1 is x, and $y\beta_2$, when there are y customers in station 2. We have two possible actions, serve station 1 or serve station 2. We shall call these actions 1 and 2. So we have the following transition rates

$$q_{xy,x'y'} = \begin{cases} \lambda_1 & \text{if}(x',y') = (x+1,y), \\ \lambda_2 & \text{if}(x',y') = (x,y+1), \\ \mu \mathbf{1}_{\{a=1\}} + x\beta_1 & \text{if}(x',y') = (x-1,y), x > 0, \\ \mu \mathbf{1}_{\{a=2\}} + y\beta_2 & \text{if}(x',y') = (x,y-1), x > 0, \\ 0 & \text{else.} \end{cases}$$

3.4.1 Reward model

In the reward model, every time a customer of type i ($i = 1, 2$) is served, a reward R_i is earned. So, provided that there are customers of type i in the system, under action i a reward R_i is received with rate μ. The goal is to prove the following theorem.

Theorem (priority-rule for the reward model). *Suppose we have the reward model as described above. If $R_1 \geq R_2$ and $\beta_1 \geq \beta_2$, then the optimal policy is a priority-rule: always serve station 1, except to avoid unforced idling.*

To prove this theorem we will use the monotonicity properties of the relative value function. We will apply the SRT-principle to get the desired properties. Fix $N \in \mathbb{N}$ and let $N = N_1$. Let $N_2 \in \mathbb{N}$ be such that $N_1/\lambda_1 \geq N_2/\lambda_2$ and $\lim_{N\to\infty} N_2 = \infty$. The arrival rates λ_1 and λ_2 are the only rates that need to be smoothed, for these are the only transitions that lead the system to a larger state. Define the new arrival rates:

$$\begin{aligned} \lambda_1^N(x) &= \lambda_1(1 - \frac{x}{N_1})^+, \\ \lambda_2^N(y) &= \lambda_2(1 - \frac{y}{N_2})^+. \end{aligned}$$

Then automatically the closed recurrent class of the new system becomes finite: $\mathcal{X}^N = \{0, \ldots, N_1\} \times \{0, \ldots, N_2\} \subset \mathcal{X}$. On $\mathcal{X}^N$ we have bounded rates. The transition rates are bounded from above by $D(N)$, with

$$D(N) = \lambda_1 + \lambda_2 + N_1\beta_1 + N_2\beta_2 + \mu.$$

Without loss of generality we assume that $D(N) = 1$. On $\mathcal{X}^N$ we are able to compute V^N with the successive approximations scheme. Let $V_n : \mathcal{X}^N \to \mathbb{R}$ as follows. Let $V_0 \equiv 0$. Given V_n, define V_{n+1} as

$$\begin{aligned} V_{n+1}(x,y) &= \lambda_1(1 - \frac{x}{N_1})V_n(x+1,y) + \lambda_2(1 - \frac{y}{N_2})V_n(x,y+1) \\ &\quad + x\beta_1 V_n(x-1,y) + y\beta_2 V_n(x,y-1) \\ &\quad + \begin{cases} \mu \max\{R_1 + V_n(x-1,y), R_2 + V_n(x,y-1)\} & x > 0,\ y > 0, \\ \mu(R_1 + V_n(x-1,y)) & x > 0,\ y = 0, \\ \mu(R_2 + V_n(x,y-1)) & x = 0,\ y > 0, \\ \mu V_n(x,y) & (x,y) = (0,0), \end{cases} \\ &\quad + [\lambda_1 \frac{x}{N_1} + \lambda_2 \frac{y}{N_2} + (N_1 - x)\beta_1 + (N_2 - y)\beta_2)]V_n(x,y). \end{aligned}$$

This inductive definition may look ambiguous, since it is composed of terms that are not in $\mathcal{X}^N$. This is not the case, since all transitions directing outwards $\mathcal{X}^N$ naturally have rate 0.

To show that the priority-rule for station 1 is optimal, by the optimality equation we need that for $x > 0,\ y > 0$, $\max\{R_1 + V(x-1,y), R_2 + V(x,y-1)\} = R_1 + V(x-1,y)$. Or equivalently that

$$V(x,y-1) - V(x-1,y) \leq R_1 - R_2.$$

We will prove this using the successive approximations scheme. To this end we need the following theorem.

Theorem (monotonicity for the reward model). *Let $R_1 \geq R_2$ and $\beta_1 \geq \beta_2$. Then it holds for all n and $(x,y) \in \mathcal{X}^N$*

1) $V_n(x+1,y) - V_n(x,y) \geq 0$ *for* $x \leq N_1 - 1$,

2) $V_n(x,y+1) - V_n(x,y) \geq 0$ *for* $y \leq N_2 - 1$,

3) $V_n(x+1,y) - V_n(x,y+1) \leq R_1 - R_2,$ *for* $x \leq N_1 - 1, y \leq N_2 - 1$.

Proof. For $V_0 \equiv 0$, the claims 1), 2) and 3) hold trivially. Suppose now that 1), 2) and 3) hold for V_n. Then we can make two remarks. First, by induction hypothesis 3), for $x > 0, y > 0, (x,y) \in \mathcal{X}^N$

$$\max\{R_1 + V_n(x-1,y), R_2 + V_n(x,y-1)\} = R_1 + V_n(x-1,y).$$

And second, on the states $y = 0,\ 0 < x \leq N_1$ it is only possible to serve station 1, so we have the following term corresponding to service completion: $R_1 + V_n(x-1,0)$. With this knowledge it is possible to fill in the maximum and it is not necessary to make case distinctions.

Hence for all $0 < x \leq N_1 - 1, (x,y) \in \mathcal{X}^N$ we get:

$$\begin{aligned}
& V_{n+1}(x+1,y) - V_{n+1}(x,y) \\
= \; & \lambda_1(1-\frac{x+1}{N_1})V_n(x+2,y) - \lambda_1(1-\frac{x}{N_1})V_n(x+1,y) \\
& +\lambda_2(1-\frac{y}{N_2})V_n(x+1,y+1) - \lambda_2(1-\frac{y}{N_2})V_n(x,y+1) \\
& +\beta_1(x+1)V_n(x,y) - \beta_1 x V_n(x-1,y) \\
& +\beta_2 y V_n(x+1,y-1) - \beta_2 y V_n(x,y-1) \\
& +\mu(R_1 + V_n(x,y)) - \mu(R_1 + V_n(x-1,y)) \\
& +\Big(\lambda_1\frac{x+1}{N_1} + \lambda_2\frac{y}{N_2} + (N_1-x-1)\beta_1 + (N_2-y)\beta_2\Big)V_n(x+1,y) \\
& -\Big(\lambda_1\frac{x}{N_1} + \lambda_2\frac{y}{N_2} + (N_1-x)\beta_1 + (N_2-y)\beta_2\Big)V_n(x,y) \\
\geq \; & \lambda_1(1-\frac{x+1}{N_1})\Big[V_n(x+2,y) - V_n(x+1,y)\Big] - \frac{\lambda_1}{N_1}V_n(x+1,y) \\
& +\beta_1 x\Big[V_n(x,y) - V_n(x-1,y)\Big] + \beta_1 V_n(x,y) \\
& +\lambda_1\frac{x}{N_1}\Big[V_n(x+1,y) - V_n(x,y)\Big] + \frac{\lambda_1}{N_1}V_n(x+1,y) \\
& +\beta_1(N_1-x-1)\Big[V_n(x+1,y) - V_n(x,y)\Big] - \beta_1 V_n(x,y) \\
\geq \; & 0.
\end{aligned}$$

The maxima are attained because of induction hypothesis 3). Both inequalities are a result of induction hypothesis 1).
To complete the argument we will have a closer look at the remaining boundaries. On $(0,y),\ 0 < y \leq N_2$ it is only possible to serve type 2 customers, hence for a service completion we get the following difference term: $R_1 + V_n(0,y) - (R_2 + V_n(0,y-1))$. This is greater than zero, because of induction hypothesis 2) and $R_1 - R_2 \geq 0$. If we look at the difference in state $(0,0):\ V_{n+1}(1,0) - V_{n+1}(0,0)$, then we will get the factor R_1, which is clearly greater than zero. We do not have to look at other states on the boundary, because of the smoothed truncation, we have a very natural boundary with no transitions outwards $\mathcal{X}^N$. We can conclude that 1) also holds for V_{n+1}.

Analogously to the reasoning above, we can derive that $V_{n+1}(x,y+1) - V_{n+1}(x+1,y) \geq 0$. The proof of 3) is a bit more complex, so we will do this next.

For $0 < x < N_1, 0 \leq y < N_2$ we have

$$
\begin{aligned}
& V_{n+1}(x+1,y) - V_{n+1}(x,y+1) \\
= \; & \lambda_1(1-\frac{x+1}{N_1})V_n(x+2,y) - \lambda_1(1-\frac{x}{N_1})V_n(x+1,y+1) \\
& +\lambda_2(1-\frac{y}{N_2})V_n(x+1,y+1) - \lambda_2(1-\frac{y+1}{N_2})V_n(x,y+2) \\
& +(x+1)\beta_1 V_n(x,y) - x\beta_1 V_n(x-1,y+1) \\
& +y\beta_2 V_n(x+1,y-1) - (y+1)\beta_2 V_n(x,y) \\
& +\mu(R_1 + V_n(x,y)) - \mu(R_1 + V_n(x-1,y+1)) \\
& +\Big(\lambda_1\frac{x+1}{N_1} + \lambda_2\frac{y}{N_2} + (N_1-x-1)\beta_1 + (N_2-y)\beta_2\Big)V_n(x+1,y) \\
& -\Big(\lambda_1\frac{x}{N_1} + \lambda_2\frac{y+1}{N_2} + (N_1-x)\beta_1 + (N_2-y-1)\beta_2\Big)V_n(x,y+1) \\
= \; & \lambda_1(1-\frac{x+1}{N_1})\Big[V_n(x+2,y) - V_n(x+1,y+1)\Big] - \frac{\lambda_1}{N_1}V_n(x+1,y+1) \\
& +\lambda_2(1-\frac{y+1}{N_2})\Big[V_n(x+1,y+1) - V_n(x,y+2)\Big] + \frac{\lambda_2}{N_2}V_n(x+1,y+1) \\
& +x\beta_1\Big[V_n(x,y) - V_n(x-1,y+1)\Big] + \beta_1 V_n(x,y) \\
& +y\beta_2\Big[V_n(x+1,y-1) - V_n(x,y)\Big] - \beta_2 V_n(x,y) \\
& +(N_1-x)\beta_1\Big[V_n(x+1,y) - V_n(x,y+1)\Big] - \beta_1 V_n(x+1,y) \\
& +(N_2-y-1)\beta_2\Big[V_n(x+1,y) - V_n(x,y+1)\Big] + \beta_2 V_n(x+1,y) \\
& +\lambda_1\frac{x}{N_1}\Big[V_n(x+1,y) - V_n(x,y+1)\Big] + \frac{\lambda_1}{N_1}V_n(x+1,y) \\
& +\lambda_2\frac{y+1}{N_2}\Big[V_n(x+1,y) - V_n(x,y+1)\Big] - \frac{\lambda_2}{N_2}V_n(x+1,y) \\
& +\mu\Big[V_n(x,y) - V_n(x-1,y+1)\Big] \\
\leq \; & \underbrace{\left(\frac{\lambda_1}{N_1} - \frac{\lambda_2}{N_2}\right)}_{\geq 0}\underbrace{\Big[V_n(x+1,y) - V_n(x+1,y+1)\Big]}_{\leq 0} \\
& +\underbrace{(\beta_1-\beta_2)}_{\geq 0}\underbrace{\Big[V_n(x,y) - V_n(x+1,y)\Big]}_{\leq 0} + D(N)(R_1 - R_2) \\
\leq \; & R_1 - R_2.
\end{aligned}
$$

Because of induction hypothesis 3) the maximization could be replaced by the maximizing term. The first inequality is a result of induction hypothesis 3) applied to the terms between square brackets, and adding up all rates in front of the square brackets to $D(N) = 1$. The second inequality follows from induction hypotheses 1) and 2), combined with the ratio in N_1 and N_2, and $\beta_1 \geq \beta_2$. On the boundaries it is easy to check that the inequalities hold. On the states where $x = 0,\ 0 \leq y \leq N_2$, we get the terms $\mu(R_1 + V_n(0,y) - (R_2 + V_n(0,y))) = \mu(R_1 - R_2)$. We can conclude that V_{n+1} satisfies condition 3).
V_n satisfies 1), 2) and 3) for all n. □

The following corollary finishes the proof of the priority-rule.

Corollary 3. *The value function of the smoothed reward model has the following property*

$$V^N(x+1,y) - V^N(x,y+1) \leq R_1 - R_2.$$

The value function of the original reward model has the following property

$$V(x+1,y) - V(x,y+1) \leq R_1 - R_2.$$

The first part of the corollary follows directly since $\lim_{n\to\infty} V_n^N(x,y) - V_n^N(0,0) = V^N(x,y) - V^N(0,0)$. The second claim follows by Corollary 1 of Section 3.2.

The priority-rule for the reward model follows from this last Corollary 3.

3.4.2 Cost model

In the cost model we have the same dynamics as in the reward model, but instead of maximizing the reward, the costs are object of minimization. There is a class-dependent penalty P_i if a customer leaves queue i. Further, there are holding costs h_i for each customer. This gives per time unit, independent of the policy, average costs of $(h_1+\beta_1 P_1)x_1+(h_2+\beta_2 P_2)x_2$. The theorem we wish to prove for this model is the following.

Theorem (priority-rule for cost model). *Suppose we have the cost model as described above, if $h_1 + \beta_1 P_1 \geq h_2 + \beta_2 P_2$ and $\beta_2 \geq \beta_1$, then the optimal policy is a priority-rule: always serve station 1, except to avoid unforced idling.*

Again, we will prove this using properties of the relative value function, which we shall derive with the SRT-principle. We will use a smoothed rate truncation that is slightly different as before. We fix N and let $N = N_1$ and $N_2 \in \mathbb{N}$ be such that $N_1/\lambda_1 \leq N_2/\lambda_2$. Further, we introduce the smoothed arrival rates as before:

$$\begin{aligned} \lambda_1^N(x) &= \lambda_1(1 - \frac{x}{N_1})^+, \\ \lambda_2^N(y) &= \lambda_2(1 - \frac{y}{N_2})^+. \end{aligned}$$

In the model induced by the smoothed rates, the transition rates are bounded from above by $D(N)$ on the closed recurrent class $\mathcal{X}^N$. Hence on $\mathcal{X}^N$ we can write the successive approximations scheme. Let $V_0 \equiv 0$, and

$$
\begin{aligned}
V_{n+1}(x,y) &= x(h_1+\beta_1 P_1)+y(h_2+\beta_2 P_2) \\
&+\lambda_1(1-\frac{x}{N_1})V_n(x+1,y)+\lambda_2(1-\frac{y}{N_2})V_n(x,y+1) \\
&+x\beta_1 V_n(x-1,y)+y\beta_2 V_n(x,y-1) \\
&+\begin{cases} \mu\min\{V_n(x-1,y),V_n(x,y-1)\} & x>0,\ y>0, \\ \mu V_n(x-1,y) & x>0,\ y=0, \\ \mu V_n(x,y-1) & x=0,\ y>0, \\ \mu V_n(0,0) & (x,y)=(0,0), \end{cases} \\
&+[\lambda_1\frac{x}{N_1}+\lambda_2\frac{y}{N_2}+(N_1-x)\beta_1+(N_2-y)\beta_2)]V_n(x,y).
\end{aligned}
$$

If we can show that $\min\{V(x-1,y),V(x,y-1)\}=V(x-1,y)$, for all $x>0,\ y>0$, then the priority-rule for station 1 is optimal. So we must show that

$$V^N(x+1,y)-V^N(x,y+1)\geq 0.$$

Therefore we will prove the following theorem.

Theorem (monotonicity for the cost model). *Let $h_1+\beta_1 P_1\geq h_2+\beta_2 P_2$ and $\beta_2\geq\beta_1$. Then for all n and $(x,y)\in\mathcal{X}^N$*

1) $V_n(x+1,y)-V_n(x,y)\geq 0,$ *for* $x\leq N_1-1,$

2) $V_n(x,y+1)-V_n(x,y)\geq 0,$ *for* $y\leq N_2-1,$

3) $V_n(x+1,y)-V_n(x,y+1)\geq 0,$ *for* $x\leq N_1-1, y\leq N_2-1.$

Proof. We will prove this similarly to the reward model. Suppose 1), 2) and 3) hold for n. We will prove that 1) holds for $n+1$. First notice that if 3) holds for n, then the server gives priority to station 1, i.e. $\min\{V_n(x-1,y),V_n(x,y-1)\}=V_n(x-1,y)$. We can fill this in, and for $0<x\leq N_1-1,\ 0\leq y\leq N_2$ we obtain

$$
\begin{aligned}
& V_{n+1}(x+1,y) - V_{n+1}(x,y) \\
= \quad & h_1 + \beta_1 P_1 \\
& + \lambda_1 (1 - \frac{x+1}{N_1}) V_n(x+2,y) - \lambda_1 (1 - \frac{x}{N_1}) V_n(x+1,y) \\
& + \lambda_2 (1 - \frac{y}{N_2}) \Big[V_n(x+1,y+1) - V_n(x,y+1) \Big] \\
& + \beta_1 (x+1) V_n(x,y) - \beta_1 x V_n(x-1,y) \\
& + \beta_2 y \Big[V_n(x+1,y-1) - V_n(x,y-1) \Big] \\
& + \mu \Big(V_n(x,y) - V_n(x-1,y) \Big) \\
& + \Big(\lambda_1 \frac{x+1}{N_1} + \lambda_2 \frac{y}{N_2} + (N_1 - x - 1))\beta_1 + (N_2 - y)\beta_2) \Big) V_n(x+1,y) \\
& - \Big(\lambda_1 \frac{x}{N_1} + \lambda_2 \frac{y}{N_2} + (N_1 - x)\beta_1 + (N_2 - y)\beta_2) \Big) V_n(x,y) \\
\geq \quad & \lambda_1 (1 - \frac{x+1}{N_1}) \Big[V_n(x+2,y) - V_n(x+1,y) \Big] - \frac{\lambda_1}{N_1} V_n(x+1,y) \\
& + \beta_1 x \Big[V_n(x,y) - V_n(x-1,y) \Big] + \beta_1 V_n(x,y) \\
& + \lambda_1 \frac{x}{N_1} \Big[V_n(x+1,y) - V_n(x,y) \Big] + \frac{\lambda_1}{N_1} V_n(x+1,y) \\
& + \beta_1 (N_1 - x - 1) \Big[V_n(x+1,y) - V_n(x,y) \Big] - \beta_1 V_n(x,y) \\
\geq \quad & 0.
\end{aligned}
$$

For all terms between the square brackets the inequality follows from induction hypothesis 1). On the boundary states $x = 0,\ 0 \leq y \leq N_2$ the inequality is trivial. So we conclude that 1) also holds for $n+1$. Non-decreasingness in the second variable is similar and very straightforward, so we skip this proof of 2) and continue with the proof of 3). For $0 < x \leq N_1 - 1,\ 0 \leq y \leq N_2 - 1$, we get

$$
\begin{aligned}
&V_{n+1}(x+1,y) - V_{n+1}(x,y+1) \\
&\quad = \quad h_1 + \beta_1 P_1 - (h_2 + \beta_2 P_2) \\
&\qquad +\lambda_1(1-\frac{x+1}{N_1})V_n(x+2,y) - \lambda_1(1-\frac{x}{N_1})V_n(x+1,y+1) \\
&\qquad +\lambda_2(1-\frac{y}{N_2})V_n(x+1,y+1) - \lambda_2(1-\frac{y+1}{N_2})V_n(x,y+2) \\
&\qquad +(x+1)\beta_1 V_n(x,y) - x\beta_1 V_n(x-1,y+1) \\
&\qquad +y\beta_2 V_n(x+1,y-1) - (y+1)\beta_2 V_n(x,y) \\
&\qquad +\mu V_n(x,y) - \mu V_n(x-1,y+1) \\
&\qquad +\Big(\lambda_1\frac{x+1}{N_1} + \lambda_2\frac{y}{N_2} + (N_1-x-1))\beta_1 + (N_2-y)\beta_2)\Big)V_n(x+1,y) \\
&\qquad -\Big(\lambda_1\frac{x}{N_1} + \lambda_2\frac{y+1}{N_2} + (N_1-x)\beta_1 + (N_2-y-1)\beta_2)\Big)V_n(x,y+1) \\
&\quad \geq \quad \lambda_1(1-\frac{x+1}{N_1})\Big[V_n(x+2,y) - V_n(x+1,y+1)\Big] - \frac{\lambda_1}{N_1}V_n(x+1,y+1) \\
&\qquad +\lambda_2(1-\frac{y+1}{N_2})\Big[V_n(x+1,y+1) - V_n(x,y+2)\Big] + \frac{\lambda_2}{N_2}V_n(x+1,y+1) \\
&\qquad +x\beta_1\Big[V_n(x,y) - V_n(x-1,y+1)\Big] + \beta_1 V_n(x,y) \\
&\qquad +y\beta_2\Big[V_n(x+1,y-1) - V_n(x,y)\Big] - \beta_2 V_n(x,y) \\
&\qquad +(N_1-x)\beta_1\Big[V_n(x+1,y) - V_n(x,y+1)\Big] - \beta_1 V_n(x+1,y) \\
&\qquad +(N_2-y-1)\beta_2\Big[V_n(x+1,y) - V_n(x,y+1)\Big] + \beta_2 V_n(x+1,y) \\
&\qquad +\mu\Big[V_n(x,y) - V_n(x-1,y+1)\Big] \\
&\qquad +\lambda_1\frac{x}{N_1}\Big[V_n(x+1,y) - V_n(x,y+1)\Big] + \frac{\lambda_1}{N_1}V_n(x+1,y) \\
&\qquad +\lambda_2\frac{y+1}{N_2}\Big[V_n(x+1,y) - V_n(x,y+1)\Big] - \frac{\lambda_2}{N_2}V_n(x+1,y) \\
&\quad \geq \quad \underbrace{\Big(\frac{\lambda_1}{N_1} - \frac{\lambda_2}{N_2}\Big)}_{\leq 0}\underbrace{\Big[V_n(x+1,y) - V_n(x+1,y+1)\Big]}_{\leq 0} \\
&\qquad + \underbrace{\Big(\beta_1 - \beta_2\Big)}_{\leq 0}\underbrace{\Big[V_n(x,y) - V_n(x+1,y)\Big]}_{\leq 0} \\
&\quad \geq \quad 0.
\end{aligned}
$$

The first inequality follows from the assumption on the costs, $h_1 + \beta_1 P_1 \geq h_2 + \beta_2 P_2$. The second inequality is due to induction hypothesis 3). The third inequality follows from the ratio between N_1 and N_2 and the departure rates, together with induction hypotheses 1) and 2) on the terms between the square brackets. On the boundary states $x = 0,\ 0 \leq y \leq N_2$ the inequality is similar. This completes the proof of 3). $\square$

The following corollary finishes the proof of the priority-rule.

Corollary 4. *The value function of the smoothed cost model has the following property*

$$V^N(x+1,y) - V^N(x,y+1) \geq 0.$$

The value function of the original cost model has the following property

$$V(x+1,y) - V(x,y+1) \geq 0.$$

The first claim of the corollary follows directly since $\lim_{n\to\infty} V_n^N(x,y) - V_n^N(0,0) = V^N(x,y) - V^N(0,0)$. The second claim follows by the limit theorems of Section 3.2.
The priority-rule for the cost model follows form this corollary.

Both in the reward model as in cost model we needed the same conditions on the input parameters as in the original literature. So far the smoothed rate truncation principle has not helped us in solving new problems. What we have seen is that regularly, the SRT method is not very straightforward, sometimes we need strong conditions on the smoothing parameters N_1 and N_2. But that did not stand in the way of obtaining the results relatively easy.
In the next chapter we will not look at a specific model, but will look at the method more general.

Chapter 4

Event-based dynamic programming

4.1 Motivation and definition

In the previous chapter we proved for several models that the corresponding relative value functions possess some structural properties, like increasingness and convexity in the state variable. A crucial step in the proofs is to show that the desired properties are propagated in the successive approximations scheme. This approach has some drawbacks; in the first place the formulas in the induction step of the proof become very large. Secondly, in these large formulas, for different models we have to repeat the same arguments over and over again.

In his monograph [8] Koole describes an approach where this repetition is not necessary. The concept of *event-based dynamic programming* uses the following observation. When we prove that an inequality propagates in the successive approximations scheme, we actually prove this inequality for each event separately. When all terms corresponding to the events in the model satisfy a certain inequality, then certainly the sum of all terms satisfies the inequality.

Event-based DP combines a number of events with a number of properties. Every event is represented by an event operator. For the framework of event-based DP there is a list developed which tells if a particular operator propagates the property of interest. These operators form the building blocks of the recursion in the successive approximations scheme. When all events that occur in a model propagate that same property then a composition of the event operators does also propagate this property. The formal set-up of event-based dynamic programming is as follows.

The framework. *Define the state space* $\mathcal{X} = \mathbb{N}_0^m$, $m \in \mathbb{N}$. *Let* $\mathcal{V} = \{f : \mathcal{X} \to \mathbb{R}\}$ *be the set of real-valued functions on* $\mathcal{X}$. *For some* $k \in \mathbb{N}$, *there are* k *operators* $T_0, \ldots, T_{k-1}$ *defined as follows*

$$T_i : \mathcal{V}^{l_i} \to \mathcal{V}, \quad l_i \in \mathbb{N} \textit{ for } i = 0, \ldots k-1.$$

The value function V_n for $n \in \mathbb{N}$ is then recursively defined by using $V_n^{(0)}, \ldots, V_n^{(k)}$

$$V_n^{(k)} = V_{n-1};$$

$$V_n^{(j)} = T_j(V_n^{(k_1^j)}, \ldots, V_n^{(k_{l_j}^j)}), \quad j = 0, \ldots, k-1, \ \textit{for } k_1^j, \ldots, k_{l_j}^j \ \textit{such that } j < k_1^j, \ldots, k_{l_j}^j \leq k;$$

$$V_n = V_n^{(0)}.$$

To illustrate this definition, let us look at the $M|M|1$-queue with holding costs. After uniformization we get the following recursion in the successive approximations scheme

$$V_{n+1}(x) = \frac{x}{\lambda + \mu} + \frac{\lambda}{\lambda + \mu} V_n(x+1) + \frac{\mu}{\lambda + \mu} V_n((x-1)^+).$$

We want to model this in the event-based DP setting. To this end, we let the dimension be $m = 1$ and the number of operators be $k = 4$. Define the following operators. The operator that represents the holding costs

$$T_0 f(x) = \frac{x}{\lambda + \mu} + f(x).$$

The uniformization operator equals

$$T_1(f_1, f_2)(x) = \frac{\lambda}{\lambda + \mu} f_1(x) + \frac{\mu}{\lambda + \mu} f_2(x).$$

The operator that represents an arrival is given by

$$T_2 f(x) = f(x+1).$$

Finally, the operator that represents a departure by

$$T_3 f(x) = f((x-1)^+).$$

To give some insight in the quite technical definition of the framework, we will write down all indices for this example. We give the composition of the above operators participating in it the explicitly.

j	l_j	$k_1^j, \ldots, k_{l_j}^j$	$V_n^{(j)}$
4	-	-	$V_{n-1} = V_n^{(4)}$
3	1	$k_1^3 = 4$	$T_3(V_n^{(4)}) = V_n^{(3)}$
2	1	$k_1^2 = 4$	$T_2(V_n^{(4)}) = V_n^{(2)}$
1	2	$k_1^1 = 2, k_2^1 = 3$	$T_1(V_n^{(2)}, V_n^{(3)}) = V_n^{(1)}$
0	1	$k_1^0 = 1$	$T_0(V_n^{(1)})$

For $j = 0, \ldots 4$, $V_n^{(j)}$ is defined. Further we have that $V_n = V_n^{(0)}$. If we combine this then the recursion of successive approximations scheme appears, and we see that we have chosen the operators correctly

$$V_n(x) = T_0(T_1(T_2(V_{n-1}(x)), T_3(V_{n-1}(x)))) = \frac{x}{\lambda + \mu} + \frac{\lambda}{\lambda + \mu} V_{n-1}(x+1) + \frac{\mu}{\lambda + \mu} V_{n-1}((x-1)^+).$$

The next theorem gives the central idea of the strength of the operators in relation to the relative value function.

Theorem. *Let $\mathcal{F}$ be a class of functions from $\mathcal{X}$ to $\mathbb{R}$, V_n as defined above and $V_0 \in \mathcal{F}$. If, for all i, for $f_i, \ldots, f_{l_i} \in \mathcal{F}$ it holds that $T_i(f_i, \ldots, f_{l_i}) \in \mathcal{F}$, then also $V_n \in \mathcal{F}$ for all n.*

Suppose we take $\mathcal{F}$ to be the set of increasing and convex functions. Then it can be shown that $\mathcal{F}$ propagates through the operators $T_0, \ldots, T_3$ in the previous example. Hence by this theorem we can conclude that the relative value function of the $M|M|1$-queue is increasing and convex.

[8] of Koole contains a comprehensive list of operators and properties that are propagated through these operators. We have appended this list in Section 4.4.2. All event operators that are treated have been uniformized. This means that models that have unbounded transition rates do not fit in this framework. We will describe next how to deal with this problem.

4.2 New operators

Let a model with unbounded transition rates be given. When we apply smoothed rate truncation to such a model the transition rates become bounded and thus uniformizable. Then the smoothed model fits in the event-based DP framework, but we may have created events that are not treated in the literature. These new smoothed events generate new operators, and so we need new propagation results. Smoothing is only needed for events that make the state larger in one or more directions. So we have set up a list of events, for which we derive a smoothed operator, together with some propagation results. This list is everything but complete, but we believe that it covers the most relevant cases. Furthermore, we think the events that are treated are typical for other events.

1. Arrivals
2. Transfers
3. Increasing transfers
4. Double arrivals
5. Controlled arrivals

Before introducing the smoothed operators, recall the properties we are interested in. Notice that it is sufficient to prove these properties on the closed class $\mathcal{X}^N$, for $1 \le i, j \le m$.

$$\begin{array}{rccl}
f \in I(i) \text{ if} & f(x+e_i) \ge f(x) & x_i \le N_i - 1; \\
f \in C(i) \text{ if} & f(x+2e_i) - f(x+e_i) \ge f(x+e_i) - f(x) & x_i \le N_i - 2; \\
f \in \text{Super}(i,j) \text{ if} & f(x) + f(x+e_i+e_j) \ge f(x+e_i) + f(x+e_j) & x_i \le N_i - 1,\ x_j \le N_j - 1; \\
f \in DI(i,j) \text{ if} & f(x+e_i) \ge f(x+e_j) & x_i \le N_i - 1.
\end{array}$$

4.2.1 Arrivals

Smoothed rate truncation w.r.t. arrivals is straightforward. Assume that customers arrive at queue i with rate λ. In event-based DP, this rate λ shows up in the uniformization operator, not in the arrivals operator. So the arrivals operator has a unit transition rate

$$T_{A(i)}f(x) = f(x+e_i).$$

If we apply SRT we get the smoothed arrivals operator

$$T^S_{A(i)}f(x) := \begin{cases} (1-\frac{x_i}{N_i})f(x+e_i) + \frac{x_i}{N_i}f(x) & x_i \leq N_i, \\ f(x) & \text{else.} \end{cases}$$

Theorem (propagation results for the smoothed arrivals operator). *The following propagation results hold for $T^S_{A(i)}$, for $i, j \in \{1 \ldots m\}$:*

I. $I(i) \to I(i)$;

II. $C(i) \to C(i)$;

III. $\text{Super}(i,j) \to \text{Super}(i,j)$;

IV. $DI(j,i) \cap I(j) \to DI(j,i)$.

We start with the proof of propagation of non-decreasingness through the operator: $T^S_{A(i)}$: $I(i) \to I(i)$.

Proof. I. Suppose $f \in I(i)$. Then we need to show that $T^S_{A(i)}f(x+e_i) - T^S_{A(i)}f(x) \geq 0$. For convenience we multiply this by N_i. We get for $x_i \leq N_i - 1$

$$\begin{aligned} & N_i(T^S_{A(i)}f(x+e_i) - T^S_{A(i)}f(x)) \\ = \; & (N_i - x_i - 1)f(x+2e_i) + (x_i+1)f(x+e_i) \\ & -(N_i - x_i)f(x+e_i) - x_i f(x) \\ = \; & (N_i - x_i - 1)[f(x+2e_i) - f(x+e_i)] - f(x+e_i) \\ & + x_i[f(x+e_i) - f(x)] + f(x+e_i) \\ \geq \; & 0. \end{aligned}$$

Inside the square brackets we gather all similar terms, but they have slightly different rates. Hence we have to compensate these terms. We can choose the rates in the right way, such that the compensation terms cancel. The terms inside the brackets are non-negative because $f \in I(i)$. In the boundary states $x_i = N_i - 1$, we have $(N_i - x_i - 1)[f(x+2e_i) - f(x+e_i)] = 0$, so the inequality holds as well. This reasoning for the boundary states holds for every proof from now on, and so we will omit this in the following proofs.
This completes the proof. □

Next we will prove the propagation of convexity through the operator, i.e. $T^S_{A(i)}$: $C(i) \to C(i)$.

Proof. II. Assume $f \in C(i)$. Then we need to show for $0 \leq x_i \leq N_i - 1$ that $N_i(T^S_{A(i)}f(x+$

$2e_i) - 2T^S_{A(i)}f(x+e_i) + T^S_{A(i)}f(x)) \geq 0$. For $x_i \leq N_i - 2$ we get

$$\begin{aligned}
&N_i(T^S_{A(i)}f(x+2e_i) - 2T^S_{A(i)}f(x+e_i) + T^S_{A(i)}f(x)) \\
&\quad= (N_i - x_i - 2)f(x+3e_i) + (x_i+2)f(x+2e_i) \\
&\qquad -2(N_i - x_i - 1)f(x+2e_i) - 2(x_i+1)f(x+e_i) \\
&\qquad +(N_i - x_i)f(x+e_i) + x_i f(x) \\
&\quad= (N_i - x_i - 2)[f(x+3e_i) - 2f(x+2e_i) + f(x+e_i)] - 2f(x+2e_i) + 2f(x+e_i) \\
&\qquad +x_i[f(x+2e_i) - 2f(x+e_i) + f(x)] + 2f(x+2e_i) - 2f(x+e_i) \\
&\quad\geq 0.
\end{aligned}$$

We use a similar method of ordering the terms as before. The inequality follows from convexity of f. This finishes the proof. □

We continue with the proof of supermodularity. We would like to prove that $T^S_{A(i)} : \text{Super}(i,j) \to \text{Super}(i,j)$.

Proof. III. Assume that $f \in \text{Super}(i,j)$, then we have for $x_i \leq N_i - 1$ that

$$\begin{aligned}
&N_i(T^S_{A(i)}f(x) + T^S_{A(i)}f(x+e_i+e_j) - T^S_{A(i)}f(x+e_i) - T^S_{A(i)}f(x+e_j)) \\
&\quad= (N_i - x_i)f(x+e_i) + x_i f(x) \\
&\qquad +(N_i - x_i - 1)f(x+2e_i+e_j) + (x_i+1)f(x+e_i+e_j) \\
&\qquad -(N_i - x_i - 1)f(x+2e_i) - (x_i+1)f(x+e_i) \\
&\qquad -(N_i - x_i)f(x+e_i+e_j) - x_i f(x+e_j) \\
&\quad= (N_i - x_i - 1)[f(x+e_i) + f(x+2e_i+e_j) - f(x+2e_i) - f(x+e_i+e_j)] \\
&\qquad +f(x+e_i) - f(x+e_i+e_j) \\
&\qquad +x_i[f(x) + f(x+e_i+e_j) - f(x+e_i) - f(x+e_j)] \\
&\qquad +f(x+e_i+e_j) - f(x+e_i) \\
&\quad\geq 0.
\end{aligned}$$

The compensation terms cancel each other nicely. The inequality follows from the assumption on f. □

The last propagation property that we prove for the smoothed arrivals operator is directional increasingness. In other words, we will prove that $T^S_{A(i)} : DI(j,i) \cap I(j) \to DI(j,i)$.

Proof. IV. Suppose that $f \in DI(j,i) \cap I(j)$. Then for $x_i \leq N_i - 1$ it holds that

$$\begin{aligned}
&N_i(T^S_{A(i)}f(x+e_j) - T^S_{A(i)}f(x+e_i)) \\
&\quad= (N_i - x_i)f(x+e_i+e_j) + x_i f(x+e_j) \\
&\qquad -(N_i - x_i - 1)f(x+2e_i) - (x_i+1)f(x+e_i) \\
&\quad= (N_i - x_i - 1)[f(x+e_i+e_j) - f(x+2e_i)] \\
&\qquad +x_i[f(x+e_j) - f(x+e_i)] \\
&\qquad +f(x+e_i+e_j) - f(x+e_i) \\
&\quad\geq 0.
\end{aligned}$$

The inequality follows because $f \in DI(j,i) \cap I(j)$. □

4.2.2 Transfers

The second event for we which we develop a smoothed event operator is a transfer. A transfer is an event where a customer moves form queue i to queue j. The transfer operator is

$$T_{T(i,j)}f(x) = f(x - e_i + e_j).$$

We need to smooth the rates in the direction of the j-th variable, for that is the variable in which the state increases when a transition occurs. We get the following smoothed event operator

$$T^S_{T(i,j)}f(x) := \begin{cases} (1 - \frac{x_j}{N_j})f(x - e_i + e_j) + \frac{x_j}{N_j}f(x) & x_i > 0,\ x_j \le N_j, \\ f(x) & \text{else.} \end{cases}$$

Theorem (propagation results for the smoothed transfers operator). *For the operator $T^S_{T(i,j)}$, for $i \neq j \in \{1 \dots m\}$ we derived the following results:*

I. $I(i) \cap I(j) \to I(i) \cap I(j)$;

II. $C(j) \cap \text{Super}(i,j) \to C(j)$;

III. $C(i) \cap \text{Super}(i,j) \cap DI(i,j) \to C(i)$;

IV. $C(i) \cap \text{Super}(i,j) \cap DI(i,j) \to \text{Super}(i,j)$;

V. $DI(i,j) \to DI(i,j)$ and $DI(j,i) \to DI(j,i)$.

We start with the proof of $T^S_{T(i,j)} : I(i) \cap I(j) \to I(i) \cap I(j)$.

Proof. I. Assume that $f \in I(i) \cap I(j)$, first we prove that $T^S_{T(i,j)}f \in I(j)$. For $x_i > 0, x_j \le N_j - 1$ we have

$$\begin{aligned} & N_j(T^S_{T(i,j)}f(x + e_j) - T^S_{T(i,j)}f(x)) \\ = \; & (N_j - x_j - 1)f(x - e_i + 2e_j) + (x_j + 1)f(x + e_j) \\ & -(N_j - x_j)f(x - e_i + e_j) - x_j f(x) \\ = \; & (N_j - x_j - 1)[f(x - e_i + 2e_j) - f(x - e_i + e_j)] - f(x - e_i + e_j) \\ & +x_j[f(x + e_j) - f(x)] + f(x + e_j) \\ \ge \; & 0. \end{aligned}$$

The inequality follows from the assumption on f that it is increasing in variable i and j. For the proof of $I(i)$, the only non-trivial case is when $x_i = 0$, then we have:

$$\begin{aligned} & N_j(T^S_{T(i,j)}f(x + e_i) - T^S_{T(i,j)}f(x)) \\ = \; & (N_j - x_j)f(x + e_j) + x_j f(x + e_i) \\ & -N_j f(x) \\ = \; & (N_j - x_j)[f(x + e_j) - f(x)] \\ & +x_j[f(x + e_i) - f(x)] \\ \ge \; & 0. \end{aligned}$$

The inequality follows from $f \in I(i) \cap I(j)$. $\square$

We continue with the proof of $T^S_{T(i,j)} : C(j) \cap \text{Super}(i,j) \to C(j)$.

Proof. II. Assume $f \in C(j) \cap \text{Super}(i,j)$, then for $x_j \leq N_j - 2$ we get

$$\begin{aligned}
&N_j(T^S_{T(i,j)}f(x+2e_j) - 2T^S_{T(i,j)}f(x+e_j) + T^S_{T(i,j)}f(x)) \\
&\quad = (N_j - x_j - 2)f(x - e_i + 3e_j) + (x_j + 2)f(x + 2e_j) \\
&\quad\quad -2(N_j - x_j - 1)f(x - e_i + 2e_j) - 2(x_j + 1)f(x + e_j) \\
&\quad\quad +(N_j - x_j)f(x - e_i + e_j) + x_j f(x) \\
&\quad = (N_j - x_j - 2)[f(x - e_i + 3e_j) - 2f(x - e_i + 2e_j) + f(x - e_i + e_j)] \\
&\quad\quad -2f(x - e_i + 2e_j) + 2f(x - e_i + e_j) \\
&\quad\quad +x_j[f(x + 2e_j) - 2f(x + e_j) + f(x)] + 2f(x + 2e_j) - 2f(x + e_j) \\
&\quad \geq 2[f(x - e_i + e_j) + f(x + 2e_j) - f(x - e_i + 2e_j) - f(x + e_j)] \\
&\quad \geq 0.
\end{aligned}$$

The first inequality follows from convexity of f, the second inequality from supermodularity. □

Next the proof of $T^S_{T(i,j)} : C(i) \cap \text{Super}(i,j) \cap DI(i,j) \to C(i)$.

Proof. III. Assume $f \in C(i) \cap \text{Super}(i,j) \cap DI(i,j)$. The proof of $C(i)$ is only complicated on the boundary states, when $x_i = 0$. When $x_i > 0$ propagation follows trivial. For $x_i = 0$ we have

$$\begin{aligned}
&N_j(T^S_{T(i,j)}f(x+2e_i) - 2T^S_{T(i,j)}f(x+e_i) + T^S_{T(i,j)}f(x)) \\
&\quad = (N_j - x_j)f(x + e_i + e_j) + x_j f(x + 2e_i) \\
&\quad\quad -2(N_j - x_j)f(x + e_j) - 2x_j f(x + e_i) \\
&\quad\quad +N_j f(x) \\
&\quad = (N_j - x_j)(f(x + e_i + e_j) - 2f(x + e_j) + f(x)) \\
&\quad\quad +x_j[f(x + 2e_i) - 2f(x + e_i) + f(x)] \\
&\quad \geq (N_j - x_j)[f(x + e_i + e_j) - f(x + e_j) - f(x + e_i) + f(x)] \\
&\quad \geq 0.
\end{aligned}$$

The first inequality follows by $DI(i,j$ and $C(i)$. The second inequality follows from $\text{Super}(i,j)$. □

Next we prove that $T^S_{T(i,j)} : C(i) \cap \text{Super}(i,j) \cap DI(i,j) \to \text{Super}(i,j)$.

Proof. IV. Suppose $f \in C(i) \cap \text{Super}(i,j)$. Then for $x_i > 0,\ x_j \leq N_j - 1$ we obtain the following inequality

$$\begin{aligned}
&N_j(T^S_{T(i,j)}f(x) + T^S_{T(i,j)}f(x+e_i+e_j) - T^S_{T(i,j)}f(x+e_i) - T^S_{T(i,j)}f(x+e_j)) \\
&\quad = (N_j - x_j)\, f(x-e_i+e_j) + x_j f(x) \\
&\qquad + (N_j - x_j - 1)\, f(x+2e_j) + (x_j+1) f(x+e_i+e_j) \\
&\qquad - (N_j - x_j)\, f(x+e_j) - x_j f(x+e_i) \\
&\qquad - (N_j - x_j - 1)\, f(x-e_i+2e_j) - (x_j+1) f(x+e_j) \\
&\quad = (N_j - x_j - 1)\, [f(x-e_i+e_j) + f(x+2e_j) - f(x+e_j) - f(x-e_i+2e_j)] \\
&\qquad + x_j[f(x) + f(x+e_i+e_j) - f(x+e_i) - f(x+e_j)] \\
&\qquad + f(x-e_i+e_j) - f(x+e_j) + f(x+e_i+e_j) - f(x+e_j) \\
&\quad \geq 0.
\end{aligned}$$

The inequality follows by the assumption that f is both $C(i)$ and $\text{Super}(i,j)$. We have to look at the boundary states, when $x_i = 0$. Then the following inequality holds

$$\begin{aligned}
&N_j(T^S_{T(i,j)}f(x) + T^S_{T(i,j)}f(x+e_i+e_j) - T^S_{T(i,j)}f(x+e_i) - T^S_{T(i,j)}f(x+e_j)) \\
&\quad = N_j f(x) \\
&\qquad + (N_j - x_j - 1)\, f(x+2e_j) + (x_j+1) f(x+e_i+e_j) \\
&\qquad - (N_j - x_j)\, f(x+e_j) - x_j f(x+e_i) \\
&\qquad - N_j f(x+e_j) \\
&\quad = (N_j - x_j)\, [f(x) + f(x+2e_j) - 2f(x+e_j)] \\
&\qquad + x_j[f(x) + f(x+e_i+e_j) - f(x+e_i) - f(x+e_j)] \\
&\qquad + f(x-e_i+e_j) - f(x+2e_j) \\
&\quad \geq 0.
\end{aligned}$$

The inequality follows from $f \in C(i) \cap \text{Super}(i,j) \cap DI(i,j)$. □

As a final propagation result for this operator we prove that: $T^S_{T(i,j)} : DI(i,j) \to DI(i,j)$ and $DI(j,i) \to DI(j,i)$.

Proof. V. Let us prove $T^S_{T(i,j)} : DI(i,j) \to DI(i,j)$ first. Suppose $f \in DI(i,j)$, then for $x_i > 0,\ x_j \leq N_j - 1$ we have

$$\begin{aligned}
&N_j(T^S_{T(i,j)}f(x+e_i) - T^S_{T(i,j)}f(x+e_j)) \\
&\quad = (N_j - x_j) f(x+e_j) + x_j f(x+e_i) \\
&\qquad - (N_j - x_j - 1) f(x-e_i+2e_j) - (x_j+1) f(x+e_j) \\
&\quad = (N_j - x_j - 1)[f(x+e_j) - f(x-e_i+2e_j)] - f(x+e_j) \\
&\qquad + x_j[f(x+e_j) - f(x-e_i)] + f(x+e_j) \\
&\quad \geq 0.
\end{aligned}$$

We get the inequality since $f \in DI(i,j)$. For $x_i = 0$ we obtain

$$\begin{aligned}
&N_j(T^S_{T(i,j)}f(x+e_i) - T^S_{T(i,j)}f(x+e_j)) \\
&\quad = (N_j - x_j) f(x+e_j) + x_j f(x+e_i) \\
&\qquad - N_j f(x+e_j) \\
&\quad = x_j[f(x+e_j) - f(x-e_i)] \\
&\quad \geq 0.
\end{aligned}$$

The inequality follows from $DI(i,j)$.
The proof of $T^S_{T(i,j)} : DI(j,i) \to DI(j,i)$ follows by symmetry. $\square$

4.2.3 Increasing transfers

In the processor sharing retrial queue as described in Bhulai et al. [2] transfer events arise that increase proportionally to the size of the queue that customers come from. We investigate whether it is possible to incorporate such an event into the event-based DP framework.
The initial event operator should be this. Since this operator is not uniformized, it does not fit in the event-based dp framework

$$\beta x_i f(x - e_i + e_j). \tag{4.1}$$

The SRT principle provides that on the essential states $x_i \leq N_i$, and so $\beta x_i \leq \beta N_i$. Hence we can divide (4.1) by βN_i and add a uniformization term to obtain the following

$$\frac{x_i}{N_i} f(x - e_i + e_j) + \left(1 - \frac{x_i}{N_i}\right) f(x) \qquad x_i \leq N_i.$$

Since the SRT method requires a truncation in the j-th variable, we get the following operator

$$\frac{x_i}{N_i}\left(1 - \frac{x_j}{N_j}\right) f(x - e_i + e_j) + \left(1 - \frac{x_i}{N_i}\left(1 - \frac{x_j}{N_j}\right)\right) f(x) \qquad x_i \leq N_i, x_j \leq N_j.$$

Unfortunately, this operator does not propagate the supermodularity property. For the processor sharing queue, where the same problem arises, Bhulai et al. [2] have been able to solve this problem by using the fact that there are also increasing transfers in the opposite direction. A useful observation here is that the relative size of the rates of these opposite transfers does not matter. This gives us the idea of adding small artificial transfers in the opposite direction of the transfers. When SRT is applied with the right proportions of N_i and N_j, these transitions compensate enough to preserve the supermodularity property through the operator.

Hence we add artificial transfers in the opposite direction with a small rate ϵx_j. To get the desired propagation results we demand that $0 < \epsilon < 1$ and $N_i = \epsilon N_j,\ N_i, N_j \in \mathbb{N}$. Define the increasing transfers operator on $\mathcal{X}^N$ as

$$\begin{aligned} T^S_{IT(i,j)} f(x) \quad &:= \quad \frac{x_i}{N_i}\left(1 - \frac{x_j}{N_j}\right) f(x - e_i + e_j) + \frac{\epsilon x_j}{N_i}\left(1 - \frac{x_i}{N_i}\right) f(x + e_i - e_j) \\ &+ \quad \left(1 - \frac{x_i}{N_i}\left(1 - \frac{x_j}{N_j}\right) - \frac{\epsilon x_j}{N_i}\left(1 - \frac{x_i}{N_i}\right)\right) f(x). \end{aligned}$$

Let $T^S_{IT(i,j)} f(x) = f(x)$, for x not in $\mathcal{X}^N$. We will need to show two things. The first one that if we take the limit of $\epsilon \to 0$, that we have convergence to the original event of increasing transfers. The second thing is that the operator has nice propagation properties.

If we take the limit $\epsilon \to 0$, then we get the following operator.

$$\begin{aligned} T^S_{IT(i,j)}f(x) &= \frac{x_i}{N_i}\left(1-\frac{x_j}{N_j}\right)f(x-e_i+e_j)+\frac{\epsilon x_j}{N_i}\left(1-\frac{x_i}{N_i}\right)f(x+e_i-e_j) \\ &+ \left(1-\frac{x_i}{N_i}\left(1-\frac{x_j}{N_j}\right)-\frac{\epsilon x_j}{N_i}\left(1-\frac{x_i}{N_i}\right)\right)f(x) \\ &\underset{\epsilon\to 0}{\longrightarrow} \frac{x_i}{N_i}\left(1-\frac{x_j}{N_j}\right)f(x-e_i+e_j)+\left(1-\frac{x_i}{N_i}\left(1-\frac{x_j}{N_j}\right)\right)f(x) \end{aligned}$$

Then we have that, just as for all event operators, by the limit theorems of Section 3.2 this smoothed event operator converges to the original increasing transfers.

Theorem (propagation results for the smoothed increasing transfers operator). *The following propagation results hold for $T^S_{IT(i,j)}$, for $i,j \in \{1,\dots,m\}$*

I. $I(i)\cap I(j) \to I(i)\cap I(j)$;

II. $C(i)\cap C(j)\cap \text{Super}(i,j) \to C(i)\cap C(j)\cap \text{Super}(i,j)$;

III. $DI(i,j) \to DI(i,j)$ and $DI(j,i) \to DI(j,i)$.

The proof of $T^S_{IT(i,j)} : I(i)\cap I(j) \to I(i)\cap I(j)$ is as follows.

Proof. I. Suppose $f \in I(i)\cap I(j)$. Then $T^S_{IT(i,j)}f \in I(i)$ because for $x_i \le N_i - 1$ we have

$$\begin{aligned} &N_iN_j\big(T^S_{IT(i,j)}f(x+e_i) - T^S_{IT(i,j)}f(x)\big) \\ &= (N_j-x_j)(x_i+1)f(x+e_j)+(N_i-x_i-1)x_jf(x+2e_i-e_j) \\ &\quad +\big(N_iN_j-(N_j-x_j)(x_i+1)-(N_i-x_i-1)x_j\big)f(x+e_i) \\ &\quad -(N_j-x_j)x_if(x-e_i+e_j)-(N_i-x_i)x_jf(x+e_i-e_j) \\ &\quad -\big(N_iN_j-(N_j-x_j)x_i-(N_i-x_i)x_j\big)f(x) \\ &= (N_j-x_j)x_i[f(x+e_j)-f(x-e_i+e_j)] \\ &\quad +(N_i-x_i-1)x_j[f(x+2e_i-e_j)-f(x+e_i-e_j)] \\ &\quad +\big(N_iN_j-(N_j-x_j)(x_i+1)-(N_i-x_i)x_j\big)[f(x+e_i)-f(x)] \\ &\quad +(N_j-x_j)[f(x+e_j)-f(x)] \\ &\quad +x_j[f(x+e_i)-f(x+e_i-e_j)] \\ &\ge 0. \end{aligned}$$

The inequality follows from increasingness of f in both variables. Further, since the operator is symmetric in i and j, we can also conclude that $T^S_{IT(i,j)}f \in I(j)$. □

Now we have arrived to the proof of the most difficult propagation result. The difficulty is in the ordering of organizing of the terms, especially in the proof of supermodularity, to get the desired inequality. As said before the propagation only works if we combine the transfers in opposite directions. We want to prove

$$T^S_{IT(i,j)} : C(i)\cap C(j)\cap \text{Super}(i,j) \to C(i)\cap C(j)\cap \text{Super}(i,j).$$

Proof. II. Suppose $f \in C(i) \cap C(j) \cap \text{Super}(i,j)$. First we prove $T^S_{IT(i,j)} f \in C(i)$. For $x_i \le N_i - 2$ we have

$$\begin{aligned}
& N_i N_j \big(T^S_{IT(i,j)} f(x+2e_i) - 2T^S_{IT(i,j)} f(x+e_i) + T^S_{IT(i,j)} f(x)\big) \\
= \; & (N_j - x_j)(x_i+2) f(x+e_i+e_j) + (N_i - x_i - 2) x_j f(x+3e_i - e_j) \\
& + \big(N_i N_j - (N_j - x_j)(x_i+2) - (N_i - x_i - 2)x_j\big) f(x+2e_i) \\
& - 2(N_j - x_j)(x_i+1) f(x+e_j) - 2(N_i - x_i - 1) x_j f(x+2e_i - e_j) \\
& - 2\big(N_i N_j - (N_j - x_j)(x_i+1) - (N_i - x_i - 1)x_j\big) f(x+e_i) \\
& + (N_j - x_j) x_i f(x - e_i + e_j) + (N_i - x_i) x_j f(x + e_i - e_j) \\
& + \big(N_i N_j - (N_j - x_j) x_i - (N_i - x_i) x_j\big) f(x) \\
= \; & (N_j - x_j) x_i [f(x+e_i+e_j) - 2f(x+e_j) + f(x-e_i+e_j)] \\
& + (N_i - x_i - 2) x_j [f(x+3e_i-e_j) - 2f(x+2e_i-e_j) + f(x+e_i-e_j)] \\
& + \big(N_i N_j - (N_j - x_j)(x_i+2) - (N_i - x_i)x_j\big)[f(x+2e_i) - 2f(x+e_i) + f(x)] \\
& + 2(N_j - x_j)[f(x) + f(x+e_i+e_j) - f(x+e_i) - f(x+e_j)] \\
& + 2x_j [f(x+2e_i) + f(x+e_i-e_j) - f(x+2e_i-e_j) - f(x+e_i)] \\
\ge \; & 0.
\end{aligned}$$

The inequality follows by convexity in i and supermodularity. The proof of $T^S_{IT(i,j)} f \in C(j)$ follows from symmetry. We will continue with proving that $T^S_{IT(i,j)} f \in \text{Super}(i,j)$. For $x_i \le N_i - 1,\ x_j \le N_j - 1$ we get

$$
\begin{aligned}
&N_iN_j\big(T^S_{IT(i,j)}f(x)+T^S_{IT(i,j)}f(x+e_i+e_j)-T^S_{IT(i,j)}f(x+e_i)-T^S_{IT(i,j)}f(x+e_j)\big)\\
&\quad= x_i\,(N_j-x_j)\,f(x-e_i+e_j)+x_j\,(N_i-x_i)\,f(x+e_i-e_j)\\
&\qquad+\big(N_iN_j-x_i\,(N_j-x_j)-x_j\,(N_i-x_i)\big)f(x)\\
&\qquad+(x_i+1)\,(N_j-x_j-1)\,f(x+2e_j)+(x_j+1)\,(N_i-x_i-1)\,f(x+2e_i)\\
&\qquad+\big(N_iN_j-(x_i+1)\,(N_j-x_j-1)-(x_j+1)\,(N_i-x_i-1)\big)f(x+e_i+e_j)\\
&\qquad-(x_i+1)\,(N_j-x_j)\,f(x+e_j)-x_j\,(N_i-x_i-1)\,f(x+2e_i-e_j)\\
&\qquad-\big(N_iN_j-(x_i+1)\,(N_j-x_j)-x_j\,(N_i-x_i-1)\big)f(x+e_i)\\
&\qquad-x_i\,(N_j-x_j-1)\,f(x-e_i+2e_j)-(x_j+1)\,(N_i-x_i)\,f(x+e_i)\\
&\qquad-\big(N_iN_j-x_i\,(N_j-x_j-1)-(x_j+1)\,(N_i-x_i)\big)f(x+e_j)\\
&\quad= x_i\,(N_j-x_j-1)\,[f(x-e_i+e_j)+f(x+2e_j)-f(x+e_j)-f(x-e_i+2e_j)]\\
&\qquad+x_j\,(N_i-x_i-1)\,[f(x+e_i-e_j)+f(x+2e_i)-f(x+2e_i-e_j)-f(x+e_i)]\\
&\qquad+\big(N_iN_j-(x_i+1)\,(N_j-x_j)-(x_j+1)\,(N_i-x_i)\big)[f(x)+f(x+e_i+e_j)-f(x+e_i)-f(x+e_j)]\\
&\qquad+x_if(x-e_i+e_j)+(N_j-x_j-1)f(x+2e_j)-(N_j+x_i-x_j)f(x+e_j)\\
&\qquad+x_jf(x+e_i-e_j)+(N_i-x_i-1)f(x+2e_i)-(N_j-x_i+x_j)f(x+e_i)\\
&\qquad+(N_j-x_j)f(x)+(x_i+1)f(x+e_i+e_j)-(N_j+x_i-x_j)f(x+e_j)\\
&\qquad+(N_i-x_i)f(x)+(x_j+1)f(x+e_i+e_j)-(N_j-x_i+x_j)f(x+e_i)\\
&\quad\geq x_j[f(x-e_i+e_j)-2f(x+e_j)+f(x+e_i+e_j)]\\
&\qquad+x_i[f(x+e_i-e_j)-2f(x+e_i)+f(x+e_i+e_j)]\\
&\qquad+(N_i-x_i-1)[f(x)-2f(x+e_i)+f(x+2e_i)]\\
&\qquad+(N_j-x_j-1)[f(x)-2f(x+e_j)+f(x+2e_j)]\\
&\qquad+2[f(x)+f(x+e_i+e_j)-f(x+e_i)-f(x+e_j)]\\
&\quad\geq 0.
\end{aligned}
$$

The first inequality follows from supermodularity, the second inequality follows from convexity in the components i and j, and again supermodularity. This completes the proof. □

Next we continue with the proof of $T^S_{IT(i,j)} : DI(i,j) \to DI(i,j)$ and $DI(j,i) \to DI(j,i)$.

Proof. III. We only prove $T^S_{IT(i,j)} : DI(i,j) \to DI(i,j)$, the other result follows by symmetry.

Suppose $f \in DI(i,j)$, then for $x_i \leq N_i - 1,\ x_j \leq N_j - 1$ we have

$$\begin{aligned}
&N_i N_j \big(T^S_{IT(i,j)} f(x+e_i) - T^S_{IT(i,j)} f(x+e_j)\big) \\
&= (N_j - x_j)(x_i+1) f(x+e_j) + (N_i - x_i - 1) x_j f(x+2e_i - e_j) \\
&\quad + \big(N_i N_j - (N_j - x_j)(x_i+1) - (N_i - x_i - 1)x_j\big) f(x+e_i) \\
&\quad - (N_j - x_j - 1) x_i f(x - e_i + 2e_j) - (N_i - x_i)(x_j+1) f(x+e_i) \\
&\quad - \big(N_i N_j - (N_j - x_j - 1)x_i - (N_i - x_i)(x_j+1)\big) f(x+e_j) \\
&= (N_j - x_j - 1)x_i [f(x+e_j) - f(x - e_i + 2e_j)] \\
&\quad + (N_i - x_i - 1)x_j [f(x+2e_i - e_j) - f(x+e_i)] \\
&\quad + \big(N_i N_j - (N_j - x_j)(x_i+1) - (N_i - x_i)(x_j+1)\big)[f(x+e_i) - f(x+e_j)] \\
&\quad + (N_j - x_j)[f(x+e_j) - f(x)] \\
&\quad + x_j [f(x+e_i) - f(x+e_i - e_j)] \\
&\quad + (N_j - x_j - 1) f(x+e_j) + (x_i+1) f(x+e_j) \\
&\quad - (N_j - x_j - 1) f(x+e_j) - (x_i+1) f(x+e_j) \\
&\quad + (N_i - x_i - 1) f(x+e_i) + (x_j+1) f(x+e_i) \\
&\quad - (N_i - x_i - 1) f(x+e_i) - (x_j+1) f(x+e_i) \\
&\geq 0.
\end{aligned}$$

All compensation terms cancel. The inequality follows since $f \in DI(i,j)$, hence the propagation result is proved. □

4.2.4 Double arrivals

Another event where the SRT method is not very straightforward is when 2 customers arrive simultaneously in queue i and j. The event operator of double arrivals is

$$T_{DA(i,j)} f(x) := f(x + e_i + e_j).$$

We need to smooth this event in two variables, i and j, the transition rates are decreased proportional to the sum of both variables. The first idea is to do is it like this

$$\left(1 - \frac{x_i + x_j}{N}\right) f(x + e_i + e_j) + \frac{x_i + x_j}{N} f(x) \qquad \text{if } x_i + x_j \leq N.$$

Unfortunately, this operator does not propagate any of the properties that we are interested in. Therefore, just as in the case of the increasing transfers, we try to add extra artificial transitions. We add increasing arrivals in the i-th and the j-th variable and get

$$\left(1 - \frac{x_i + x_j}{N}\right) f(x + e_i + e_j) + \frac{x_j}{N} f(x + e_i) + \frac{x_i}{N} f(x + e_j) \qquad \text{if } x_i + x_j \leq N.$$

Now there are still transitions moving outside the area $x_i + x_j \leq N$, these transition rates need to be smoothed too. These transitions rates are smoothed such that no transitions move outwards the square $x_i, x_j \leq N$. The smoothed version of the double arrivals operator becomes

$$T^S_{DA(i,j)}f(x) := \begin{cases} \left(1-\frac{x_i+x_j}{N}\right)f(x+e_i+e_j)+\frac{x_j}{N}f(x+e_i)+\frac{x_i}{N}f(x+e_j) & \text{if } x_i+x_j\le N,\\ \left(1-\left(1-\frac{x_i}{N}\right)^+-\left(1-\frac{x_j}{N}\right)^+\right)f(x) & \\ \quad +\left(1-\frac{x_i}{N}\right)^+f(x+e_i)+\left(1-\frac{x_j}{N}\right)^+f(x+e_j) & \text{if } x_i+x_j>N. \end{cases}$$

Theorem (propagation results for the smoothed double arrivals operator). *The propagation results for $T^S_{DA(i,j)}$, for $i\neq j\in\{1,\dots m\}$*

I. $I(i)\to I(i)$;

II. $C(i)\cap \text{Super}(i,j)\to C(i)$;

III. $\text{Super}(i,j)\to\text{Super}(i,j)$;

IV. $DI(i,j)\to DI(i,j)$.

First the proof of $T^S_{DA(i,j)}: I(i)\to I(i)$.

Proof. I. Suppose $f\in I(i)$, then for $x_i+x_j\le N-1$ we have

$$\begin{aligned} &N\big(T^S_{DA(i,j)}f(x+e_i)-T^S_{DA(i,j)}f(x)\big)\\ &\quad= (N-x_i-x_j-1)f(x+2e_i+e_j)+x_jf(x+2e_i)+(x_i+1)f(x+e_i+e_j)\\ &\qquad -(N-x_i-x_j)f(x+e_i+e_j)-x_jf(x+e_i)-x_if(x+e_j)\\ &\quad= (N-x_i-x_j-1)[f(x+2e_i+e_j)-f(x+e_i+e_j)]-f(x+e_i+e_j)\\ &\qquad +x_j[f(x+2e_i)-f(x+e_i)]\\ &\qquad +x_i[f(x+e_i+e_j)-f(x+e_j)]+f(x+e_i+e_j)\\ &\quad\ge 0. \end{aligned}$$

The inequality follows from $f\in I(i)$. For $x_i+x_j\ge N,\ x_i\le N-1, x_j\le N$ we get

$$\begin{aligned} &N\big(T^S_{DA(i,j)}f(x+e_i)-T^S_{DA(i,j)}f(x)\big)\\ &\quad= (N-x_i-1)f(x+2e_i)+(N-x_j)f(x+e_i+e_j)+(x_i+x_j-N+1)f(x+e_i)\\ &\qquad -(N-x_i)f(x+e_i)-(N-x_j)f(x+e_j)-(x_i+x_j-N)f(x)\\ &\quad= (N-x_i-1)[f(x+2e_i)-f(x+e_i)]-f(x+e_i)\\ &\qquad +(N-x_j)[f(x+e_i+e_j)-f(x+e_j)]\\ &\qquad (x_i+x_j-N)[f(x+e_i)-f(x)]+f(x+e_i)\\ &\quad\ge 0. \end{aligned}$$

The inequality follows by increasingness of f. □

Next the proof of $T^S_{DA(i,j)}: C(i)\cap\text{Super}(i,j)\to C(i)$.

Proof. II.Suppose $f \in C(i) \cap \text{Super}(i,j)$. Then for $x_i + x_j \leq N-2$, we have

$$\begin{aligned}
&N\big(T^S_{DA(i,j)}f(x+2e_i) - 2T^S_{DA(i,j)}f(x+e_i) + T^S_{DA(i,j)}f(x)\big)\\
&= (N-x_i-x_j-2)f(x+3e_i+e_j) + x_j f(x+3e_i) + (x_i+2)f(x+2e_i+e_j)\\
&\quad -2(N-x_i-x_j-1)f(x+2e_i+e_j) - 2x_j f(x+2e_i) - 2(x_i+1)f(x+e_i+e_j)\\
&\quad +(N-x_i-x_j)f(x+e_i+e_j) + x_j f(x+e_i) + x_i f(x+e_j)\\
&= (N-x_i-x_j-2)[f(x+3e_i+e_j) - 2f(x+2e_i+e_j) + f(x+e_i+e_j)]\\
&\quad +x_j[f(x+3e_i) - 2f(x+2e_i) + f(x+e_i)]\\
&\quad +x_i[f(x+2e_i+e_j) - 2f(x+e_i+e_j) + f(x+e_j)]\\
&\quad -2f(x+2e_i+e_j) + 2f(x+e_i+e_j)\\
&\quad +2f(x+2e_i+e_j) - 2f(x+e_i+e_j)\\
&\geq 0.
\end{aligned}$$

The inequality follows from $C(i)$. For $x_i + x_j = N-1$, $x_i \leq N-2$ we have

$$\begin{aligned}
&N\big(T^S_{DA(i,j)}f(x+2e_i) - 2T^S_{DA(i,j)}f(x+e_i) + T^S_{DA(i,j)}f(x)\big)\\
&= (x_j-1)f(x+3e_i) + (N-x_j)f(x+2e_i+e_j) + f(x+2e_i)\\
&\quad -2x_j f(x+2e_i) - 2(N-x_j)f(x+e_i+e_j)\\
&\quad +f(x+e_i+e_j) + x_j f(x+e_i) + (N-x_j-1)f(x+e_j)\\
&= (x_j-1)[f(x+3e_i) - 2f(x+2e_i) + f(x+e_i)]\\
&\quad +(N-x_j-1)[f(x+2e_i+e_j) - 2f(x+e_i+e_j) + f(x+e_j)]\\
&\quad +f(x+2e_i+e_j) - 2f(x+e_i+e_j)\\
&\quad -2f(x+2e_i) + f(x+e_i)\\
&\quad +f(x+2e_i) + f(x+e_i+e_j)\\
&\geq f(x+e_i) + f(x+2e_i+e_j) - f(x+2e_i) - f(x+e_i+e_j)\\
&\geq 0.
\end{aligned}$$

The first inequality comes from convexity of f, the second inequality follows from supermodularity.

For $x_i + x_j \geq N,\ x_i \leq N-1, x_j \leq N$ we get

$$\begin{aligned}
&N\big(T^S_{DA(i,j)}f(x+2e_i) - 2T^S_{DA(i,j)}f(x+e_i) + T^S_{DA(i,j)}f(x)\big)\\
&= (N-x_i-2)f(x+3e_i) + (N-x_j)f(x+2e_i+e_j) + (x_i+x_j-N+2)f(x+2e_i)\\
&\quad -2(N-x_i-1)f(x+2e_i) - 2(N-x_j)f(x+e_i+e_j) - 2(x_i+x_j-N+1)f(x+e_i)\\
&\quad +(N-x_i)f(x+e_i) + (N-x_j)f(x+e_j) + (x_i+x_j-N)f(x)\\
&= (N-x_i-2)[f(x+3e_i) - 2f(x+2e_i) + f(x+e_i)]\\
&\quad +(N-x_j)[f(x+2e_i+e_j) - 2f(x+e_i+e_j) + f(x+e_j)]\\
&\quad +(x_i+x_j-N)[f(x+2e_i) - 2f(x+e_i) + f(x)]\\
&\quad -2f(x+2e_i) + 2f(x+e_i)\\
&\quad +2f(x+2e_i) - 2f(x+e_i)\\
&\geq 0.
\end{aligned}$$

The inequality follows from convexity of f. □

The proof of Super(i, j)

Proof. III. Suppose $f \in$ Super(i, j): For $x_i + x_j \leq N - 2$ we have

$$\begin{aligned}
&N\big(T^S_{DA(i,j)}f(x) + T^S_{DA(i,j)}f(x+e_i+e_j) - T^S_{DA(i,j)}f(x+e_i) - T^S_{DA(i,j)}f(x+e_j)\big)\\
&= (N_j - x_i - x_j)f(x+e_i+e_j) + x_j f(x+e_i) + x_i f(x+e_j)\\
&\quad +(N - x_i - x_j - 2)f(x+2e_i+2e_j) + (x_j+1)f(x+2e_i+e_j) + (x_i+1)f(x+e_i+2e_j)\\
&\quad -(N - x_i - x_j - 1)f(x+2e_i+e_j) - x_j f(x+2e_i) - (x_i+1)f(x+e_i+e_j)\\
&\quad -(N - x_i - x_j - 1)f(x+e_i+2e_j) - (x_j+1)f(x+e_i+e_j) - x_i f(x+2e_j)\\
&= (N - x_i - x_j - 2)[f(x+e_i+e_j) + f(x+2e_i+2e_j) - f(x+2e_i+e_j) - f(x+e_i+2e_j)]\\
&\quad +x_j[f(x+e_i) + f(x+2e_i+e_j) - f(x+2e_i) - f(x+e_i+e_j)]\\
&\quad +x_i[f(x+e_j) + f(x+e_i+2e_j) - f(x+e_i+e_j) - f(x+2e_j)]\\
&\quad +2f(x+e_i+e_j) - f(x+2e_i+e_j) - f(x+e_i+2e_j)\\
&\quad +f(x+2e_i+e_j) - f(x+e_i+e_j) + f(x+e_i+2e_j) - f(x+e_i+e_j)\\
&\geq 0.
\end{aligned}$$

The compensation terms cancel, the inequality follows from supermodularity. If $x_i + x_j = N - 1$ then we get

$$\begin{aligned}
&N\big(T^S_{DA(i,j)}f(x) + T^S_{DA(i,j)}f(x+e_i+e_j) - T^S_{DA(i,j)}f(x+e_i) - T^S_{DA(i,j)}f(x+e_j)\big)\\
&= f(x+e_i+e_j) + x_j f(x+e_i) + (N - x_j - 1)f(x+e_j)\\
&\quad x_j f(x+2e_i+e_j) + (N - x_j - 1)f(x+e_i+2e_j) + f(x+e_i+e_j)\\
&\quad -x_j f(x+2e_i) - (N - x_j)f(x+e_i+e_j)\\
&\quad -(x_j+1)f(x+e_i+e_j) + (N - x_j - 1)f(x+2e_j)\\
&= x_j[f(x+e_i) + f(x+2e_i+e_j) - f(x+2e_i) - f(x+e_i+e_j)]\\
&\quad +(N - x_j - 1)[f(x+e_j) + f(x+e_i+2e_j) - f(x+e_i+e_j) - f(x+2e_j)]\\
&\geq 0.
\end{aligned}$$

The inequality follows from supermodularity.

For $x_i + x_j \geq N,\ x_i \leq N-1, x_j \leq N-1$ we get

$$\begin{aligned}
&N\big(T^S_{DA(i,j)}f(x) + T^S_{DA(i,j)}f(x+e_i+e_j) - T^S_{DA(i,j)}f(x+e_i) - T^S_{DA(i,j)}f(x+e_j)\big)\\
&= (N-x_i)f(x+e_i) + (N-x_j)f(x+e_j) + (x_i+x_j-N)f(x)\\
&\quad (N-x_i-1)f(x+2e_i+e_j) + (N-x_j-1)f(x+e_i+2e_j) + (x_i+x_j-N+2)f(x+e_i+e_j)\\
&\quad -(N-x_i-1)f(x+2e_i) - (N-x_j)f(x+e_i+e_j) - (x_i+x_j-N+1)f(x+e_i)\\
&\quad -(N-x_i)f(x+e_i+e_j) + (N-x_j-1)f(x+2e_j) - (x_i+x_j-N+1)f(x+e_j)\\
&= (N-x_i-1)[f(x+e_i) + f(x+2e_i+e_j) - f(x+2e_i) - f(x+e_i+e_j)]\\
&\quad +(N-x_j-1)[f(x+e_j) + f(x+e_i+2e_j) - f(x+e_i+e_j) - f(x+2e_j)]\\
&\quad +(x_i+x_j-N)[f(x) + f(x+e_i+e_j) - f(x+e_i) - f(x+e_j)]\\
&\quad +f(x+e_i) - f(x+e_i+e_j)\\
&\quad +f(x+e_j) - f(x+e_i+e_j)\\
&\quad +2f(x+e_i+e_j) - f(x+e_i) - f(x+e_j)\\
&\geq 0.
\end{aligned}$$

The inequality follows from supermodularity. □

The proof of directional increasingness $T^S_{DA(i,j)} : DI(i,j) \to DI(i,j)$ and $DI(j,i) \to DI(j,i)$.

Proof. IV. Suppose $f \in DI(i,j)$, then for $x_i + x_j \leq N-1$

$$\begin{aligned}
&N\big(T^S_{DA(i,j)}f(x+e_i) - T^S_{DA(i,j)}f(x+e_j)\big)\\
&= (N-x_i-x_j-1)f(x+2e_i+e_j) + x_j f(x+2e_i) + (x_i+1)f(x+e_i+e_j)\\
&\quad -(N-x_i-x_j-1)f(x+e_i+2e_j) - (x_j+1)f(x+e_i+e_j) - x_i f(x+2e_j)\\
&= (N-x_i-x_j-1)[f(x+2e_i+e_j) - f(x+e_i+e_j)]\\
&\quad +x_j[f(x+2e_i) - f(x+e_i+e_j)] - f(x+e_i+e_j)\\
&\quad +x_i[f(x+e_i+e_j) - f(x+2e_j)] + f(x+e_i+e_j)\\
&\geq 0.
\end{aligned}$$

And for $x_i + x_j \geq N,\ x_i, x_j \leq N-1$

$$\begin{aligned}
&N\big(T^S_{DA(i,j)}f(x+e_i) - T^S_{DA(i,j)}f(x+e_j)\big)\\
&= (N-x_i-1)f(x+2e_i) + (N-x_j)f(x+e_i+e_j) + (x_i+x_j-N+1)f(x+e_i)\\
&\quad -(N-x_i)f(x+e_i+e_j) - (N-x_j-1)f(x+2e_j) - (x_i+x_j-N+1)f(x+e_j)\\
&= (N-x_i-1)[f(x+2e_i) - f(x+e_i+e_j)] - f(x+e_i+e_j)\\
&\quad +(N-x_j-1)[f(x+e_i+e_j) - f(x+2e_j)] + f(x+e_i+e_j)\\
&\quad (x_i+x_j-N+1)[f(x+e_i) - f(x+e_j)]\\
&\geq 0.
\end{aligned}$$

The inequalities in both formulas follow directly from the directional increasingness of f. The proof of $DI(j,i)$ follows by symmetry. □

4.2.5 Controlled arrivals

We also investigate events where control plays a role. An important form of control is admission control. When a customer arrives in queue i, we can accept or reject with associated costs c and c' respectively. The controlled arrival operator is

$$T_{CA}f(x) = \min\{c + f(x),\ c' + f(x + e_i)\}.$$

We want the smoothed operator to propagate the properties we are interested in. Therefore it turns out to be necessary to take as a uniformization term $c + f(x)$, instead of $f(x)$. Hence we introduce the following operator

$$T^S_{CA(i)}f(x) := \left(1 - \frac{x_i}{N_i}\right)^+ \min\{c + f(x),\ c' + f(x + e_i)\} + \left(1 - \left(1 - \frac{x_i}{N_i}\right)^+\right)(c + f(x)).$$

Theorem (propagation results for the smoothed controlled arrivals operator). *For the operator $T^S_{CA(i)}$, the propagation results are, for $i \neq j \in \{1, \dots, m\}$:*

I. $I(i) \to I(i)$;

II. $C(i) \to C(i)$;

III. $\text{Super}(i, j) \to \text{Super}(i, j)$;

IV. $DI(i, j) \to DI(i, j)$.

Proof of $T^S_{CA(i)} : I(i) \to I(i)$.

Proof. I. Suppose $f \in I(i)$. Then for $x_i \leq N_i - 1$ it holds that

$$\begin{aligned}
&N_i(T^S_{CA(i)}f(x + e_i) - T^S_{CA(i)}f(x)) \\
&= (N_i - x_i - 1) \min\{c + f(x + e_i), c' + f(x + 2e_i)\} + (x_i + 1)(c + f(x + e_i)) \\
&\quad - (N_i - x_i) \min\{c + f(x), c' + f(x + e_i)\} - x_i(c + f(x)) \\
&= (N_i - x_i - 1)[\underbrace{\min\{c + f(x + e_i), c' + f(x + 2e_i)\}}_{(1)} - \underbrace{\min\{c + f(x), c' + f(x + e_i)\}}_{(2)}] \\
&\quad - \underbrace{\min\{c + f(x), c' + f(x + e_i)\}}_{(3)} \\
&\quad + x_i[f(x + e_i) - f(x)] + (c + f(x + e_i)) \\
&\geq 0.
\end{aligned}$$

To prove that this is greater or equal than zero, we have to make some case distinctions. This is due to the terms with a minimization. First note the following. Minus the minimum is always greater or equal than minus any term inside the minimization (i.e. $-\min_{b \in B}(b) \geq -b,\ \forall b \in B$). This means that we have a certain freedom of choice for the terms with a minus sign in front of it.

If the minimum of (1) is $c + f(x + e_i)$ (reject), then for (2) we also choose reject (this only

makes the expression smaller), then the terms inside the square brackets are greater or equal than 0 by increasingness in the i-th component.
If the minimum of (1) is accept, then for (2) choose accept and the inequality between the square brackets also follows by increasingness of f.
For (3) choose reject, the inequality of the compensation terms follows by increasingness in x_i. □

Proof of $T^S_{CA(i)} : C(i) \to C(i)$.

Proof. II. Suppose $f \in C(i)$, then for $x_i \leq N_i - 2$ we obtain

$$\begin{aligned}
&N_i(T^S_{CA(i)}f(x+2e_i) - 2T^S_{CA(i)}f(x+e_i) + T^S_{CA(i)}f(x)) \\
&= (N_i - x_i - 2)\min\{c + f(x+2e_i), c' + f(x+3e_i)\} + (x_i+2)(c + f(x+2e_i)) \\
&\quad -2(N_i - x_i - 1)\min\{c + f(x+e_i), c' + f(x+2e_i)\} - 2(x_i+1)(c + f(x+e_i)) \\
&\quad +(N_i - x_i)\min\{c + f(x), c' + f(x+e_i)\} + x_i(c + f(x)) \\
&= (N_i - x_i - 2)[\underbrace{\min\{c + f(x+2e_i), c' + f(x+3e_i)\}}_{(1)} - 2\underbrace{\min\{c + f(x+e_i), c' + f(x+2e_i)\}}_{(2)} \\
&\qquad + \underbrace{\min\{c + f(x), c' + f(x+e_i)\}}_{(3)}] \\
&\quad -2\underbrace{\min\{c + f(x+e_i), c' + f(x+2e_i)\}}_{(4)} + 2\underbrace{\min\{c + f(x), c' + f(x+e_i)\}}_{(5)} \\
&\quad + x_i[f(x+2e_i) - 2f(x+e_i) + f(x)] + 2f(x+2e_i) - 2f(x+e_i) \\
&\geq 0.
\end{aligned}$$

If the minimum of (1) and (3) is accept, then for (2) choose accept two times, the inequality between the square brackets follows from convexity of f.
If the minimums of both (1) and (3) are reject, then in (2) we choose two times reject and the inequality follows by convexity.
If the minimum of (1) is accept and (3) is reject, then in (2) we choose one time accept and once reject, the inequality follows by convexity.
If the minimum of (1) is reject and (3) is accept, then in (2) we choose one time accept and once reject, the terms between the square brackets are 0.
If the minimum of (5) is accept, choose for (4) accept, the compensation terms cancel.
If the minimum of (5) is reject, choose for (4) reject, the inequality follows by convexity on the compensation terms. □

Proof of $T^S_{CA(i)} : \text{Super}(i,j) \to \text{Super}(i,j)$.

Proof. III. Suppose $f \in \text{Super}(i,j)$, then for $x_i \leq N_i - 1$

$$\begin{aligned}
&N_i(T^S_{CA(i)}f(x) + T^S_{CA(i)}f(x+e_i+e_j) - T^S_{CA(i)}f(x+e_i) - T^S_{CA(i)}f(x+e_j)) \\
&= (N_i - x_i)\min\{c + f(x), c' + f(x+e_i)\} + x_i(c + f(x)) \\
&\quad +(N_i - x_i - 1)\min\{c + f(x+e_i+e_j), c' + f(x+2e_i+e_j)\} + (x_i+1)(c + f(x+e_i+e_j)) \\
&\quad -(N_i - x_i - 1)\min\{c + f(x+e_i), c' + f(x+2e_i)\} - (x_i+1)(c + f(x+e_i)) \\
&\quad -(N_i - x_i)\min\{c + f(x+e_j), c' + f(x+e_i+e_j)\} - x_i(c + f(x+e_j)) \\
&= (N_i - x_i - 1)[\underbrace{\min\{c + f(x), c' + f(x+e_i)\}}_{(1)} + \underbrace{\min\{c + f(x+e_i+e_j), c' + f(x+2e_i+e_j)\}}_{(2)} \\
&\qquad - \underbrace{\min\{c + f(x+e_i), c' + f(x+2e_i)\}}_{(3)} - \underbrace{\min\{c + f(x+e_j), c' + f(x+e_i+e_j)\}}_{(4)}] \\
&\quad - \underbrace{\min\{c + f(x), c' + f(x+e_i)\}}_{(5)} - \underbrace{\min\{c + f(x+e_j), c' + f(x+e_i+e_j)\}}_{(6)} \\
&\quad + x_i[f(x) + f(x+e_i+e_j) - f(x+e_i) - f(x+e_j)] \\
&\quad + f(x+e_i+e_j) - f(x+e_i) \\
&\geq 0.
\end{aligned}$$

For (3) copy the minimum of (2), for (4) copy the minimum of (1) and for (6) copy the minimum of (5). Then the inequality follows by supermodularity of f. □

Proof of $T^S_{CA(i)} : DI(i,j) \to DI(i,j)$.

Proof. IV. Suppose $f \in DI(i,j)$. Then for $x_i \leq N_i - 1$ it holds that

$$\begin{aligned}
&N_i(T^S_{CA(i)}f(x+e_i) - T^S_{CA(i)}f(x+e_j)) \\
&= (N_i - x_i - 1)\min\{c + f(x+e_i), c' + f(x+2e_i)\} + (x_i+1)(c + f(x+e_i)) \\
&\quad -(N_i - x_i)\min\{c + f(x+e_j), c' + f(x+e_i+e_j)\} - x_i(c + f(x+e_j)) \\
&= (N_i - x_i - 1)[\underbrace{\min\{c + f(x+e_i), c' + f(x+2e_i)\}}_{(1)} - \underbrace{\min\{c + f(x+e_j), c' + f(x+e_i+e_j)\}}_{(2)}] \\
&\quad - \underbrace{\min\{c + f(x+e_j), c' + f(x+e_i+e_j)\}}_{(3)} \\
&\quad + x_i[f(x+e_i) - f(x+e_j)] + (c + f(x+e_i)) \\
&\geq 0.
\end{aligned}$$

Choose for (2) the same action as (1), for (3) choose reject. Then the inequality follows by directional increasingness. □

4.3 Convergence and composition of the new operators

The composition of the new operators works similarly as for the existing operators, though sometimes some extra caution is necessary. For some operators extra conditions are necessary on the rates of N_i and N_j. When many new operators are used at the same time, these extra conditions may conflict and the results can not be copied automatically to the model that is studied. This is not very likely to happen.

For the convergence of the new operators to the events that they are representing we need the conditions described in Theorem 1 of Section 3.2. When these conditions are satisfied, then the theorems of that section imply that the properties derived for the value function associated to the smoothed events, also hold for the value function of the original events.

4.4 Propagation results

We will give an overview of the new operators, together with the propagation results that have been derived. After that a comprehensive list of operators and properties with their propagation results follows. This list is made by Koole [8].

4.4.1 New results

We have introduced the following new operators. The smoothed arrivals operator

$$T^S_{A(i)}f(x) = \left(1 - \frac{x_i}{N_i}\right)^+ f(x+e_i) + \left(1 - \left(1 - \frac{x_i}{N_i}\right)^+\right) f(x).$$

The smoothed transfers operator

$$T^S_{T(i,j)}f(x) = \begin{cases} (1 - \frac{x_j}{N_j})f(x - e_i + e_j) + \frac{x_j}{N_j} f(x) & x_i > 0,\ x_j \leq N_j, \\ f(x) & \text{else.} \end{cases}$$

The smoothed increasing transfers operator, for $0 < \epsilon < 1$ and $N_i = \epsilon N_j$, the operator is

$$T^S_{IT(i,j)}f(x) = \begin{cases} \frac{x_i}{N_i}\left(1 - \frac{x_j}{N_j}\right) f(x - e_i + e_j) + \frac{\epsilon x_j}{N_i}\left(1 - \frac{x_i}{N_i}\right) f(x + e_i - e_j) & \\ \quad + \left(1 - \frac{x_i}{N_i}\left(1 - \frac{x_j}{N_j}\right) - \frac{\epsilon x_j}{N_i}\left(1 - \frac{x_i}{N_i}\right)\right) f(x) & \text{if } x_i \leq N_i, x_j \leq N_j, \\ f(x) & \text{else.} \end{cases}$$

The smoothed double arrivals operator

$$T^S_{DA(i,j)}f(x) = \begin{cases} \left(1 - \frac{x_i + x_j}{N}\right) f(x + e_i + e_j) + \frac{x_j}{N} f(x+e_i) + \frac{x_i}{N} f(x + e_j) & \text{if } x_i + x_j \leq N, \\ \left(1 - \left(1 - \frac{x_i}{N}\right)^+ - \left(1 - \frac{x_j}{N}\right)^+\right) f(x) & \\ \quad + \left(1 - \frac{x_i}{N}\right)^+ f(x + e_i) + \left(1 - \frac{x_j}{N}\right)^+ f(x + e_j) & \text{if } x_i + x_j > N. \end{cases}$$

The smoothed controlled arrivals operator

$$T^S_{CA(i)}f(x) = \left(1 - \frac{x_i}{N}\right)^+ \min\{c + f(x),\ c' + f(x + e_i)\} + \left(1 - \left(1 - \frac{x_i}{N}\right)^+\right)(c + f(x)).$$

The following results have been derived for these operators, for $i \neq j \in \{1 \dots m\}$ we have

$T^S_{A(i)} : I(i) \to I(i),\ C(i) \to C(i),\ \text{Super}(i,j) \to \text{Super}(i,j),\ DI(j,i) \cap I(j) \to DI(j,i);$

$T^S_{T(i,j)} : I(i) \cap I(j) \to I(i) \cap I(j),\ C(j) \cap \text{Super}(i,j) \to C(j),\ C(i) \cap \text{Super}(i,j) \cap DI(i,j) \to C(i) \cap \text{Super}(i,j),\ DI(i,j) \to DI(i,j),\ DI(j,i) \to DI(j,i);$

$T^S_{IT(i,j)} : I(i) \cap I(j) \to I(i) \cap I(j),\ C(i) \cap C(j) \cap \text{Super}(i,j) \to C(i) \cap C(j) \cap \text{Super}(i,j),\ DI(i,j) \to DI(i,j),\ DI(j,i) \to DI(j,i);$

$T^S_{DA(i,j)} : I(i) \to I(i),\ C(i) \cap \text{Super}(i,j) \to C(i),\ \text{Super}(i,j) \to \text{Super}(i,j),\ DI(i,j) \to DI(i,j);$

$T^S_{CA(i)} : I(i) \to I(i),\ C(i) \to C(i),\ \text{Super}(i,j) \to \text{Super}(i,j),\ DI(i,j) \to DI(i,j).$

4.4.2 Known results

First we define a list of operators, grouped in different categories.

Environmental operators:

$$T_{disc} f(x) = C(x) + \alpha f(x) \text{ for some real-valued function } C;$$

$$T_{env}(f_1, \dots, f_l)(x) = \sum_{y \in \mathbb{N}_0} \lambda(x_0, y) \sum_{j=1}^{l} q^j(x_0, y) f_j(x^*);$$

$$T_{min}(f_1, \dots, f_l)(x) = \min_a \left\{ \sum_{y \in \mathbb{N}_0} \lambda(x_0, a, y) \sum_{j=1}^{l} q^j(x_0, a, y) f_j(x^*) \right\};$$

$$T_{max}(f_1, \dots, f_l)(x) = \max_a \left\{ \sum_{y \in \mathbb{N}_0} \lambda(x_0, a, y) \sum_{j=1}^{l} q^j(x_0, a, y) f_j(x^*) \right\};$$

$$T_{unif}(f_1, \dots, f_l)(x) = \sum_j p(j) f_j(x) \text{ with } p(j) > 0 \text{ for all } j.$$

Arrival operator:

$$T_{A(i)}f(x) = f(x + e_i);$$

$$T_{CA(i)}f(x) = \min\{c + f(x), c' + f(x + e_i)\}, c, c' \in \mathbb{R};$$

$$T_{FS(i)}f(x) = \begin{cases} (1 - \frac{x_i}{B})f(x + e_i) + \frac{x_i}{B}f(x) & x_i \leq B \\ f(x) & \text{else;} \end{cases}$$

$$T_{CAF}f(x) = \min\{c + f(x), f(x + \sum_{i=1}^{m} e_i)\};$$

$$T_R f(x) = \min_{1 \leq i \leq m} f(x + ei).$$

Departure operators:

$$T_{D1(i)}f(x) = f((x - e_i)^+);$$

$$T_{D(i)}f(x) = \mu(x_i)f((x - e_i)^+) + (1 - \mu(x_i))f(x) \text{ with } 0 \leq \mu(x) \leq 1 \text{ for all } x \in \mathbb{N} \text{ and } \mu(0) = 0;$$

$$T_{PD}f(x) = \sum_{1 \leq i \leq m} \mu(i)f((x - e_i)+) \text{ with } \sum_{1 \leq i \leq m} \mu(i) = 1;$$

$$T_{CD(i)}f(x) = \begin{cases} \min_{\mu \in [0,1]}\{c(\mu) + \mu f(x - e_i) + (1 - \mu)f(x)\} & \text{if } x_i > 0, \\ c(0) + f(x) & \text{otherwise,} \end{cases}$$

with $c(\mu) \in \mathbb{R}$ for all $\mu \in [0, 1]$, assuming that the minimum always exists;

$$T_{MS}f(x) = \begin{cases} \min_{j \in I: x_j > 0}\{\mu(j)f(x - e_j) + (1 - \mu(j))f(x)\} & \text{if } \sum_{j \in I} x_j > 0, \\ f(x) & \text{otherwise,} \end{cases}$$

for $\mu(j) \leq 1$;

$$T_{MMS}f(x) = \begin{cases} \frac{1}{s} \min_{i_1,\ldots,i_s \in I: \sum_{k \in I} I\{i_k = j\} \leq x_j} \{\sum_{k=1}^{s}(\mu(i_k)f(x - e_{i_k}) + (1 - \mu(i_k))f(x))\} \\ \qquad \text{if } \sum_{j \in I} x_j \geq s, \\ \frac{1}{s}\sum_{j \in I} x_j(\mu(j)f(x - e_j) + (1\mu(j))f(x)) + \frac{s - \sum_{j \in I} x_j}{s} f(x) \\ \qquad \text{otherwise ,} \end{cases}$$

for $\mu(j) \leq 1$.

Tandem operators:

$$T_{TD1(i)}f(x) = \begin{cases} f(x - e_i + e_{i+1(\text{mod } m)}) & \text{if } x_i > 0, \\ f(x) & \text{otherwise;} \end{cases}$$

$$T_{CTD(i)}f(x) = \begin{cases} \min_{\mu\in[0,1]}\{c(\mu) + \mu f(x - e_i + e_{i+1(\text{mod } m)}) + (1-\mu)f(x)\} & \text{if } x_i > 0, \\ c(0) + f(x) & \text{otherwise;} \end{cases}$$

$$T_{TD(i)}f(x) = \begin{cases} \frac{x_i}{S} f(x - e_i + e_{i+1(\text{mod } m)}) + \frac{S-x_i}{S} f(x) & \text{if } x_i < S, \\ f(x - e_i + e_{i+1(\text{mod } m)}) & \text{otherwise;} \end{cases}$$

$$T_{MTS}f(x) = \begin{cases} \min_{j\in I: x_j>0} \{\sum_{k=0}^{m} \mu(i,k) f(x - e_i + e_k)\} & \text{if } \sum_{j\in I} x_j > 0, \\ f(x) & \text{otherwise,} \end{cases}$$

where $\sum_{k=0}^{m} \mu(i,k) = 1$ for all i, $\mu(i,j) = 0$ for all i and $0 < j < i-1$ and $e_0 = 0$.

Properties

Next, define the properties of interest.

First order properties:

$f \in I(i)$ if $f(x) \leq f(x + e_i)$ for $1 \leq i \leq m$;

$$I = \bigcap_{1 \leq i \leq m} I(i);$$

$f \in UI(i)$ if $f(x + e_{i+1}) \geq f(x + e_i)$ for $1 \leq i \leq m$;

$$UI = \bigcap_{1 \leq i \leq m-1} UI(i);$$

$$f \in wUI(i) \text{ if } \mu(i) f(x + e_{i+1}) + (1 - \mu(i)) f(x + e_i + e_{i+1}) \leq \mu(i+1) f(x + e_i) + (1 - \mu(i+1)) f(x + e_i + e_{i+1})$$

for $1 \leq i \leq m$, for given constants $0 < \mu(j) \leq 1, 1 \leq j \leq m$;

$$wUI = \bigcap_{1 \leq i \leq m-1} wUI(i);$$

$$f \in gUI(i) \text{ if } \sum_{k=0}^{m} \mu(i,k) f(x + e_{i+1} + e_k) \leq \sum_{k=0}^{m} \mu(i+1,k) f(x + e_i + e_k)$$

for $1 \leq i \leq m,\ e_0 = 0$, constants $\mu(j,k)$, such that $\sum_{k=1}^{m} \mu(j,k) = 1, 1 \leq j \leq m$;

$$gUI = \bigcap_{1 \leq i \leq m-1} gUI(i).$$

Schur convexity:

$f \in SC$ if
$f(x + e_i) \leq f(x + e_j)$
for all x and i, j with $i \neq j$ and $x_i \leq x_j$ and
$f(x + ke_i) = f(x + ke_j)$
for all x and i, j with $i \neq j$,$x_i = x_j$ and $k > 0$;

$f \in ASC$ if
$f(x + e_i) \leq f(x + e_j)$
for all x and i, j with $i < j$ and $x_i \leq x_j$ and
$f(x + ke_i) = f(x + ke_j)$
for all x and i, j with $i < j$,$x_i = x_j$ and $k > 0$.

Convexity:

$f \in Cx(i)$ if
$2f(x + e_i) \leq f(x) + f(x + 2e_i)$
for all x and $1 \leq i \leq m$;

$$Cx = \bigcap_{1 \leq i \leq m} Cx(i);$$

$f \in Cv(i)$ if
$f(x) + f(x + 2e_i) \leq 2f(x + e_i)$
for all x and $1 \leq i \leq m$;

$$Cv = \bigcap_{1 \leq i \leq m} Cv(i);$$

$f \in Super(i, j)$ if
$f(x + e_i) + f(x + e_j) \leq f(x) + f(x + e_i + e_j)$
for all x and $1 \leq i < j \leq m$;

$$Super = \bigcap_{1 \leq i < j \leq m} Super(i, j);$$

$f \in Sub(i,j)$ if

$$f(x) + f(x + e_i + e_j) \leq f(x + e_i) + f(x + e_j)$$

for all x and $1 \leq i < j \leq m$;

$$Sub = \bigcap_{1 \leq i < j \leq m} Sub(i,j);$$

$f \in SuperC(i,j)$ if

$$f(x + e_i) + f(x + e_j + e_j) \leq f(x + e_j) + f(x + 2e_i)$$

for all x and $1 \leq i, j \leq m,\ i \neq j$;

$$SuperC = \bigcap_{1 \leq i,j \leq m,\ i \neq j} SuperC(i,j);$$

$f \in MM(i,j)$ if

$$f(x) + f(x + d_i + d_j) \leq f(x + d_i) + f(x + d_j)$$

for all x and $1 \leq i < j \leq m$, such that $x + d_i, x + d_j \in \mathbb{N}_0^{d+1}$, with $d_0 = e_1,\ d_k = -e_k + e_{k+1},\ k = 1, \ldots, m-1$, and $d_m = -e_m$;

$$MM = \bigcap_{1 \leq i < j \leq m} MM(i,j);$$

Combination of operators and properties:

The following propagation results hold for environmental operators:

$T_{disc}, T_{env} : I \to I,\ UI \to UI,\ wUI \to wUI,\ gUI \to gUI,\ SC \to SC,\ ASC \to ASC,$
$\quad Cx \to Cx,\ Super \to Super,\ Sub \to Sub,\ SuperC \to SuperC,\ Sub \to Sub,\ MM \to MM;$

$T_{min} : I \to I,\ UI \to UI,\ wUI \to wUI$ when $\mu(1) \le \ldots \le \mu(m),\ SC \to SC,\ ASC \to ASC;$

$T_{max} : I \to I,\ UI \to UI,\ wUI \to wUI$ when $\mu(1) \le \ldots \le \mu(m),\ SC \to SC,\ ASC \to ASC,$
$\quad Cx \to Cx.$

The following results hold for the arrival operators, $1 \le i, j \le m, i \ne j$:

$T_{A(i)} : I \to I,\ UI \to UI,\ wUI \to wUI,\ gUI \to gUI,\ Cx \to Cx,\ Super \to Super,\ Sub \to Sub,$
$\quad SuperC \to SuperC,\ Sub \to Sub,\ MM \to MM;$

$T_{CA(i)} : I \to I,\ UI \to UI,\ wUI \to wUI$ when $\mu(1) \le \ldots \le \mu(m),\ Cx(i) \to Cx(i),$
$\quad Super(i,j) \to Super(i,j),\ Sub \to Sub,\ Super(i,j) \cap SuperC(i,j) \to SuperC(i,j),$
$\quad Super(i,j) \cap SuperC(j,i) \to SuperC(j,i),\ Sub(i,j) \cap SubC(i,j) \to SubC(i,j),$
$\quad Sub(i,j) \cap SubC(j,i) \to SubC(j,i),\ MM \to MM$ for $i = 1$;

$T_{FS(i)} : I \to I,\ Cx \to Cx, Super \to Super, Sub \to Sub;$

$T_{CAF} : I \to I,\ Sub(i,j) \cap SubC(i,j) \to Sub(i,j)$ if $m = 2$, $SubC(i,j) \to SubC(i,j)$ if $m = 2$;

$T_R : I \to I,\ SC \to SC,\ Super(i,j) \cap SuperC(i,j) \cap SuperC(j,i) \to Super(i,j)$ if $m = 2$,
$\quad ASC \to ASC,\ Super(i,j) \cap SuperC(i,j) \to SuperC(i,j)$ if $m = 2$.

For departure operators the following progation results hold, for $1 \le i, j \le m$:

$T_{D(i)} : I \to I,\ I \cap UI \to UI$ for $i = m,\ I(i) \cap Cx(i) \to Cx(i)$ if $\mu(x) \in I \cap Cv$,
$Cx(j) \to Cx(j)$ for $j \ne i,\ Super \to Super,\ Sub \to Sub$;

$T_{D1(i)} : I \to I,\ I \cap UI \to UI$ for $i = m,\ I(i) \cap Cx(i) \to Cx(i)$,
$Cx(j) \to Cx(j)$ for $j \ne i,\ Super \to Super,\ Sub \to Sub,\ SuperC(j,k) \to SuperC(j,k)\ (j,k \ne i)$,
$I(i) \cap SuperC(i,j) \to SuperC(i,j)\ (j \ne i),\ Cx(j) \cap SuperC(j,i) \to SuperC(j,i)$,
$SubC(j,k) \to SubC(j,k)\ (j,k \ne i),\ I(i) \cap SubC(i,j) \to SubC(i,j)\ (j \ne i)$,
$Cx(j) \cap SubC(j,i) \to SubC(j,i),\ UI \cap MM \to MM$ for $i = m$;

$T_{PD} : I \to I,\ UI \to UI,\ I \cap SC \to SC$ for $\mu(i) = \mu(j)$,
$I \cap ASC \to ASC$ if $\mu(i) \ge \mu(j)$ if $i < j,\ I(i) \cap Cx(i) \to Cx(i),\ Cx(j) \to Cx(j)$ for $j \ne i$,
$Super \to Super,\ Sub \to Sub,\ SuperC(j,k) \to SuperC(j,k)\ (j,k \ne i)$,
$I(i) \cap SuperC(i,j) \to SuperC(i,j)\ (j \ne i),\ Cx(j) \cap SuperC(j,i) \to SuperC(j,i)$,
$SubC(j,k) \to SubC(j,k)\ (j,k \ne i),\ I(i) \cap SubC(i,j) \to SubC(i,j)\ (j \ne i)$,
$Cx(j) \cap SubC(j,i) \to SubC(j,i),\ UI \cap MM \to MM$ for $i = m$;

$T_{CD(i)} : I \to I$ if $c(0) = \min_{\mu \in [0,1]} c(\mu),\ I \cap UI \to UI$ for $i = m$ if $c(0) = \min_{\mu \in [0,1]} c(\mu)$,
$Cx(i) \to Cx(i),\ Super(i,j) \to Super(i,j),\ Sub \to Sub$ if $c(0) = \min_{\mu \in [0,1]} c(\mu)$,
$Cx(i) \cap SuperC(i,j) \to SuperC(i,j),\ Super(i,j) \cap SuperC(j,i) \to SuperC(j,i)$,
$Cx(i) \cap SubC(i,j) \to SubC(i,j),\ Sub(i,j) \cap SubC(j,i) \to SubC(j,i),\ MM \to MM$ for $i = m$;

$T_{MS} : I \to I,\ wUI \to wUI$ for μ as in TMS,
$Super \cap SuperC \to Super$ for $m = 2$ and $\mu(1) = \mu(2)$,
$SuperC \to SuperC$ for $m = 2$ and $\mu(1) = \mu(2)$;

$T_{MMS} : I \to I,\ I \cap wUI \to wUI$ for μ as in $TMMS$ and $\mu(i) \le \mu(j), i < j \in I$.

For transfer operators the following propagation results hold, $1 \le i \le m$:

$$T_{TD1(i)} : I \to I,\ UI \to UI \text{ for } i < m,\ UI \cap MM \to MM \text{ for } i < m,$$
$$UI \cap Cx \cap Super \to Cx \text{ for } i < m,\ UI \cap Cx \cap Super \to Super \text{ for } i < m;$$

$$T_{CTD(i)} : I \to I \text{ if } c(0) = \min_{\mu \in [0,1]} c(\mu),\ UI \to UI \text{ for } i < m,\ MM \to MM;$$

$$T_{TD(i)} : I \to I,\ UI \to UI \text{ for } i < m,\ UI \cap Cx \cap Super \to Cx \text{ for } i < m,$$
$$UI \cap Cx \cap Super \to Super \text{ for } i < m;$$

$$T_{MTS} : I \to I,\ gUI \to gUI \text{ for } \mu \text{ as in } TMTS.$$

Chapter 5

Conclusion and further research

The smoothed rate truncation can be applied on the single server queue. For both the cost as the reward model it is possible to prove properties of the value function. In the available literature it was pretty hard work to give a priority-rule for this model, with the newly developed SRT principle it is a relatively straightforward result.
The propagation results for the new operators in event-based dynamic programming are hopeful. It is nice to see that for the smoothed operators the same results can be acomplished as for the normal operators. The results for analyzed operators can be used to study models with unbounded transition rates.
A lot more work can be done here. There is a large number of operators and properties that can be studied. The methods used in the propagation proofs may be used to try on other problems in this field.

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