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MASTER THESIS

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Backward allocation of the diversification effect in insurance risk

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Abstract:

The determination of the sufficient amount of economic capital and its allocation to the business lines is the key issue for insurance companies. In this thesis we introduce two methods of aggregating economic capital- one is based on linear correlation and the second deals with copulas. A multitude of allocation principles have been proposed in the literature. We chose some of them which are the most used in practice and compared their advantages and disadvantages. Numerical examples of capital aggregation and its allocation to business units are additionaly provided in the last chapter.

Keywords:

dependence measures, risk measures, copula, capital aggregation, diversification, allocation principles

Contents

Introduction

Evaluating the total capital requirement of an insurance company is an important risk management issue, as well as the allocation of this capital to its various business units.

Regarding economic capital, it has become a topic discussed at various industry conferences, received attention by regulators and rating agencies, and has shown up over the years in various other disciplines, in particular in the banking and insurance industry.

It is well known that insurance companies are obligated to hold a sufficient amount of capital to remain solvent. Holding this capital protects the company from insolvency, and ensures the future of the company as a going-concern. While it is desirable that it holds large amounts of capital, usually this does not come without cost. Investors demand a premium for lending capital and this cost of capital can indirectly be passed to the policyholders in the form of a higher premium loading. The capital required by the insurer is often viewed by ratings agencies as a measure of the company's capacity to bear risks. There has been a recent surge in the literature in developing a framework of risk measurements for computing capital requirements which is an important part of the risk management process for the insurance companies.

Further, a fundamental question in actuarial science is how to allocate a given amount of capital between the different busines lines of the company. This task is called the capital allocation problem. The term "capital allocation" has been used in finance literature where a similar concept of fair division of capital in a diversified portfolio of investments has been investigated.

Capital allocation is generally not an end in itself, but rather a step in a decision-making process. There are more reasons for allocating the economic capital to the business units. Firstly, as was mentioned, there is a cost associated with holding capital and the insurance company may wish to accurately determine this cost by line of business and thereby redistribute this cost equitably across the lines. Secondly, capital is often viewed as a measure of the level of risk inherent in the company and division of the capital therefore provides a division of the level of risks inherent across the business units. This division of total company risk can be useful to the insurance company wishing to allocate expenses across the lines of business, prioritizing new capital budgeting projects, or even deciding which lines of business to expand or to contract. Last, capital allocation formulas provide a useful device for fair assessment of performance of managers of various business units. Salaries and bonuses may be linked to performance. In summary, the richer information often derived from capital allocation improves management of the insurance enterprise.

This master thesis is devoted to economic capital aggregation in the first place and then its allocation to the business lines.

In the first chapter we introduce dependence measures. Besides the known Pearson correlation coeficient we mention dependence measures which are connected to copulas. We also provide an introduction to the copula theory.

In the second chapter we derive the aggregated economic capital using the correlation matrix. Because individual risks are dependent, we can see the advantages resulting from the diversification effect. It means that the total economic capital is always less than the sum of capitals of individual business units.

The third chapter is devoted to economic capital aggregation using copulas and we give a brief overview of copulas used in practice.

In the fourth chapter we discuss the capital allocation principles. Because in the literature one can find a numerous ways how to allocate the capital we choose the main of them.

In the fifth chapter we apply the allocation principles to the exact numerical examples. We will demonstrate on these examples differences between using individual principles.

1. Dependence measures

There are many ways how to measure dependence. In this chapter we introduce essential dependence measures and their advantages and disadvantages. The first one is the well-known linear correlation and the second one is the class of measures based on copulas. Both of these measures give a scalar measurement for two random variables (X,Y).

1.1 Linear correlation

The Pearson coefficient of linear correlation measures the linear dependence between pair of random variables (X,Y) and is easily countable. It is defined by

$$\rho(X,Y) = \frac{cov(X,Y)}{\sqrt{var(X)var(Y)}},$$

where

$$cov(X,Y) = E(XY) - E(X)E(Y)$$
$$var(X) = E(X^2) - E(X)^2.$$

Correlation takes values in $[-1,1]$ while the frontier values ± 1 need not to be reached for some marginal distributions. If the random variables X,Y are independent, then $\rho(X,Y) = 0$, but the reverse implication does not hold.
Moreover, $|\rho(X,Y)| = 1$ responds to *perfect linear dependence* between X, Y, i.e. $Y = a + bX$ almost surely for some $a \in R$, $b \neq 0$, with $b > 0$ for the positive linear dependence and $b < 0$ for the negative linear dependence. Correlation is invariant under strictly increasing linear transformation as well, so

$$\rho(a_1X + b_1, a_2Y + b_2) = sgn(a_1a_2)\rho(X,Y).$$

Let us look at the main disadvantages of using the Pearson correlation coeficient as dependence measure:

- Correlation is defined for the random variables with finite variances only. For instance, this property can cause problems when we work with heavy-tailed distributions. It does not deliberate the tail dependencies.
- If the random variables X,Y are independent then $\rho(X,Y) = 0$, but the converse is false. Also $\rho(X,Y) \approx 0$ does not mean weak dependence between random variables.

- Correlation is not invariant under nonlinear strictly increasing transformations,
$$\rho(t_1(X), t_2(Y)) \neq \rho(X, Y)$$
for t_1, t_2 strictly increasing functions.

Despite of these shortcomings correlation still plays a crucial role in financial theory. It is the canonical measure in the case of multivariate normal distributions, and more generally for elliptical distributions. In insurance industry the losses often have the lognormal distribution which belong to this category.

1.2 Copula-based dependence measures

The study and applications of copulas in statistics and probability have extended in the last years. The interest in copulas grows for two main reasons: At first, as a way of studying scale-free measures of dependence; and secondly, as a starting point for constructing families of bivariate distributions.

1.2.1 Introduction to the theory of copulas

We focus on bivariate copulas of continuous random variables with distribution functions X_1, X_2. In advance we introduce the definition of copula:

Definition 1.2.1. (bivariate case) A two-dimensional copula C is a joint distribution function of standard uniform distributed random variables (U_1, U_2) defined on $[0, 1]^2$
$$C(u_1, u_2) = P(U_1 < u_1, U_2 < u_2), u_1, u_2 \in [0, 1].$$

We can also use an alternative definition of a copula which is more formal:

Definition 1.2.2. A bivariate copula is any function $[0, 1]^2 \to [0, 1]$ which has the following three properties:

1. $C(u_1, 0) = C(0, u_2) = 0$
2. $C(u_1, 1) = u_1$, $C(1, u_2) = u_2$
3. $\forall u_1, u_2, v_1, v_2 \in [0, 1], u_1 \leq v_1, u_2 \leq v_2$
   $C(v_1, v_2) - C(v_1, u_2) - C(u_1, v_2) + C(u_1, u_2) \geq 0.$

Both definitions are equivalent.

We already described the concept of copulas. There are some basic terms used in connection with copulas.
Survival copula: $\bar{C}(u_1, u_2) = C(1 - u_1, 1 - u_2) + u_1 + u_2 - 1$
Dual copula: $\tilde{C}(u_1, u_2) = u_1 + u_2 - C(u_1, u_2)$

Co-copula: $C^*(u_1, u_2) = 1 - C(1 - u_1, 1 - u_2)$.
Though, only the *survival copula* fulfills the definition of copulas. In the other two terms is name *copula* used just formally.

The following theorem is essential to many applications of copulas in statistical theory and explains the role of copulas in the relationship between multivariate distribution functions and their univariate margins. It also shows that any distribution function can be described by a copula and vice versa.

Theorem 1.2.1. (Sklar)
Let F be the bivariate joint distribution function with continuous marginal distribution functions F_1 and F_2. Then there exists a copula C such that
$F(x_1, x_2) = C(F_1(x_1), F_2(x_2))$ for all $x_1, x_2 \in [-\infty, \infty]$.

As we have seen, a copula fully explains the dependence structure of continuous random variables without reference to their marginal distribution. We define the case of the perfect dependence:

Definition 1.2.3. X_1, X_2 are comonotonic if for their copula holds
$C(u_1, u_2) = C_U(u_1, u_2) = min(u_1, u_2)$.
X_1, X_2 are countermonotonic if for their copula holds
$C(u_1, u_2) = C_L(u_1, u_2) = max(u_1 + u_2 - 1, 0)$.

Comonotonicity, resp.countermonotonicity is the strongest dependence structure which can occur between two random variables.
The useful property of copulas is that they are invariant under strictly increasing transformations of random variables:

Theorem 1.2.1. For t_1, t_2 strictly increasing functions have that the random vectors (X_1, X_2) and $(t_1(X_1), t_2(X_2))$ have the same copula $C(u_1, u_2)$.
Furthermore, we obtain:

- For t_1 strictly increasing and t_2 strictly decreasing $(t_1(X_1), t_2(X_2))$ has a copula $u_1 - C(u_1, 1 - u_2)$.
- For t_1 strictly decreasing and t_2 strictly increasing $(t_1(X_1), t_2(X_2))$ has a copula $u_2 - C(1 - u_1, u_2)$.
- For t_1, t_2 strictly decreasing $(t_1(X_1), t_2(X_2))$ has a copula $u_1 + u_2 - 1 + C(1 - u_1, 1 - u_2)$.

Consequence 1.2.1. Continuous random variables are comonotonic (countermonotonic) if and only if $X_2 = t(X_1)$ almost surely, where t is an increasing (decreasing) function.

The next two theorems concern with partial derivatives of copulas.

Theorem 1.2.2. Let C be a copula. For any $v \in [0,1]$ the partial derivative $\frac{\partial C(u,v)}{\partial u}$ exists for almost all u, and for such v and u

$$0 \leq \frac{\partial}{\partial u} C(u,v) \leq 1.$$

Similarly, for any $u \in [0,1]$ the partial derivative $\frac{\partial C(u,v)}{\partial v}$ exists for almost all v, and for such u and v

$$0 \leq \frac{\partial}{\partial v} C(u,v) \leq 1.$$

Furthermore, the functions $u \to \frac{\partial C(u,v)}{\partial v}$ and $v \to \frac{\partial C(u,v)}{\partial u}$ are defined and nondecreasing almost everywhere on $[0,1]$.

Theorem 1.2.3. Let C be a copula. If $\frac{\partial C(u,v)}{\partial v}$ and $\frac{\partial^2 C(u,v)}{\partial u \partial v}$ are continous on $[0,1]^2$ and $\frac{\partial C(u,v)}{\partial u}$ exists $\forall u \in (0,1)$ when $v=0$, then $\frac{\partial C(u,v)}{\partial u}$ and $\frac{\partial^2 C(u,v)}{\partial v \partial u}$ exist in $(0,1)^2$ and

$$\frac{\partial^2 C(u,v)}{\partial u \partial v} = \frac{\partial^2 C(u,v)}{\partial v \partial u.}$$

Often we are more interested in the conditional distribution of a copula.
Because the copula is increasing in each argument,

$$C_{U_2|U_1}(u_2|u_1) = P(U_2 \leq u_2|U_1 = u_1) = \lim_{\delta \to 0} \frac{C(u_1+\delta, u_2) - C(u_1,u_2)}{\delta} = \frac{\partial}{\partial u_1} C(u_1,u_2),$$

and the partial derivative exists almost everywhere. The conditional distribution can be interpreted the following way: Suppose that continuous risks (X_1, X_2) have the (unique) copula C. Then $1 - C_{U_2|U_1}(q|p)$ is the probability that X_2 exceeds its qth quantile under the condition that X_1 attains its pth quantile.

Next we give some examples of the most useful copulas.

Fundamental copulas

The *independence copula* is

$$C_I(u_1,u_2) = u_1 u_2.$$

It is obvious that continuous random variables are independent if and only if their dependence structure is given by the independence copula.
The *Fréchet upper bound* (or comonotonicity copula) is

$$C_U(u_1,u_2) = min(u_1,u_2).$$

It represents the perfect positive dependence; it is a distribution function of (U,U).
The *Fréchet lower bound* (or countermonotonicity copula) is defined by

$$C_L(u_1,u_2) = max(u_1+u_2-1,0).$$

This is the case of perfectly negative dependent random variables; it is a joint distribution function of $(U, 1-U)$. Every copula is bounded by Fréchet lower and upper bound copulas

$$C_L(u_1,u_2) \leq C(u_1,u_2) \leq C_U(u_1,u_2), \forall (u_1,u_2) \in [0,1]^2.$$

Elliptical copulas

Elliptical copulas are the copulas of elliptical distributions. First we define the elliptical distributions and the notion of the special case of spherical distributions.

Definition 1.2.4. A random vector $\mathbf{X} = (X_1,\ldots,X_d)'$ has a spherical distribution if for every orthogonal map $U \in R^{dxd}$ (i.e., maps satisfying $UU' = U'U = I_d$)

$$U\mathbf{X} =^d \mathbf{X}.$$

The characteristic function $\psi(t) = E[exp(it'X)]$ of such distributions takes a particularly simple form. There exists a function $\phi : R_{>0} \to R$ such that $\psi(t) = \psi(t't) = \psi(t_1^2 + \ldots + t_d^2)$. This function is the *characteristic generator* of the spherical distribution and the notation $X \sim S_d(\psi)$ is used.

Definition 1.2.5. $\mathbf{X}$ has an elliptical distribution if

$$\mathbf{X} =^d \mu + A\mathbf{Y},$$

where $\mathbf{Y} \sim S_k(\psi)$ and $\mathbf{A} \in R^{dxk}$ and $\mu \in R^d$ are a matrix and vector of constants, respectively.

Mathematically the elliptical distributions are the *affine maps* of spherical distributions in R^d.
The most used distributions from this family are multivariate (in our case bivariate) normal and the Student $t-$distribution. They do not have simple closed forms and are restricted to have a radial symmetry.
The *Gauss (normal) copula* is given by

$$C_\rho^{Ga}(u_1,u_2) = \int_{-\infty}^{\Phi^{-1}(u_1)} \int_{-\infty}^{\Phi^{-1}(u_2)} \frac{1}{2\pi(1-\rho^2)^{1/2}} \exp\left\{ \frac{-(s_1^2 - 2\rho s_1 s_2 + s_2^2)}{2(1-\rho^2)} \right\} ds_1 ds_2,$$

where Φ^{-1} is the inverse of the univariate standard normal distribution function and $|\rho| < 1$, the linear correlation coefficient, is the copula parameter.
The *student $t-$copula* with ν degrees of freedom and correlation coefficient ρ is an elliptical copula defined as:

$$C_{\rho,\nu}^t(u_1,u_2) = \int_{-\infty}^{t_\nu^{-1}(u_1)} \int_{-\infty}^{t_\nu^{-1}(u_2)} \frac{1}{2\pi(1-\rho^2)^{1/2}} \exp\left\{ 1 + \frac{(s_1^2 - 2\rho s_1 s_2 + s_2^2)}{\nu(1-\rho^2)} \right\}^{-\frac{\nu+2}{2}} ds_1 ds_2,$$

where t_ν^{-1} denotes the inverse of the distribution function of the standard univariate t-distribution with ν degrees of freedom and ν and ρ are the copula parameters.

Archimedean copulas

The Archimedean copulas are the important class of copulas for many reasons. They can be easily constructed, a lot of families of copulas belong to this class and they possess many nice properties.

Definition 1.2.6. Let φ be a continuous, strictly decreasing function from $[0,1]$ to $[0,\infty]$ such that $\varphi(1)=0$, and let the $\varphi^{[-1]}$ be the pseudo-inverse of φ.Then the function C from $[0,1]^2$ to $[0,1]$ given by

$$C(u,v)=\varphi^{[-1]}(\varphi(u)+\varphi(v)),$$

C satisfies the Fréchet boundary conditions for copulas and is called the *Archimedean copula*. The function φ is called *the additive generator of the copula*. Moreover, if $\varphi(0)=\infty$ then φ is a strict generator and $C(u,v)=\varphi^{[-1]}(\varphi(u)+\varphi(v))$ is said to be a *strict Archimedean copula*.

There are some properties which the Archimedean copulas possess. As Lei Hua mentioned in [8], they behave like the binary operation:

- **Commutative**
  $C(u,v)=C(v,u), \forall u,v\in[0,1]$
- **Associative**
  $C(C(u,v),w)=C(u,C(v,w)), \forall u,v,w\in[0,1]$
- **Order preserving**
  $C(u_1,v_1)\leq C(u_2,v_2), u_1\leq u_2, v_1\leq v_2\in[0,1]$
- If $c>0$ is any constant, then $c\varphi$ is also a generator of C.

For instance the Fréchet lower bound and the independent copula belong to this family of copulas. Other well-known representatives are
Gumbel copula:
$C_\theta^{Gu}(u_1,u_2)=exp(-((-lnu_1)^\theta+(-lnu_2)^\theta)^{\frac{1}{\theta}}), 1\leq\theta<\infty$
Clayton copula:
$C_\theta^{Cl}(u_1,u_2)=(u_1^{-1}+u_2^{-1}-1)^{\frac{-1}{\theta}}$.

1.2.2 Measures of concordance

This section is devoted to dependence measures related to copulas, which are more suitable than the Pearson coefficient of linear correlation in some cases. We denote the common risk measure $\rho(X,Y)$ and require to have these properties:

1. $\rho(X,Y)=\rho(Y,X)$ (symmetry)

2. $-1 \leq \rho(X, Y) \leq 1$ (normalisation)

3. $\rho(X, Y) = 1 \Leftrightarrow$ X,Y are comonotonic
   $\rho(X, Y) = -1 \Leftrightarrow$ X,Y are countermonotonic

4. $\rho(t(X), Y) = \rho(X, Y) \Leftrightarrow t$ a is strictly increasing function
   $\rho(t(X), Y) = -\rho(X, Y) \Leftrightarrow t$ a is strictly decreasing function

The Pearson coefficient merely satisfies the properties 1,2. That is why we establish dependence measures fulfilling properties 1-4 called *measures of concordance.* We can also require the property of independence
5. $\rho(X, Y) = 0 \Leftrightarrow X, Y$ are independent.
Unfortunately, the 4. and 5. property are mutually contradicting so there is no dependence measure satisfying both properties.

Definition 1.2.7. Two observations (x_1, y_1) and (x_2, y_2) of a pair (X, Y) of continuous random variables are concordant if $(x_1 - x_2)(y_1 - y_2) > 0$ and discordant if $(x_1 - x_2)(y_1 - y_2) < 0$.

In other words, X, Y are concordant if both values of one pair are greater than the corresponding values of the other pair; it happens when $x_1 < x_2$ and $y_1 < y_2$ or $x_1 > x_2$ and $y_1 > y_2$. Alike (x_1, y_1) and (x_2, y_2) are said to be discordant if for one pair one value is greater and the second value is smaller than for the other pair, that is if $x_1 < x_2$ and $y_1 > y_2$ or $x_1 > x_2$ and $y_1 < y_2$. *Concordance function* Q is then defined by

$$Q = P[(X_1 - X_2)(Y_1 - Y_2) > 0] - P[(X_1 - X_2)(Y_1 - Y_2) < 0],$$

the difference between the probabilities of concordance and discordance between two random vectors (X_1, Y_1) and (X_2, Y_2). Measures of concordance are discussed in [11] more closely.

1.2.3 Rank correlation

Rank correlations are scalar measures which are derived from the concordance function. They are appropriate for identification of copulas from data by looking at the ranks of the data alone. Moreover, they depend only on the copula and not on the marginal distributions. There are two main representatives of rank correlation: Kendall's tau and Spearman's rho. We are going to discuss them more closely.

Kendall's tau

Definition 1.2.8. For a vector of continuous random variables (X,Y) with joint distribution function Kendall's tau is given by

$$\begin{aligned} \rho_t &= P[(X_1 - X_2)(Y_1 - Y_2) > 0] - P[(X_1 - X_2)(Y_1 - Y_2) < 0] \quad (1.1) \\ &= Esgn[(X_1 - X_2)(Y_1 - Y_2)], \quad (1.2) \end{aligned}$$

where (X_1, Y_1) and (X_2, Y_2) are independent identically distributed random variables with the joint distribution function H.

Spearman'rho

Let (X_1, Y_1),(X_2, Y_2) and (X_3, Y_3) be three independent random vectors with a joint distribution function H. The Spearman's rho is defined to be proportional to the probability of concordance minus the probability of discordance for a pair of random vectors (X_1, Y_1) and (X_2, Y_3) with the same margins, but one vector has distribution function H, while the components of the other are independent:

$$\rho_s = 3(P[(X_1 - X_2)(Y_1 - Y_3) > 0] - P[(X_1 - X_2)(Y_1 - Y_3) < 0]).$$

There is also the other definition of ρ_s which we are interested in because it involves the concept of copulas:

Definition 1.2.9. For random variables X and Y with marginal distribution functions F and G Spearman's rho is given by $\rho_s = \rho(F(X), G(Y))$.

We can see that Spearman's rho is the linear correlation of transformed random variables by means of marginal distribution functions.

As we said, Kendall's tau and Spearman's rho depend only on copulas. Therefore in the next proposition the alternative definitions are given.

Proposition 1. Suppose X and Y have continuous marginal distributions and the unique copula C. Then the rank correlations are given by

$$\rho_t(X, Y) = 4 \int_0^1 \int_0^1 C(u_1, u_2) dC(u_1, u_2) - 1,$$

$$\rho_s(X, Y) = 12 \int_0^1 \int_0^1 (C(u_1, u_2) - u_1 u_2) du_1 du_2.$$

Altough both Kendall's tau and Spearman's rho measure the probability of concordance between random variables with a given copula, their values often differ. We subscribe relationship between $\rho_t(X, Y) = \rho_t$ and $\rho_s(X, Y) = \rho_s$:

- $-1 \leq 3\rho_t - 2\rho_s \leq 1$

- $\frac{1+\rho_s}{2} \geq \left(\frac{1+\rho_t}{2}\right)^2$,
  $\frac{1-\rho_s}{2} \geq \left(\frac{1-\rho_t}{2}\right)^2$

- $\frac{3\rho_t-1}{2} \leq \rho_s \leq \frac{1+2\rho_t-\rho_t^2}{2}, \rho_t \geq 0$
  $\frac{\rho_t^2+2\rho_t-1}{2} \leq \rho_s \leq \frac{3\rho_t+1}{2}, \rho_t \leq 0.$

For more information we refer to [12].

1.2.4 Coefficients of tail dependence

As well as the rank correlation, coefficients of tail dependence belong to dependence measures expressed by copulas. They measure the strength of dependence in the tails of bivariate distributions. These coefficients are defined by limiting conditional probabilities of quantile excesses. We distinguish two cases of the tail dependence; the upper and the lower tail dependence. In the first case we look at the probability that Y exceeds its $\alpha - quantile$ under the condition that X exceeded its $\alpha - quantile$, and then consider the limit as α goes to infinity. The roles of X and Y are obviously interchangeable. By [9] we have definition:

Definition 1.2.10. Let X and Y are random variables with distribution functions F and G. The coefficient of the upper tail dependence of X and Y is

$$\lambda_u := \lambda_u(X,Y) = \lim_{\alpha \to 1^-} P(Y > G^{-1}(\alpha) | X > F^{-1}(\alpha)),$$

provided a limit $\lambda_u \in [0,1]$ exists. If $\lambda_u \in (0,1]$, then X and Y are said to show upper tail dependence or extremal dependence in the upper tail; if $\lambda_u = 0$, they are asymptotically independent in the upper tail.
Analogously, the coefficient of the lower tail dependence is

$$\lambda_l := \lambda_l(X,Y) = \lim_{\alpha \to 0^+} P(Y \leq G^{-1}(\alpha) | X \leq F^{-1}(\alpha)),$$

provided a limit $\lambda_l \in [0,1]$ exists.

Because F and G are continuous distribution functions, we can rewrite definition 1.2.10 by using formulas for conditional probabilities in terms of copulas as

$$\lambda_l = \lim_{\alpha \to 0^+} \frac{P(Y \leq G^{-1}(\alpha), X \leq F^{-1}(\alpha))}{P(X \leq F^{-1}(\alpha))} \quad (1.3)$$

$$= \lim_{\alpha \to 0^+} \frac{C(\alpha,\alpha)}{\alpha} \quad (1.4)$$

for the lower tail dependence and

$$\lambda_u = \lim_{\alpha \to 1^-} \frac{\hat{C}(1-\alpha, 1-\alpha)}{1-\alpha} = \lim_{\alpha \to 0^+} \frac{\hat{C}(\alpha,\alpha)}{\alpha} \quad (1.5)$$

for the upper tail dependence. $\hat{C}$ denotes the survival copula of C and for radially symetric copulas $\lambda_u = \lambda_l$. The tail dependence parameters are easily evaluated if the copula has a simple closed form. For copulas without a simple closed form, as the Gaussian copula for instance, an alternative formula is used.

2. Linear aggregation of economic capital

2.1 Introduction

In agreement with the new risk-based solvency regulations, insurance companies are required to compute their economic capital. It is still a relatively new framework in the insurance industry. It depends on distribution functions and the dependence structure between sub-risks and business units. The model for capital aggregation can be based on the simple linear aggregation between the losses or on the copulas. The linear aggregation model is based on aggregating risks $X_1, X_2, \ldots, X_n$ using correlations and the individual risk measures. In this chapter we will discuss the linear approach to the capital aggregation.

2.2 Risk measures

Definition 2.2.1. A risk measure is a mapping ρ from a set Γ of real-valued random variables defined on (Ω, F, P) to R:

$$\rho : \Gamma \to R : X \in \Gamma \to \rho[X].$$

Generally it has a nonnegative value but in some important cases this requirement would be limiting. Firstly risk measures have been related to principles for determining insurance premia in nonlife insurance. Recently, they started to be used in a risk management where $\rho[X]$ represents the amount of capital to be set aside to make the loss X an acceptable risk. The most known properties for risk measures are *requirements of coherence* and they are defined the following way:

Definition 2.2.2. A risk measure ρ is called **coherent** if it satisfies the following properties:

1. **Translation invariance**: $\rho[X + a] = \rho[X] - a$ for any $X \in \Gamma$ and a $\in R$.
2. **Positive homogeneity**: $\rho[aX] = a\rho[X]$ for any $X \in \Gamma$ and $a > 0$.
3. **Subadditivity**: $\rho[X + Y] \leq \rho[X] + \rho[Y]$ for any $X, Y \in \Gamma$.
4. **Monotonicity**: $X \leq Y \Rightarrow \rho[X] \leq \rho[Y]$ for any $X, Y \in \Gamma$.

Sometimes we are also interested in the other property of risk measure:

5. **Law invariance**: For any $X_1, X_2 \in \Gamma$ with $P[X_1 \leq x] = P[X_2 \leq x]$ for all $x \in R$, $\rho[X_1] = \rho[X_2]$.

Coherent risk measures are mentioned in [1].
Now we mention two risk measures which are frequently used in practice.

- **Value at Risk**
  Value at Risk (VaR) is the most widely used risk measure in financial institutions. It is usually chosen in situations where we want to avoid the default event but the information about size of the shortfall is not so important.

  Definition 2.2.3. Suppose it is given some confidence level $\alpha \in (0,1)$. The VaR of a portfolio at the confidence level α is defined by the smallest number x such that the probability that the loss X exceeds x is no larger than $(1-\alpha)$. Formally,
  $VaR_\alpha = \inf\{x \in R : P(X > x) \leq 1-\alpha\} = \inf\{x \in R : F_X(x) > \alpha\}$.

  In other words, VaR is a $\alpha - quantile$ of the loss distribution. The typical value of the confidence level in the insurance industry is 99.5%. VaR is not a coherent risk measure because it does not satisfy the requirement of subadditivity. It is highly criticized for violating this property because then there are no benefits from the diversification effect. However, there is a known case where VaR satisfies this property. For *jointly elliptically distributed random variables* the VaR is a coherent risk measure.

- **Expected shortfall**
  As we said, VaR does not give us any information about the severity of default. Therefore we introduce the next risk measure the Expected shortfall, also called the Conditional Tail Expectation (CTE) or Tail Value-at-Risk (TVaR) at probability level α. The expected shortfall is defined as the average of all losses which are greater than or equal to VaR; it is the average loss in the worst $(1-\alpha)$% cases. We denote it ES_α and define as

  $$ES_\alpha(X) = \frac{1}{1-\alpha}\int_\alpha^1 VaR_X(x)dx = E[X|X > F_X^{-1}(\alpha)].$$

  CTE_α is the coherent risk measure.

2.3 Economic capital

At its most basic level, economic capital can be defined as sufficient surplus to cover potential losses, at a given risk tolerance level, over a specified time horizon. In other words, it is the amount of capital which a company needs to remain solvent. According to the survey presented in [14], there are three main definitions of economic capital:
Definition 1 Economic Capital is defined as sufficient surplus to meet potential negative cash flows and reductions in value of assets or increases in value of liabilities at a given level of risk tolerance, over a specified time horizon.

Definition 2 Economic Capital is defined as the excess of the market value of the assets over the fair value of liabilities required to ensure that obligations can be satisfied at a given level of risk tolerance, over a specified time horizon.
Definition 3 Economic Capital is defined as sufficient surplus to maintain solvency at a given level of risk tolerance, over a specified time horizon.

Computation of economic capital is based on the institution's financial strength and on the expected loss. Financial strength is represented by the probability that the company stays solvent over the measurement period on the confidence level. The expected loss is the average loss which can occur in the given time horizon. We will consider the business which faces the random loss S over the one - year horizon. We denote the economic capital EC[S] and define it in the following way:

Definition 2.3.1. Economic capital is given by

$$EC[S] = \rho(S) - ES,$$

where ρ is a risk measure and S is the random variable representing the loss of the company.

$\rho(S)$ is called the *total balance sheet capital requirement.* Thence we define the *economic default* as the occurence that S *exceeds* $\rho(S)$. According to Solvency II and the Swiss Solvency test, we will work with two risk measures: the Value-at-Risk and the Expected Shortfall. The standard approach is to use VaR at the confidence level 99.5% and ES is adopted at 99% as a risk measure. However, as we mentioned afore, VaR violates the property of subadditivity and therefore it is not a coherent risk measure. This property is very important because it guarantees the diversification effect between risks. By using VaR as a risk measure and considering two risks X, Y, we do not necessarily obtain

$$VaR(X + Y) \leq VaR(X) + VaR(Y).$$

On the other hand, VaR is subadditive in the ideal situation where the all losses are ellipticaly distributed.

2.4 Diversification effect

The aggregation of capital leads to the diversification effect. Consider two business units with risks X_1 and X_2. Next, we assume that the total balance sheet stand-alone capital for each unit is computed using the risk measure $\rho(X)$. If each of the units is not responsible for shortfall of the other one, the total balance sheet capital for each portfolio is given by $\rho(X_j), j = 1, 2$. If they are considered on aggregate basis the purpose is to avoid the shortfall of the aggregate risks $X_1 + X_2$.

As consistent with [7] the following inequality holds:

$$(X_1 + X_2 + \rho(X_1) + \rho(X_2))_+ \leq (X_1 - \rho(X_1))_+ + (X_2 - \rho(X_2))_+.$$

It means that the shortfall of aggregated business units is always smaller than the sum of shortfalls of the stand-alone business units. The explanation is that the shortfall of one business unit can be compensated by the gain of the other one. This also implies

$$\rho(X_1 + X_2) \leq \rho(X_1) + \rho(X_2).$$

Because taking expectations is a linear operation, for the total economic capital we get

$$EC(X_1 + X_2) \leq EC(X_1) + EC(X_2).$$

The diversification gain represents the percentage of the economic capital which an insurance company can save by the positive aggregation of more risks.

Definition 2.4.1. The diversification gain for a portfolio S aggregating the risks $X_1, X_2, \ldots, X_n$ is given by:

$$D_\rho = 100\% - \frac{EC(S)}{\sum_{i=1}^n EC(X_i)}.$$

2.5 Correlation matrix

In this approach at first we compute the required economic capital for each business unit of the company. The next and important step is aggregation of these capitals into a total capital amount. Let $X_1, X_2, \ldots X_n$ be the individual losses of the business units and $S = \sum_{i=1}^n X_i$ the total loss of the company. Further, we denote $R = \{r_{ij}\}_{i,j=1}^n$ the correlation matrix between losses X_i, where the correlation is defined as

$$r_{ij} = r(X_i, X_j) = \frac{cov(X_i, X_j)}{\sqrt{\sigma^2(X_i)\sigma^2(X_j)}}.$$

If the $EC(X_1), EC(X_2), \ldots, EC(X_n)$ are economic capitals computed by the formula given in 2.3.1, the total capital which is needed as the protection against bankrupcy according to [13] is

$$EC(S) = \sqrt{\sum_{i,j} r_{ij} EC(X_i) EC(X_j)}. \tag{2.1}$$

For instance, the diversification effect for the two risks X_1, X_2 looks like

$$EC(X_1) + EC(X_2) = \sqrt{EC(X_1)^2 + 2r_{ij} EC(X_1) EC(X_2) + EC(X_2)^2}.$$

As a special case where the subrisks are comonotonic, i.e., the extreme events happen at the same time, the aggregate capital is simply the sum of the individual capitals

$$EC(S) = \sum_{i=1}^{n} EC(X_i).$$

3. Copula approach for computing economic capital

3.1 Introduction

Copulas became more considerable in recent years, especially in finance and statistics. Risk professionals pay more attention to the choice of copulas in risk management.
Copulas were originally introduced as mathematical functions as a useful tool to model dependence. The term is derived from the latin word “copula”, contraction of *co-apula*, meaning connection, bond, tie (co means together and apere means to join).

In this chapter we will explain the role of copulas in economic capital calculations. There is an interesting comparison of correlation coefficient and copulas in [4].

3.2 Bottom up and top down approach

Several approaches can be used to risk capital aggregation. Most of them belong to the class of bottom-up aggregation methods and only few of them use the top-down approach.
Bottom-up approach
In the bottom-up aggregation approaches, one develops marginal models for the loss distribution of each business unit independently. These marginal distributions are merged to a joint distribution using correlation structure or a copula function. The dependence between business lines is modelled indirectly (on the base of historical data or expert evaluation). The simultaneous distribution of the risks is defined by the marginal loss distributions and a correlation or copula structure.
Top-down approach
Top down approaches, by contrast, do not try to identify single risks but rather start from aggregated data. The empirical panel of data allows to estimate the joint distribution of the total risk. Consequently, single losses are not required in this approach.

In both approaches a common time horizon for the parameter estimation has to be determined. In the perfect case the time horizon corresponds to the internal capital allocation cycle which is usually one year. The task of estimating joint distributions may be decomposed into two parts

1. estimation of the marginal distributions

2. estimation of the dependence structure.

Copulas may be thought of as a more flexible version of correlation matrices.

3.3 The most used copulas in finance

As copulas join risks together, they tell us how risk Y behaves if we know risk X. We can find it for all realizations of X, regardless of whether is X small, medium or large. Here are some examples:

- An independent copula means that the realizations for Y occurs independently of what happens with X.
- A comonotonic copula means the full positive dependence between X and Y, i.e., knowing X implies knowing Y.
- A Gaussian copula means that there is a linear dependence between X and Y after transformation.

In insurance and finance two families of copulas are taken into account: Archimedean and elliptical. Within Archimedean copulas the Gaussian and the Student t copula are considered and among the elliptical family we deal with the Clayton and Gumbel copula.

The Gaussian copula
The Gaussian copula is the most popular copula used in applications. It is implied by a multivariate Gaussian distribution (normal distribution). A multivariate Gaussian distribution is a set of normally distributed marginal distributions that are combined by a Gaussian copula. The Gaussian copula is often used by insurance companies to derive aggregate risk distributions without consideration of the impact when marginal risk distributions are no longer normal. If other than normal marginal distributions are combined by a Gaussian copula, the resulting joint distribution is referred to as a *meta-Gaussian distribution.* We shall use the Gaussian copula as a benchmark to which we compare the other copulas.

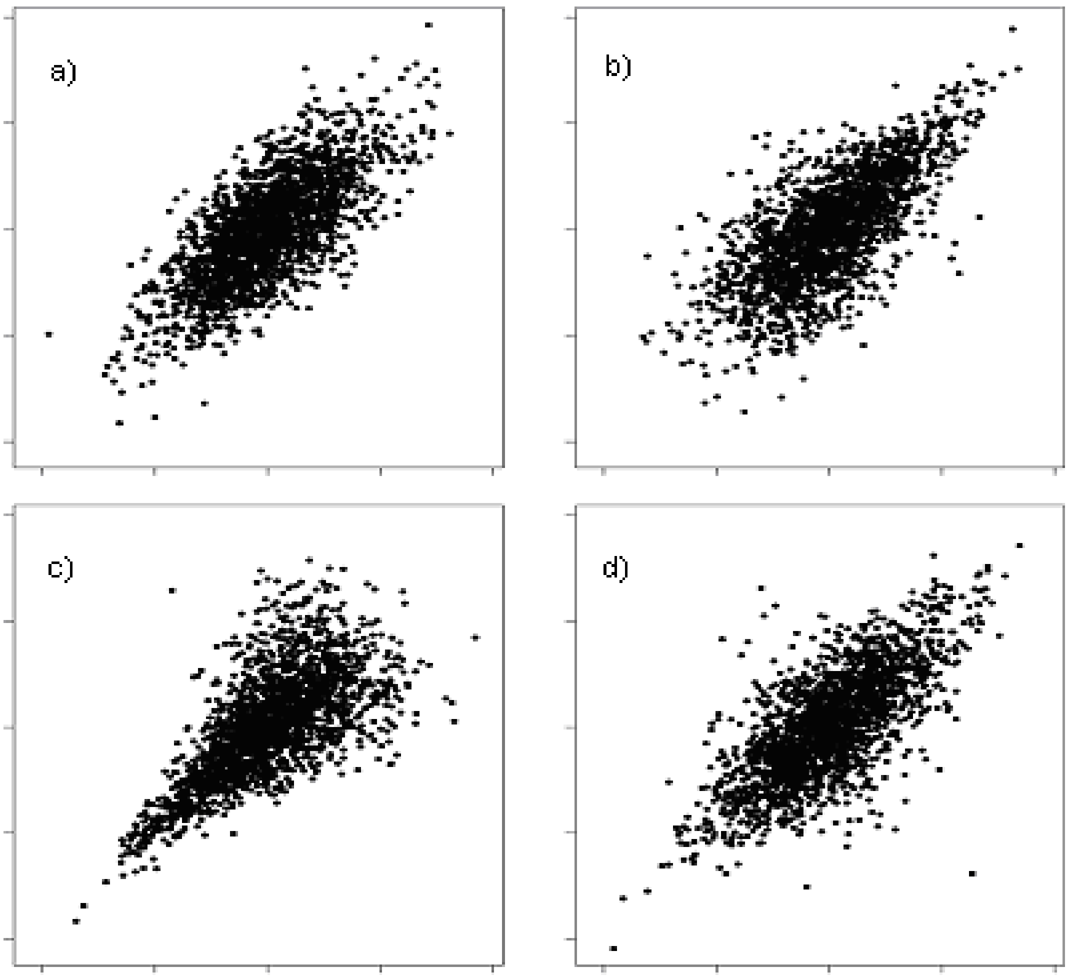
The Student t copula
The Student t copula is the copula that is implied by a multivariate Student t distribution. In the bivariate case, the Student t copula has the parameter ρ like the Gaussian copula. Additionally it has the (scalar) parameter ν which represents the degrees of freedom. With the increasing ν also increases the positive tail dependence. As the degrees of freedom of a Student t copula increase, the copula approaches a Gaussian copula, so the Gaussian copula can be regarded as a limiting case of the Student t copula, where $\nu \to \infty$.

The Gumbel copula
Contrary to the Normal and Student copulas, it is not derived from a known multivariate distribution, but it is part of the Archimedian copulas. The Gumbel copula is different from the elliptical described copulas. It can model only independence or positive dependence structures and it depends on a single parameter. The main interest for using a Gumbel copula is that it confronts the solvency of a company to unfavourable scenarios (stress scenarios), i.e., where major events tend to be linked, while the most common claims remain independent.

The Clayton copula
The Clayton copula displays lower tail dependence and zero upper tail dependence. The Clayton copula assigns a higher probability to joint extremely negative realisations as compared to the Gaussian copula, while it assigns a lower probability to joint extremely positive realisations.

Differences between copulas
Some copulas allow to model both positive and negative dependence in their standard versions by assigning appropriate copula parameters. To these copulas belong the Gaussian and the Student t copula. The Student t copula assigns a higher probability to joint extreme events than the Gaussian copula. The Student t copula displays ***symmetric tail dependence***.

Asymmetric tail dependence is prevalent if the probability of joint extreme negative realisations differs from that of joint extreme positive realisations. Further, the Clayton copula assigns a higher probability to joint extreme negative events than to joint extreme positive events. The Clayton copula is said to display ***lower tail dependence***, while it displays zero ***upper tail dependence***. The converse can be said about the Gumbel copula (displaying upper but zero lower tail dependence).
For illustration we refer to the picture 3.3 from the book [9] where the simulations of Gaussian, Clayton, Gumbel and Student t copulas are given. In the case of the Student t copula we can see the symmetric tail dependence while for the Clayton and the Gumbel copula only the lower or upper tail dependencies can be seen.



Two thousand simulated points from four distributions with standard normal margins a) Gaussian b) Gumbel c) Clayton and d) Student t.

3.4 Fitting copula to data

It is very difficult to find a good model that describes both marginal behaviour and the dependence structure of the risks effectively. This section is devoted to estimation of copula parameters from empirical data. The main method is to estimate parameters with maximum likelihood method (MLE), alternatives are the method of moments using rank correlation and the computation of the non-parametric empirical copula. Authors of [3] presented three methods and we shortly discuss.

	Gaussian	Student t	Clayton	Gumbel
ρ_S	$\frac{6}{\pi}arcsin\frac{\rho}{2}$			
ρ_t	$\frac{2}{\pi}arcsin\rho$	$\frac{2}{\pi}arcsin\rho$	$\frac{\theta}{\theta+2}$	$1-\frac{1}{\theta}$

Table 3.1: Relationships between rank correlations and copula parameters.

- **Parameter estimation using correlation measures**
  Depending on which particular copula we want to fit, it may be easier to use empirical estimates of Spearman's or Kendall's rank correlation to infer the copula parameter from these correlation measures. For some copulas a simple relationship exists between either ρ_S or ρ_t and the copula parameter, hence the copula parameter may easily be computed from the estimate of one of the two correlation measures ρ_S and ρ_t. In the Table 3.1 the relationships between Spearman's rho ρ_S and Kendall's tau ρ_t and copula parameters are given.

  The general method of computing is always similar: we look for a theoretical relationship between one of the rank correlations and the parameters of the copula and substitute empirical values of the rank correlation into this relationship to get estimates of some or all of the copula parameters.

  The advantage of this approach is that it is computationally very fast. The approach seems useful as it allows to estimate starting values for numerical parameter estimations that are based on a MLE, speeding up the copula parameter estimation.

- **Maximum likelihood method** (MLE)
  In classical statistics fitting a multivariate distribution is done by using the maximum likelihood method for a multivariate parametric family of distributions. The copula technique is different since it suggests the possibility of a two stage statistical procedure: estimate the marginal distributions and the copula function separately from each other. By the choice of the marginal distributions we determine the copula, hence the chosen dependence structure, and therefore different statistical tools for fitting the marginals may generate distinct dependencies. We mention the three approaches to fitting of the marginals:

  1. **Parametric estimation**
     We choose an appropriate parametric model for the data and fit it by MLE, i.e., we fit parametric distributions to the marginal. In insurance data it is common to consider a standard actuarial loss distribution such as Pareto or lognormal.
  2. **Non-parametric estimation with variant of empirical distribution function**

We replace the marginal distributions by their empirical distributions.

3. **Extreme value theory for the tails**
We can model the body of the distribution empirically, but for the the tails we use a semiparametrical model using a generalized Pareto distribution (GPD).

- **Empirical copulas**
Empirical copulas may be used alternatively to the parametric copulas which we presented earlier. The empirical copula asymptotically converges to the true copula for $N \to \infty$. It may be used for Monte Carlo simulations or for a visualisation of the goodness-of-fit of some parametric copula, by comparing a parameterised copula to the empirical copula.

3.5 Conclusion

Knowledge of copulas rapidly increased in the last years. They started to play an important role in insurance industry and finance as a tool for computing economic capital. They offer a flexible structure which can be used in many situations. Unfortunately, the right choice of the copula is a very difficult exercise. To the right choice of copula is for instance devoted paper [10]. The main problem is that estimating the copula requires high quality data which are very often not available.

One may think that for economic capital calculations we only need data that reveals the structure of the upper tail dependence. It is true that economic capital is most sensitive to upper tail dependence and these observations are extremely rare. In the absence of empirical data fitting a copula becomes a meaningless exercise. Typically for the aggregation across risk types (life, non-life, credit, market,...) data availability is a significant challenge. Moreover, from a computational point of view, in most cases it is an extremely complicated task.

With regards to economic capital aggregation, the challenge consists of simple but still consistent and well balanced models. This is not an easy task and requires experience and lot of training. But still, despite of these disadvantages, copulas play the unchangeable role in financial and probability theory.

4. Capital allocation to the lines of business

4.1 Introduction

The theme of this chapter is the sharing of capital between the business units of the insurance company. We call this sharing "allocation". The problem of allocation is interesting and non-trivial, because the simple sum of individual business unit capitals is usually larger than the total economic capital of the company needed. That is, there is a decline in total costs to be expected by pooling the activities of the company, and this advantage needs to be shared fairly between the constituents. There are number of reasons why companies want to allocate their total capital across the lines of business:

- There is a need to redistribute the total cost associated with holding the capital in the form of charges.
- Allocation is a necessary activity for financial reporting purposes.
- Capital allocation is a useful device for comparing performance of the business units by determining the return on allocated capital for each unit.

4.2 The allocation problem

We assume that the business lines of an insurance company face risks $X_1, X_2, \ldots, X_n$ and the total risk of the whole company is $S = \sum_{i=1}^{n} X_i$. Moreover, the aggregate level of capital $K = EC(S)$ of the insurance company has already been derived from the formula

$$K = EC(S) = \rho(S) - ES.$$

The company wishes to decompose this capital across its business units, in other words to find the nonnegative real numbers $K_1, K_2, \ldots, K_n$ such that

$$\sum_{i=1}^{n} K_i = K.$$

We rewrite it formally:

Definition 4.2.1. Denote the vector of losses by $X^T = (X_1, X_2, \ldots, X_n)$. An allocation A is the mapping

$$A : X^T \to R^n$$

such that $A(X^T) = (K_1, K_2, \ldots, K_n)^T \in R^n$ where

$$\sum_{i=1}^{n} K_i = K, \tag{4.1}$$

S is the total company loss and K is the total company capital.

4.1 is called *full allocation* requirement. If the subadditivity property of the risk capital holds, then it is obvious that

$$\sum_{i=1}^{n} [EC(X_i) - K_i] \geq 0,$$

which represents *the diversification benefit.*

Definition 4.2.2. For a company with n business units and corresponding risks $X^T = (X_1, X_2, \ldots, X_n)$, the $i - th$ business unit's diversification benefit is given by

$$\delta_i = EC(X_i) - K_i$$

for $i = 1, \ldots, n$.

4.3 Proportional allocation principles

Using the proportional allocation approach, every business line gets the same ratio as reduction because of the group diversification. The particular allocated capitals are obtained by first choosing a risk measure ρ and then attributing the capital $K_i = \alpha\rho[X_i]$ to each unit $i, i = 1, 2, \ldots, n$. The factor α is chosen such that the full allocation requirement is satisfied. This leads to the proportional allocation principle

$$K_i = \frac{K}{\sum_{j=1}^{n} \rho[X_j]} \rho[X_i], i = 1, 2, \ldots n.$$

We will discuss more closely four proportional allocation principles given in the next table. The foundation for these principles is in the paper [6].

Haircut allocation	$\rho[X_i] = F_{X_i}^{-1}(p)$
Quantile allocation	$\rho[X_i] = F_{X_i}^{-1}(F_{S^c}(K))$
Covariance allocation	$\rho[X_i] = Cov[X_i, S]$
CTE allocation	$\rho[X_i] = E[X_i \mid S > F_S^{-1}(p)]$

If the risk measure is law-invariant, the proportional allocation is not influenced by dependencies between the risks X_i.

4.3.1 The haircut allocation principle

The haircut allocation principle is based on allocating the capital $K_i, i = 1, 2, ..., n$ to business unit i, where

$$K_i = \gamma F_{X_i}^{-1}(p).$$

The value of γ is chosen such that the full allocation requirement is satisfied. It leads to the formula of the capital allocation:

$$K_i = \frac{K}{\sum_{j=1}^{n} VaR_p(X_j)} VaR_p(X_i), i = 1, \ldots, n.$$

It is evident that this principle does not make allowance for a dependence structure between the losses X_i of the individual business units. Furthermore, if we use VaR as a risk measure for computing risk capital $\rho(S) = VaR(S)$, we obtain

$$K_i = \frac{VaR_p(S) - ES}{\sum_{j=1}^{n} VaR_p(X_j)} VaR_p(X_i), i = 1, 2, \ldots, n.$$

Because VaR is not a subadditive risk measure, it may happen that the allocated amount of capital K_i exceeds the respective stand-alone capitals $VaR(X_i)$.

4.3.2 The quantile allocation principle

Before we introduce this principle, we give some definitions which we will need.

Definition 4.3.1. The α-mixed inverse distribution function $F_X^{-1(\alpha)}$ of X is defined:

$$F_X^{-1(\alpha)(p)} = \alpha F_X^{-1}(p) + (1 - \alpha) F_X^{-1+}(p)$$

where $F_X^{-1+}(p) = \sup\{x \in R | F_X(x) \leq p\}$, $p \in (0, 1), \alpha \in [0, 1]$.

Definition 4.3.2. The comonotonic sum S^c is defined as

$$S^c = \sum_{i=1}^{n} F_{X_i}^{-1}(U),$$

where U is a uniform random variable on $(0, 1)$.

Now we will consider the approach where we adopt the probability level among the business lines and determine an α-mixed inverse with $\alpha \in [0, 1]$. Again, the full allocation requirement has to be satisfied. This gives rise to the *quantile allocation principle*:

$$K_i = F_{X_i}^{-1(\alpha)}(\beta p),$$

with α and β chosen such that $\sum_{i=1}^{n} K_i = K$. The allocated capitals $K_i, i = 1, 2, \ldots, n$ do not make allowance for a dependence structure between the different risks $X_1, X_2, \ldots, X_n$. The appropriate levels of α and β are derived as a solution from

$$K = \sum_{i=1}^{n} F_{X_i}^{-1(\alpha)}(\beta p).$$

We can adopt this formula as

$$K = F_{S^c}^{-1(\alpha)}(\beta p),$$

$$\beta p = F_{S^c}(K).$$

It leads to

$$K = F_{S^c}^{-1(\alpha)}(F_{S^c}(K)).$$

The quantile allocation principle can be rewritten as

$$K_i = F_{X_i}^{-1(\alpha)}(F_{S^c}(K)), i = 1, 2, \ldots, n.$$

Note that the quantile allocation principle can be considered as a special case of the haircut allocation principle where $p = F_S^c(K)$.

4.3.3 The variance-covariance allocation principle

This ad hoc approach is widely used in the insurance industry. Unlike the previous two criteria, the variance-covariance principle takes into account the dependence structure between losses. This principle is given by

$$K_i = \frac{K}{Var[S]} Cov[X_i, S], i = 1, 2, \ldots, n.$$

$Var[S]$ is the variance of the aggregate loss and $Cov[X_i, S]$ covariance between the individual loss X_i and aggregate loss S. The lines of business facing a loss that is more correlated with the total loss S are required to hold a larger amount of capital than the less correlated ones.

4.3.4 The CTE allocation principle

As we said earlier, the Conditional Tail Expectation (CTE) defined as

$$CTE_p(S) = E[S|S > F_S^{-1}(p)],$$

where S is the total loss and $p \in (0, 1)$ given probability level, is the coherent risk measure. We define the *CTE allocation principle*:

$$K_i = \frac{K}{CTE_p[S]} E[X_i|S > F_S^{-1}(p)], i = 1, 2, \ldots, n.$$

This allocation rule also takes into account the dependencies between risks.

4.4 Euler allocation principle

The Euler allocation principle, also known as the gradient allocation principle, is an old allocation method known from game theory as the Aumann-Shapley value. Euler capital allocation considers the impact of changes of positions on the necessary risk capital. This principle is based on fairness- it means that each business unit profits from the diversification benefit. Moreover, by [18] Euler allocation principle is the only per-unit capital allocation principle suitable for performance measurement.

4.4.1 Risk contributions

We are interested in how much business unit i contributes to EC. We define the risk contribution of X_i to $\rho(S)$ by $\rho(X_i|S)$.

Definition 4.4.1. Let $\mu_i = E[X_i]$. Then
the total portfolio Return on Risk Adjusted Capital is defined by

$$RORAC(S) = \frac{E[S]}{\rho(S)} = \frac{\sum_{i=1}^{n} \mu_i}{\rho(S)},$$

the portfolio RORAC of the i-th asset is defined by

$$RORAC(X_i|S) = \frac{E[X_i]}{\rho(X_i|S)} = \frac{\mu_i}{\rho(X_i|S)}.$$

From the economic point of view, the next two properties of risk contributions are needed.

Definition 4.4.2. Let S denote total risk of the company. Then:

- Risk contributions $\rho(X_1|S), \ldots, \rho(X_n|S)$ to company risk $\rho(S)$ satisfy the full allocation property if

$$\sum_{i=1}^{n} \rho(X_i|S) = \rho(S).$$

- Risk contributions $\rho(X_i|S)$ are RORAC compatible if there are some $\epsilon_i > 0$ such that

$$RORAC(X_i|S) > RORAC(S) \Rightarrow RORAC(S + hX_i) > RORAC(S)$$

for all $0 < h < \epsilon_i$.

4.4.2 Euler allocation

First we mention the notion of homogeneous risk measures and functions and Euler's theorem, which are essential for the Euler allocation principle.

Definition 4.4.3. A risk measure ρ is called homogeneous of degree τ if for all $h > 0$ the following equation holds:

$$\rho(hX) = h^\tau \rho(X).$$

A function $f : U \subset R^n \to R$ is called homogeneous of degree τ if for all $h > 0$ and $u \in U$ with $hu \in U$ the following equation holds:

$$f(hu) = h^\tau f(u).$$

Theorem 4.4.1 (Euler's theorem). Let $U \subset R^n$ be an open set and $f : U \to R$ be a continuously differentiable function. Then f is homogeneous of degree τ if and only if it satisfies the following equation:

$$\tau f(u) = \sum_{i=1}^{n} u_i \frac{\partial f(u)}{\partial u_i}, u = (u_1, \dots, u_n) \in U.$$

Dirk Tasche in [15] shows that for a "smooth" function the only vector field which is suitable for performance measurement with the function is the gradient of the function.

Definition 4.4.4. Let ρ be a risk measure and f_ρ the function defined by $f_{\rho,S} = \rho(S)$. Assume that f_ρ is continuously differentiable. If there are risk contributions $\rho(X_1|S), \dots, \rho(X_n|S)$ that are RORAC compatible, then $\rho(X_i|S)$ is uniquely determined as

$$\rho_{Euler}(X_i|S) = \frac{d\rho}{dh}(S + hX_i)|_{h=0}. \tag{4.2}$$

If ρ is a risk measure which is homogeneous of degree 1, then the risk contributions according to 4.2 are called the Euler contributions. Euler contributions satisfy both properties of 4.4.2, i.e., they are RORAC compatible and satisfy the full allocation rule. The process of assigning capital to business units by calculating Euler contributions is called the Euler allocation.
The RORAC of the risk represents the ratio between the expected profit and the economic capital contribution necessary to run the risk. According to Euler's priciple it is guaranteed that if the RORAC of risk X_i is higher than the RORAC of total risk S containing the risk, increasing the weight of X_i will improve the RORAC of the whole portfolio.

4.4.3 Euler VaR-contributions

VaR is a risk measure that is homogeneous of degree 1, but not subadditive in general. With reference to [17], under some smoothness conditions which imply that S has a density a general formula for Euler VaR-contributions can be derived:

$$VaR(X_i|S) = \frac{dVaR(S + hX_i)}{dh}|_{h=0} = E[X_i|S = VaR_\alpha(S)].$$

In general, no closed-form representations of $VaR_\alpha(S)$ and the risk contributions $VaR_\alpha(X_i|S)$ are available. These values can often be inferred from Monte-Carlo samples. This means essentially to generate a sample $(S^{(t)}, {X_1}^{(t)}, {X_2}^{(t)}, \dots, {X_n}^{(t)})$, $t = 1, 2, \dots, T$ and then to estimate the quantities. How to generate values for VaR is quite obvious, but not for the risk contributions $VaR_\alpha(X_i|S)$ as estimating derivatives of stochastic quantities without closed-form representation is less clear. If $P[S = VaR_\alpha(S)]$ is positive, the conditional expectation is given by

$$E[X_i|S = VaR_\alpha(S)] = \frac{E_\alpha[X_i \mathbf{1}_{\{S=VaR_\alpha(S)\}}]}{P[S = VaR_\alpha(S)]}.$$

For $P[S = VaR_\alpha(S)]$ positive the magnitude will usually be very small, such as $1 - \alpha$ or less.
The effect of diversification in the case of VaR contributions is as follows:

Definition 4.4.5. Let $X_1, X_2, \dots, X_n$ be the loss variables and $S = \sum_{i=1}^n X_i$. Then

$$DI_\alpha(S) = \frac{VaR_\alpha(S) - ES}{\sum_{i=1}^n VaR_\alpha(X_i) - ES}$$

denotes the diversification index of risk S with respect to economic capital based on VaR_α. Next,

$$DI_\alpha(X_i|S) = \frac{VaR_\alpha(X_i|S) - EX_i}{VaR_\alpha(X_i) - EX_i}$$

denotes the marginal diversification index of business unit X_i with respect to economic capital based on VaR_α. DI assuming a value close to 1 indicates that there is no significant diversification in the portfolio.

4.5 Marginal allocation principle

Marginal risk contributions to the economic capital of a company are differences of the total capital amount of the company with business unit i and total capital without business unit i.

Definition 4.5.1. Marginal risk contribution of business unit $i, i = 1, 2, \dots, n$ is defined by

$$\rho_{marg}(X_i|S) = \rho(S) - \rho(S - X_i).$$

If the used risk measure is subadditive, continuously differentiable and homogeneous of degree 1, marginal risk contributions are always smaller than the corresponding Euler contributions ([16], Proposition 2.2).

Proposition 2. Let ρ be a subadditive and continuously differentiable risk measure that is homogenous of degree 1. Then the marginal risk contributions $\rho_{marg}(X_i|S)$ defined by 4.5.1 are smaller than the corresponding Euler contributions, i.e.,

$$\rho_{marg}(X_i|S) \leq \rho_{Euler}(X_i|S).$$

In particular, the sum of the marginal risk contributions underestimates total risk:

$$\sum_{i=1}^{n} \rho_{marg}(X_i|S) = \sum_{i=1}^{n} (\rho(S) - \rho(S - X_i)) \leq \rho(S).$$

The main disadvantage of this principle is that the full allocation property is not satisfied. Therefore sometimes marginal risk contributions are defined as

$$\rho^*_{marg}(X_i|S) = \frac{\rho_{marg}(X_i|S)}{\sum_{j=1}^{n} \rho_{marg}(X_j|S)} \rho(S). \tag{4.3}$$

Altough now the full allocation property is satisfied, the marginal risk contribution defined by 4.3 is not RORAC compatible.

4.6 Shapley allocation principle

Game theory provides an excellent framework for allocating capital. This approach is axiomatic; it means that we define the set of axioms which we need to be fullfilled by a fair capital allocation principle. We will consider the *coherent risk measure* and the *coherent allocation principle* only. The properties that define coherent risk measure are introduced in Chapter 1. In this section we assume all risk measures are coherent and follow the paper [5].

4.6.1 Coherence of allocation principle

We suggest a set of axioms which are necessary properties of *reasonable* allocation property. The following notation is used:

- $X_i, i = 1, 2, \ldots, n$ are the risks of business units of the company
- S represents the total loss of the company, $S = \sum_{i=1}^{n} X_i$
- N is a set of all business units of the company
- A is a set of economic capital allocation problems

- pairs (N, ρ) consist of a set of n portfolios (business units) and a coherent risk measure ρ
- $K = \rho(S) = ES$ is economic capital of the company; because of the linearity of expected value ES we will work with the risk capital only and it will be denoted $\tilde{K} = \rho(S)$

Now we can define the **coherent allocation principle**:

Definition 4.6.1. An allocation principle is a function $\Pi : A \to R^n$ that maps each allocation problem (N, ρ) into a unique allocation:

$$\Pi : (N, \rho) \longrightarrow \begin{bmatrix} \Pi_1(N,\rho) \\ \Pi_2(N,\rho) \\ \vdots \\ \Pi_n(N,\rho) \end{bmatrix} = \begin{bmatrix} \tilde{K}_1 \\ \tilde{K}_2 \\ \vdots \\ \tilde{K}_n \end{bmatrix}$$

such that $\sum_{i \in N} \tilde{K}_i = \rho(S)$.

Again the condition of full allocation has to be satisfied.

Definition 4.6.2. An allocation principle Π is coherent if for every allocation problem (N, ρ), the allocation $\Pi(N, \rho)$ satisfies the three properties:

1. **No undercut**
$$\forall M \subseteq N, \sum_{i \in M} \tilde{K}_i \leq \rho(\sum_{i \in M} X_i)$$
2. **Symmetry** If by joining any subset $M \subseteq N \backslash \{i, j\}$, units i and j both make the same contribution to the risk capital, then $\tilde{K}_i = \tilde{K}_j$.
3. **Riskless allocation**
$$\tilde{K}_n = \rho(\alpha r_f) = -\alpha.$$
Recall the n^{th} unit is riskless.

The allocation principle is nonnegative if $\tilde{K}_i \geq 0, i \in N$. The three axioms in the previous definition are necessary conditions of the fairness and credibility of allocation principle.

4.6.2 Allocation to atomic players

Game theory is the study where players use different strategies to achieve their goals. We focus on coalition games and players who are atomic, meaning that fractions of players are not allowed.

Definition 4.6.3. A coalition game (N, c) consists of

- a finite set N of n players
- a cost function c that associates a real number $c(U)$ to each subset U of N (called a coalition)

The goal of each player is to minimize the costs, and strategy consists of taking part in a coalition or not. In coalition games, the main question is how to allocate the cost $c(N)$ between the players.

Definition 4.6.4. A value is a function $\Phi : G \to R^n$ that maps each game (N, c) into a unique **allocation**:

$$\Phi : (N,c) \longrightarrow \begin{bmatrix} \Phi_1(N,c) \\ \Phi_2(N,c) \\ \vdots \\ \Phi_n(N,c) \end{bmatrix} = \begin{bmatrix} \tilde{K}_1 \\ \tilde{K}_2 \\ \vdots \\ \tilde{K}_n \end{bmatrix}$$

where $\sum_{i\in N} \tilde{K}_i = c(N)$ and G is a set of games with n players.

Because c is usually assumed to be subadditive, players form the largest coalition N since it improves the total cost. The problem is only to find a way of allocating cost $c(N)$ with minimizing cost of each player. If the $\tilde{K}_i$ of player i is higher than $c(i)$, there is a threat that this player leaves the coalition. To avoid this situation, we give a set of allocations that do not allow threat called *the core.*

Definition 4.6.5. The core of a coalition game (N, c) is the set of allocations $\tilde{K} \in R^n$ for which $\sum_{i\in U} \tilde{K}_i \leq c(U)$ for all coalitions $U \subseteq N$.

Next we introduce the notion of a balanced game and important conditions for the core to be nonempty:
Let C be the set of all coalitions of N and denote $1_U \in R^n$ the characteristic vector of the coalition U: *A balanced collection of weights* is a collection of $|C|$ numbers $\lambda_U in[0, 1]$ such that $\sum_{U\in C} \lambda_U 1_U = 1_N$.
A game is balanced if $\sum_{U\in C} \lambda_U c(U) \geq c(N)$ for all balanced collections of weights.

Theorem 4.6.1 (**Bondareva-Shapley theorem**)**.** A coalition game has a nonempty core if and only if it is balanced.

4.6.3 The Shapley value

We use the notation $\Delta_i(U) = c(U \cup i) - c(U)$ for any set $U \subset N, i \notin U$. Two players i and j are *interchangeable* in (N, c) if $\Delta_i(U) = \Delta_j(U)$ for each $U \subset N$ and $i, j \notin U$. A player is a *dummy* if $\Delta_i(U) = c(i)$.

Definition 4.6.6. We define three properties:

- **Symmetry** If players i and j are interchangeable then $\Phi(N, c)_i = \Phi(N, c)_j$

- **Dummy player** For a dummy player, $\Phi(N,c)_i = c(i)$
- **Additivity over games** For two games (N, c_1) and (N, c_2)

$$\Phi(N, c_1 + c_2) = \Phi(N, c_1) + \Phi(N, c_2),$$

where $(N, c_1 + c_2)$ is defined by $(c_1 + c_2)(U) = c_1(U) + c_2(U)$ for all $U \subseteq N$.

The Shapley value is the only value that satisfies the properties of symmetry, dummy player and additivity over games.

At last we give the algebraic definition of the Shapley value which provides an explicit computational approach.

Definition 4.6.7. The Shapley value K^{Uh} for the game (N, c) is defined as:

$$K_i^{Uh} = \sum_{U \in C_i} \frac{(u-1)!(n-u)!}{n!} \left(c(U) - c(U \backslash \{i\})\right), i \in N,$$

where $u = |U|$ and C_i represents all coalitions of N that contain i. We can notice that this computation may be very long because the evaluation of c for each of the 2^n possible coalitions is required.

4.6.4 Economic capital allocation and game theory

Finally, we introduce the concept of capital allocation as coalition games. We will associate business units of the company with the atomic players of a game, risk measure ρ with the cost function c

$$c(U) = \rho(\sum_{i \in U} X_i), U \subseteq N$$

and allocation principles became values. If ρ is coherent and thus subadditive, it implies c is subadditive in the sense $c(U \cup V) \leq c(U) + c(V)$. The nonemptiness of the core is a crucial condition for existence of the coherent allocation principle:

Theorem 4.6.2. If an economic capital allocation problem is modelled as a coalitional game whose cost function is defined with a coherent risk measure ρ, then its core is nonempty.

When we model the allocation problem by means of game theory, the Shapley value yields to the economic capital allocation principle. It is coherent but only for the *no undercut axiom.* It satisfies also symmetry by the definition and the riskless axiom is implied by the dummy player axiom.
The Shapley value is the coherent allocation principle, if it maps games to elements of the core. It holds if conditions of one of the following theorems are satisfied:

Theorem 4.6.3. If a game (N, c) is strongly subadditive, its core contains the Shapley value. (The game is strongly subadditive if it is based on a strongly subadditive cost function)

Theorem 4.6.4. If for all coalitions $U, |U| \geq 2$,

$$\sum_{T \subseteq U} (-1)^{|U|-|T|} c(T) \leq 0$$

then the core contains the Shapley value.

In the case of 4.6.3, the strong subadditivity of c implies that ρ is linear. This result is difficult to accept because it eliminates the diversification effect. If we consider the conditions of 4.6.4, it is in no way implied by the coherence of the risk measure ρ.
We can see that we do not have the convincing proof of the existence of coherent allocations. Although, if we consider the case of non-atomic players, it means that fractions of players are allowed, we get much stronger existence results. This model is called the Aumann-Shapley or the Euler allocation principle which is discussed in 4.4.

5. Numerical examples

5.1 Introduction

In the previous chapter we introduced the well known principles of capital allocation. Now we will give the practical examples of these approaches and their impact on amounts of allocated capital.
Typical distributions used for modelling insurance risks are:

- the lognormal distribution
- the Pareto distribution
- the Gamma distribution
- the Weibull distribution.

To demonstrate these principles we will consider four dependent risks corresponding to particular business units of insurance company. Dependence between them is modelled by correlation matrix

$$\mathbf{R} = \begin{pmatrix} 1 & 0.5 & 0.25 & 0.75 \\ 0.5 & 1 & 0.5 & 0.25 \\ 0.25 & 0.5 & 1 & 0.25 \\ 0.75 & 0.25 & 0.25 & 1 \end{pmatrix}$$

which will be common for all examples. First, the economic capitals $K_i, i = 1, \ldots, 4$ are computed. Then the aggregate capital is derived and allocated to the business units. The allocation principles which we are going to analyse are the haircut allocation principle, the variance-covariance allocation principle, the marginal principle and the Euler principle. The reason why we decided specially for these principles is that we want to compare different approaches to capital allocation. For instance both the haircut and variance-covariance allocation principle belong to the proportional approach while the variance-covariance allocation takes into account dependencies between risks, the haircut allocation is based on quantiles only. Further, in the case of the Euler principle we look into the fact how the small change of the subrisk influences the whole portfolio and the marginal principle calculates differences between portfolio with and without the individual risks. We give three examples of economic capital allocation. Each of them should clarify the properties of these principles from different points of view. For computing the capitals we used the computer program Wolfram Mathematica. The source code can be found in the appendix.

5.2 Example 1

In this example we consider four risks coming from the probability distributions suitable for modelling insurance risks. These distributions are placed in the table 5.1 together with their mean values and variances.

Computation of individual economic capitals which are given by formula 2.3.1

Risk	Distribution	Mean	Variance
1	Weibull(2.2,121)	107.161	2643
2	Lognormal(4.86,0.41)	140.337	3605.19
3	Pareto(88,2.17)	163.214	72 211.2
4	Gamma(15.3,13)	198.9	2585.7

Table 5.1: Distributions of risks

which we realize by means of the computer program Wolfram Mathematica. As a risk measure we use the Value-at-Risk at confidence level $\alpha = 0.05$ which is the common choice in insurance industry. Then the aggregated capital ES is derived from 2.1 and we get the following results:

$EC_1 = 75.7953$

$EC_2 = 74.6039$

$EC_3 = 73.1088$

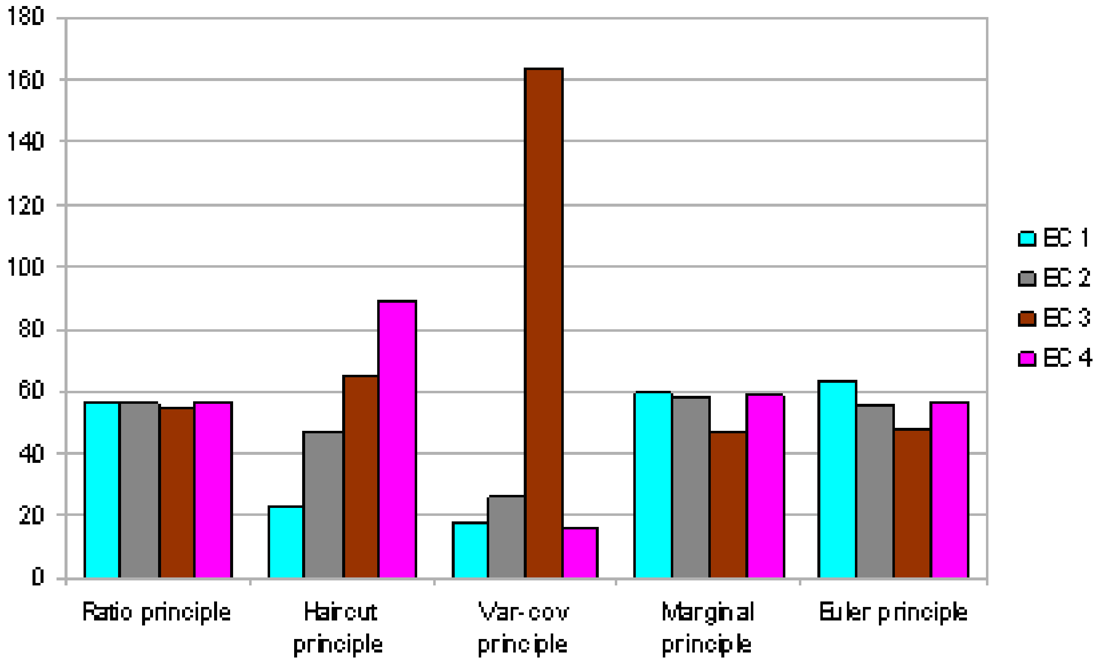
$EC_4 = 75.6276$

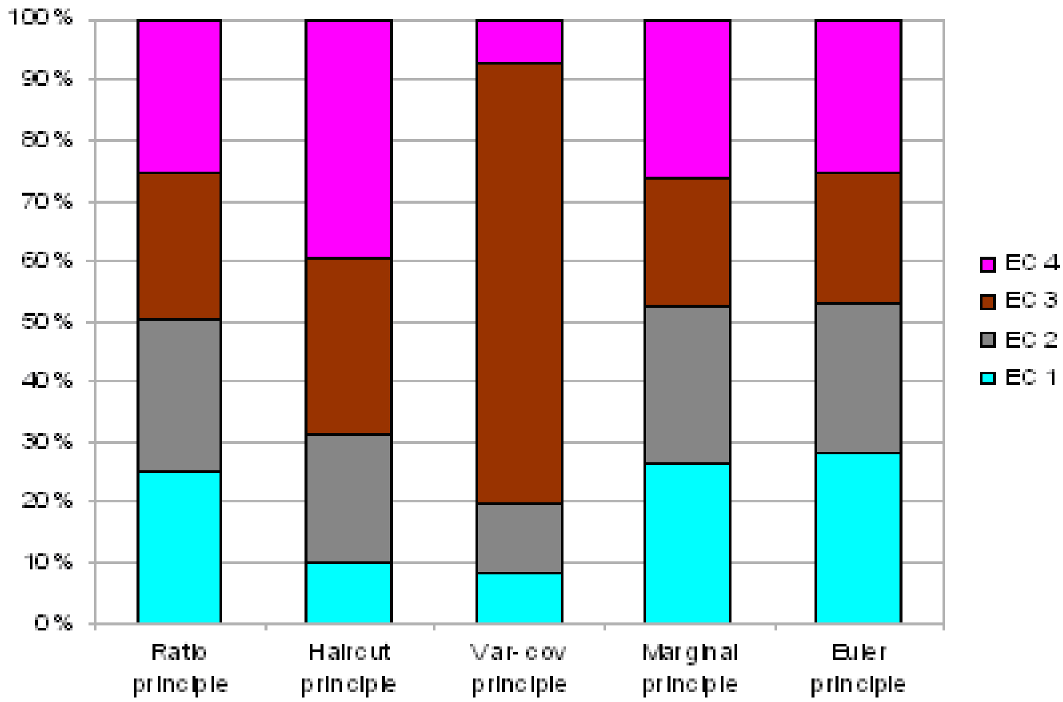
$ES = 224.585$

We can see that all capitals have the similar values. They were chosen purposely because someone could expect that the allocated capitals would have the similar values too. However, we want to find the main aspects which influence the amount of allocated economic capitals for every principle. In Table 5.2 are discussed the consequential allocated capitals by using different allocation principles. The ratio principle we add only for the comparison with capitals allocated by a simple ratio. The graphical representation can be found in plot 5.2 and 5.2, in the former are diagrammized values and in the latter percentual contribution to the total capital.

	Ratio principle	Haircut principle	Var-cov principle	Marginal principle	Euler principle
EC_1	56,9056	22,6883	18,0538	59,5296	63,3692
EC_2	56,0111	47,5489	26,2822	58,4727	55,8008
EC_3	54,8887	65,178	163,844	47,3952	48,3685
EC_4	56,7797	89,17	16,4054	59,1876	57,0466

Table 5.2: Economic capitals allocated to the business lines using of several allocation principles


180
160
140
120
100
80
60
40
20
0
Ratio principle
Haircut principle
Var-cov principle
Marginal principle
Euler principle
EC 1
EC 2
EC 3
EC 4


100 %
90 %
80 %
70 %
60 %
50 %
40 %
30 %
20 %
10 %
0 %
Ratio principle
Haircut principle
Var-cov principle
Marginal principle
Euler principle
EC 4
EC 3
EC 2
EC 1

The marginal and Euler allocation principle give us very similar results. As a quite surprising and remarkable conclusion we can consider the capitals which arise from the variance-covariance principle. It is obvious that this principle is not very consistent with the others. In this case the total economic capital consists almost exclusively of EC_3, the rest of capitals are represented by a minimal amount. The explanation can be that there is a strong influence of variance of the corresponding distribution. We notice that the variance of the Pareto distribution is much greater than the left over so this capital will be dominant. Regarding the haircut principle we consider 99.5% quantile of the distribution. Hence, the result depends on how heavy is the tail of the distribution, i.e., how high is the probability of extreme values. All the distributions we use to model insurance risks are heavy-tailed but they differ in the length of the tails.

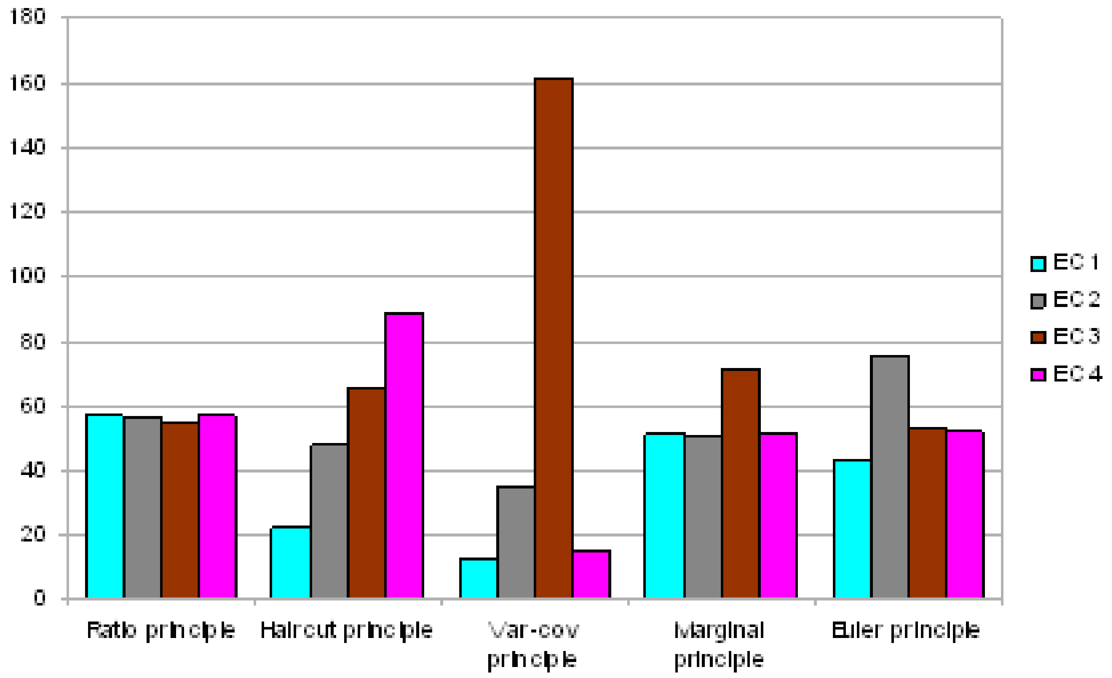
Now we will investigate how the capitals change in case of changing the correlation matrix which represents the dependence structure between risks. The distributions of losses stay the same, we only use a different correlation matrix

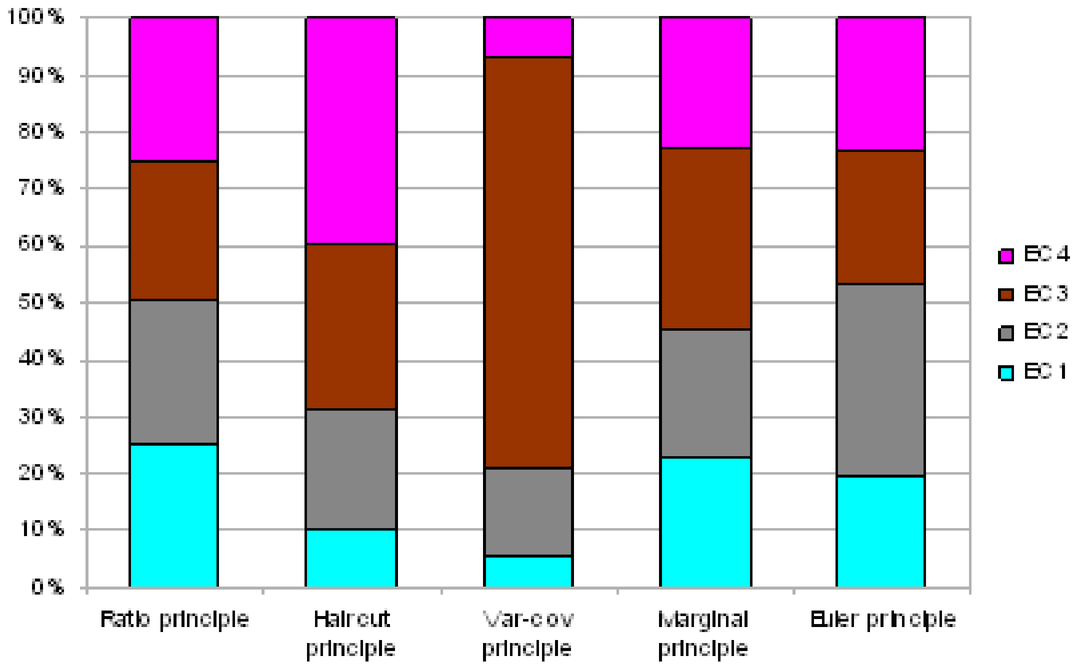
$$\mathbf{Q} = \begin{pmatrix} 1 & 0.5 & 0.2 & 0 \\ 0.5 & 1 & 0.75 & 0.8 \\ 0.2 & 0.75 & 1 & 0.25 \\ 0 & 0.8 & 0.25 & 1 \end{pmatrix}$$

We get the following results:

	Ratio principle	Haircut principle	Var-cov principle	Marginal principle	Euler principle
EC_1	56,7819	22,639	12,4041	51,4195	43,4288
EC_2	55,8894	47,4455	35,1414	50,4314	75,4844
EC_3	54,7693	65,0363	161,483	71,0663	53,2531
EC_4	56,6563	88,9761	15,0689	51,1798	51,9306

Table 5.3: Economic capitals allocated to the business lines using of several allocation principles





In the haircut and variance-covariance principle we observe almost no modifications. There is only small increase in amount of EC_2. The reason is that the second capital is now most dependent on the others so there is a need of increase. It is not so significant because these two principles are not so influenced by the dependence structure between risks, unlike the marginal and the Euler principle. We can notice also an increase of the second capital in the Euler principle which is not so small. As we said, the Euler principle is strongly influenced by the dependence structure between risks- it measures the contribution of each risk to the total capital amount.

5.3 Example 2

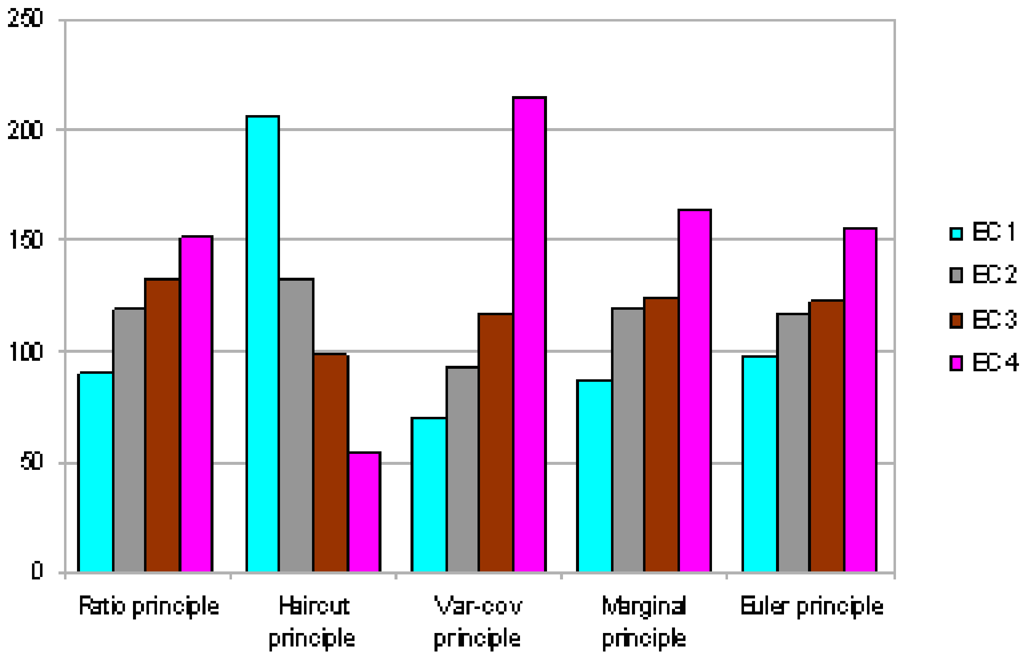
As was mentioned, the lognormal distribution is very popular for modelling insurance risks. In this example we are interested in four lognormal-distributed risks but each of them with different parameters. They are given in Table 5.3. Notice that we chose distributions similar in means but different in variances.

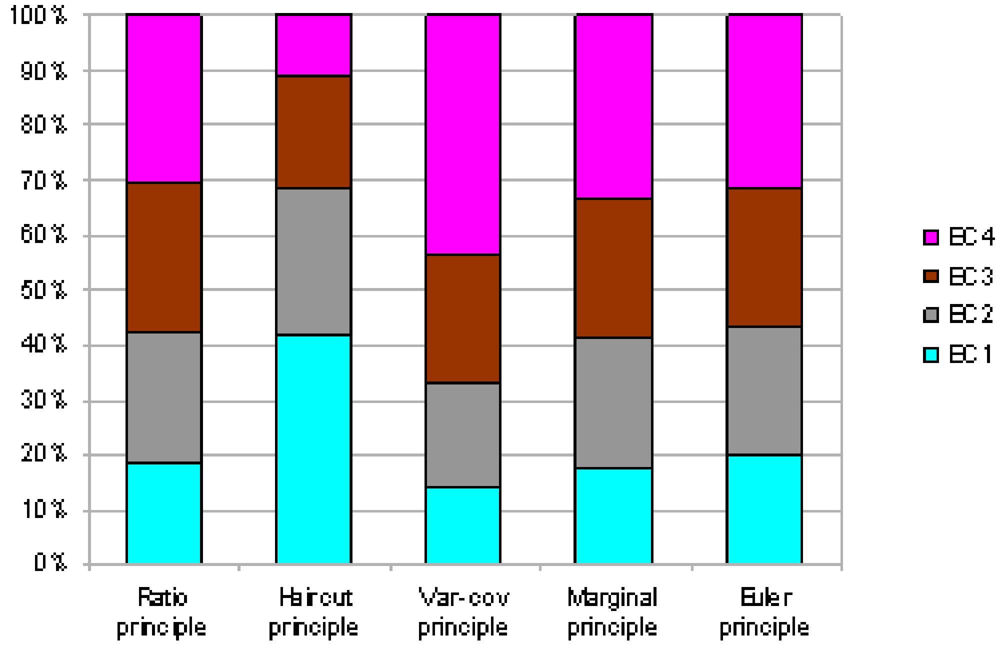
Risk	Distribution	Mean	Variance
1	Lognormal(5.37,0.4)	232.758	9400.19
2	Lognormal(5.265,0.6)	231.597	23 242.6
3	Lognormal(5.18,0.73)	231.933	37 863.1
4	Lognormal(4.98,0.97)	232.863	84 715.28

We repeat the method of computation from the previous example and the results are:
$EC_1 = 121.477$
$EC_2 = 159.495$
$EC_3 = 178.456$
$EC_4 = 203.36$
$ES = 493.449$

	Ratio principle	Haircut principle	Var-cov principle	Marginal principle	Euler principle
EC_1	90,4404	206,152	69,7384	86,3636	97,8224
EC_2	118,745	133,573	92,4345	118,559	116,454
EC_3	132,862	99,0688	116,968	124,961	122,956
EC_4	151,403	54,6555	214,308	163,565	156,217

Table 5.4: Economic capitals allocated to the business lines using of several allocation principles





We do not observe any unexpected results. As was said earlier, the biggest impact on variance-covariance capital allocation has the variance of distribution and correlation between risks. The result confirm this conjecture. The Euler and the marginal allocated capitals are stable like in previous example, so is the haircut principle.

5.4 Example 3

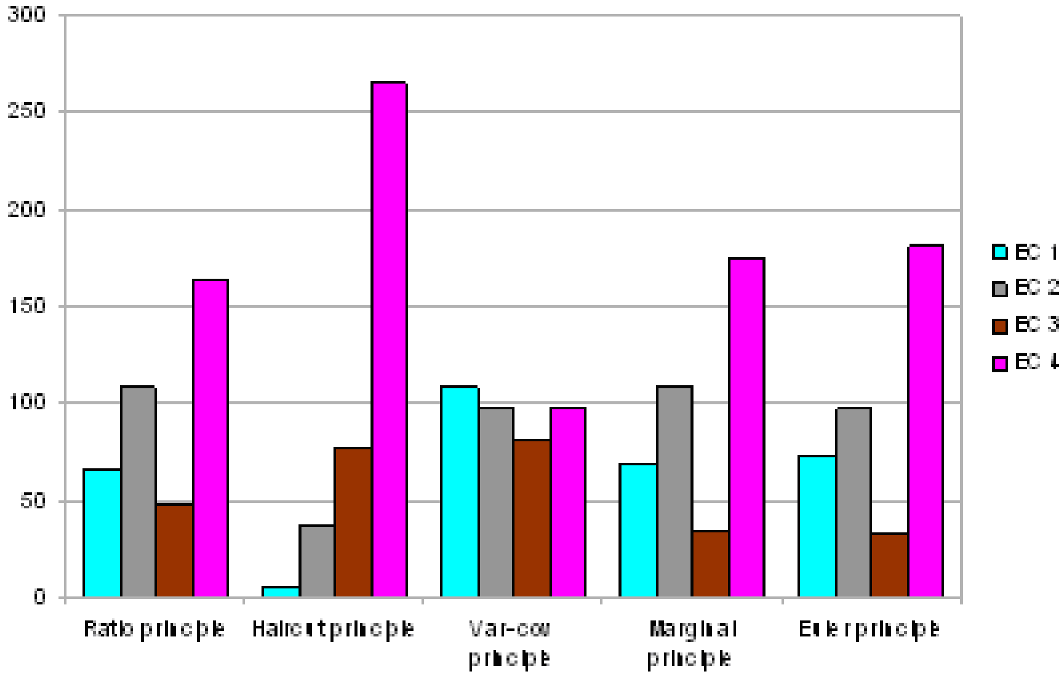
For the last demonstration we decided to choose distributions with similar variances and different mean values.

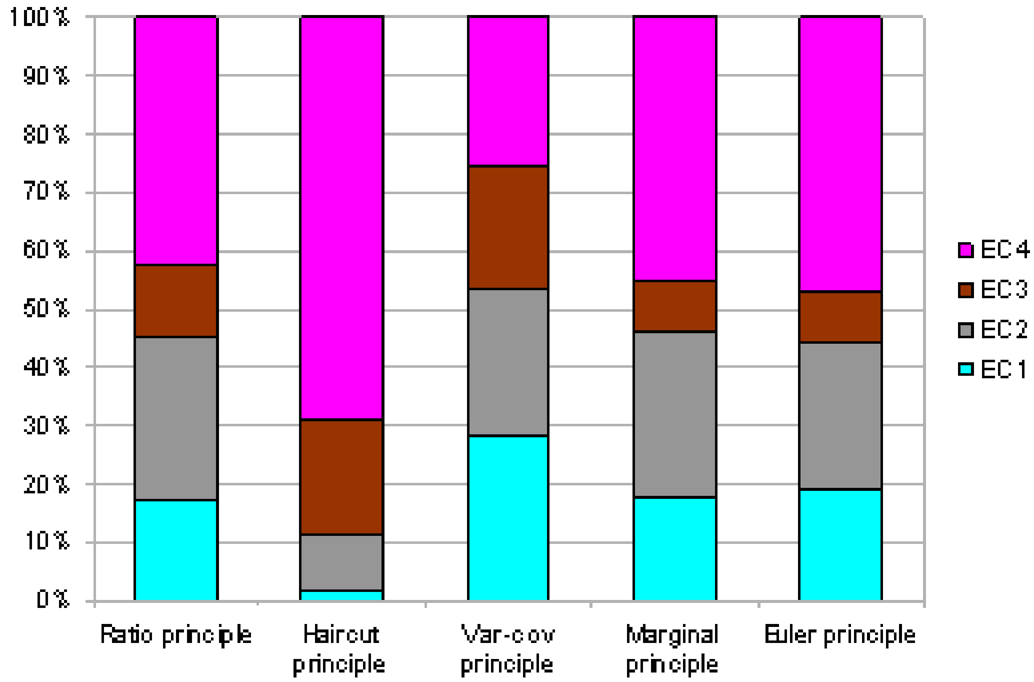
Risk	Distribution	Mean	Variance
1	Lognormal(3.95,1.09)	94.071	20 184.
2	Lognormal(5.03,0.67)	191.416	20 759.9
3	Pareto(103,2.59)	167.78	18 421.6
4	Gamma(16.2,35.6)	576.72	20 531.2

$EC_1 = 85.4249$
$EC_2 = 140.614$
$EC_3 = 62.7197$
$EC_4 = 213.789$
$ES = 502.547$

	Ratio principle	Haircut principle	Var-cov principle	Marginal principle	Euler principle
EC_1	65,6914	6,33503	108,35	68,4716	73,1235
EC_2	108,131	37,2232	98,5398	108,901	97,7716
EC_3	48,2312	76,9779	81,2666	34,078	33,7562
EC_4	164,403	265,921	98,2998	175,006	181,805

Table 5.5: Economic capitals allocated to the business lines using of several allocation principles


300
250
200
150
100
50
0
Ratio principle
Haircut principle
Var-cov principle
Marginal principle
Euler principle
EC 1
EC 2
EC 3
EC 4


100 %
90 %
80 %
70 %
60 %
50 %
40 %
30 %
20 %
10 %
0 %
Ratio principle
Haircut principle
Var-cov principle
Marginal principle
Euler principle
EC 4
EC 3
EC 2
EC 1

We begin with the variance-covariance principle. Although the various distributions are used, all the risks have very similar variances. Therefore, the amounts of allocated capital are nearly the same despite the original capitals not having identical values. The marginal and the Euler principle, again, are not so sensitive to changes in mean or variance, so the results are more - less proportional. The biggest disproportion can be seen under the haircut principle, where the EC_4 coming from the distribution with the largest mean value has the biggest value of capital.

Conclusion

The economic capital aggregation and its backward allocation to the business units is an important task in finance and insurance industry. Economic capital was originally developed by banks as a tool for capital allocation and performance measurement. This thesis is devoted to the task of capital aggregation and the allocation to the business lines of the company. For the economic capital determination two main methods were used: linear aggregation and copulas.
While linear correlation is the basic tool for economic capital derivation, copulas represent a quite new concept in risk aggregation. To advantage of linear correlation belongs easy computation. On the other hand, linear correlation does not take into account tail dependencies but copulas do. Unlike the correlation, copulas are very computationally complicated. Moreover, it is very difficult to fit the right copula to data and it requires deeper expert knowledge. That is the reason why we demonstrate the calculation of aggregated economic capital only using the linear correlation matrix. We can notice the effect of diversification in numerical examples- more correlated risks mean larger amounts of economic capital needed.
The computed aggregated capital has to be backward allocated to the individual business lines of company. In the second part of the thesis we introduced the allocation principles of economic capital. We chose the principles which do not have so common features and they are based on different computation methods. The largest group of approaches involves proportional principles. It means that economic capital is shared with the busines units by some proportional rule. The choice of this rule usually depends on statistical parameters, quantile or variances for instance. Then we discussed the Euler (or gradient) and the marginal (or incremental) allocation principle. By these principles the amount of allocated capital is highly influenced by the dependence structure between risks. For instance, we refer to [2] for a good comparison of the Euler and the haircut principle and illustration of copula approach to capital aggregation.
In the last chapter numerical examples of these principles are provided. We compared the haircut, variance-covariance, Euler and marginal allocation methods. The most significant results are:

- Because the haircut allocation principle is based on the quantile of the loss distribution, the main aspect which influences the amount of allocated capital is how heavy the tail of the loss distribution is.
- The variance-covariance principle is influenced by variance of distribution at the first place. Also covariance between risks play the role, but not so important.
- The marginal principle depends on the dependence structure between risks more than on some statistical parameters.

- The Euler allocation principle satisfies the condition of fairness, it can be derived from game theory. It is the only principle suitable for performance measurement.

We can see that the Euler principle is the most *stable* principle in all cases. It is not so highly influenced by the variance of the portfolio or by the quantile. It also satisfies the property of RORAC compatibility. Because of these advantages we consider the Euler principle to be the most appropriate approach to economic capital allocation.

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Appendix

In this section the source code from Wolfram Mathematica is given for computing aggregated economic capital using the correlation matrix and allocation principles can be found. Because we use the same method for all examples, we show only one code (Example 2). The rest is the same but with different parameters and distributions.

```
Clear[mi1,mi2,mi3,mi4,sigma1,sigma2,sigma3,sigma4]
mi1=5.37;
mi2=5.265;
mi3=5.18;
mi4=4.98;
sigma1=0.4;
sigma2=0.6;
sigma3=0.73;
sigma4=0.97;

Economic capitals
Parameters1={Mean[LogNormalDistribution[mi1,sigma1]],
Variance[LogNormalDistribution[mi1,sigma1]]}
Parameters2={Mean[LogNormalDistribution[mi2,sigma2]],
Variance[LogNormalDistribution[mi2,sigma2]]}
Parameter3={Mean[LogNormalDistribution[mi3,sigma3]],
Variance[LogNormalDistribution[mi3,sigma3]]}
Parameters4={Mean[LogNormalDistribution[mi4,sigma4]],
Variance[LogNormalDistribution[mi4,sigma4]]}
kap1=-(Quantile[LogNormalDistribution[mi1,sigma1],0.05]
-Mean[LogNormalDistribution[mi1,sigma1]])
kap2=-(Quantile[LogNormalDistribution[mi2,sigma2],0.05]
-Mean[LogNormalDistribution[mi2,sigma2]])
kap3=-(Quantile[LogNormalDistribution[mi3,sigma3],0.05]
-Mean[LogNormalDistribution[mi3,sigma3]])
kap4=-(Quantile[LogNormalDistribution[mi4,sigma4],0.05]
-Mean[LogNormalDistribution[mi4,sigma4]])
kapital=kap1+kap2+kap3+kap4
K={kap1,kap2,kap3,kap4}


Aggregated economic capital
korelacie={{1,0.5,0.25,0.75},{0.5,1,0.5,0.25},
{0.25,0.5,1,0.25},{0.75,0.25,0.25,1}}
korelacie//MatrixForm
```

```
divcapital=Sqrt[K.correlations.Transpose[{K}]]
(1 0.5 0.25 0.75
0.5 1 0.5 0.25
0.25 0.5 1 0.25
0.75 0.25 0.25 1

)
Ratio principle
simplecapital={divkapital/kapital*kap1,divkapital/kapital*kap2,
divkapital/kapital*kap3,divkapital/kapital*kap4}

Haircut allocation principle
sumakvantilov=Quantile[LogNormalDistribution[mi1,sigma1],0.05]
+ Quantile[LogNormalDistribution[mi2,sigma2],0.05]+
Quantile[LogNormalDistribution[mi3,sigma3],0.05]+
Quantile[LogNormalDistribution[mi4,sigma4],0.05]
quantilekap= {divkapital/sumakvantilov*
Quantile[LogNormalDistribution[mi1,sigma1],0.05],
divkapital/sumakvantilov*Quantile[LogNormalDistribution[mi2,sigma2],0.05],
divkapital/sumakvantilov*Quantile[LogNormalDistribution[mi3,sigma3],0.05],
divkapital/sumakvantilov*Quantile[LogNormalDistribution[mi4,sigma4],0.05]}

Variance - covariance allocation principle
variance={Variance[LogNormalDistribution[mi1,sigma1]],
Variance[LogNormalDistribution[mi2,sigma2]],
Variance[LogNormalDistribution[mi3,sigma3]],
Variance[LogNormalDistribution[mi4,sigma4]]}
covariance=Table[ korelacie[[i,j]]*Sqrt[variance[[i]]]*
Sqrt[variance[[j]]],{i,1,4},{j,1,4}]
covariance//MatrixForm
(9400.19 7390.62 4716.46 21164.6
7390.62 23242.6 14832.7 11093.4
4716.46 14832.7 37863.1 14158.9
21164.6 11093.4 14158.9 84715.2

)
mean={Mean[LogNormalDistribution[mi1,sigma1]],
Mean[LogNormalDistribution[mi2,sigma2]],
Mean[LogNormalDistribution[mi3,sigma3]],
Mean[LogNormalDistribution[mi4,sigma4]]}
meanS=Sum[mean[[i]],{i,1,4}]
varianceS=Total[Total[Table[covariance[[i,j]],{i,1,4},{j,1,4}]]]
```

```
EXiXj=Table[covariance[[i,j]]+mean[[i]]*mean[[j]],{i,1,4},{j,1,4}]
covXS={Total[Table[EXiXj[[1,i]],{i,1,4}]]-mean[[1]]*meanS,
Total[Table[EXiXj[[2,i]],{i,1,4}]]
-mean[[2]]*meanS,Total[Table[EXiXj[[3,i]],{i,1,4}]]-mean[[3]]*meanS,
Total[Table[EXiXj[[4,i]],{i,1,4}]]
-mean[[4]]*meanS}
covcapital={Table[divkapital*covXS[[i]]/varianceS,{i,1,4}]}

Euler allocation principle
delta=0.05;
eulerkap1=Sqrt[{kap1+delta*kap1,kap2,kap3,kap4}.korelacie.
Transpose[{{kap1+delta*kap1,kap2,kap3,kap4}}]]
eulerkap2=Sqrt[{kap1,kap2+delta*kap2,kap3,kap4}.korelacie.
Transpose[{{kap1,kap2+delta*kap2,kap3,kap4}}]]
eulerkap3=Sqrt[{kap1,kap2,kap3+delta*kap3,kap4}.korelacie.
Transpose[{{kap1,kap2,kap3+delta*kap3,kap4}}]]
eulerkap4=Sqrt[{kap1,kap2,kap3,kap4+delta*kap4}.korelacie.
Transpose[{{kap1,kap2,kap3,kap4+delta*kap4}}]]
ratio={eulerkap1-divkapital,eulerkap2-divkapital,
eulerkap3-divkapital,eulerkap4-divkapital}
Euler1=ratio[[1]]/Sum[ratio[[i]],{i,1,4}]*divkapital
Euler2=ratio[[2]]/Sum[ratio[[i]],{i,1,4}]*divkapital
Euler3=ratio[[3]]/Sum[ratio[[i]],{i,1,4}]*divkapital
Euler4=ratio[[4]]/Sum[ratio[[i]],{i,1,4}]*divkapital

Marginal allocation principle
korelacieABC={{1,0.5,0.25},{0.5,1,0.5},{0.25,0.5,1}};
korelacieABD={{1,0.5,0.75},{0.5,1,0.25},{0.75,0.25,1}};
korelacieACD={{1,0.25,0.75},{0.25,1,0.25},{0.75,0.25,1}};
korelacieBCD={{1,0.5,0.25},{0.5,1,0.25},{0.25,0.25,1}};
kapABC=Sqrt[Delete[K,4].korelacieABC.Transpose[{Delete[K,4]}]]
kapABD=Sqrt[Delete[K,3].korelacieABD.Transpose[{Delete[K,3]}]]
kapACD=Sqrt[Delete[K,2].korelacieABC.Transpose[{Delete[K,2]}]]
kapBCD=Sqrt[Delete[K,1].korelacieABC.Transpose[{Delete[K,1]}]]
prirABC=divkapital-kapABC
prirABD=divkapital-kapABD
prirACD=divkapital-kapACD
prirBCD=divkapital-kapBCD
prirastok=prirABC+prirABD+prirACD+prirBCD
upravenyprirD=prirABC/prirastok*divkapital
upravenyprirC=prirABD/prirastok*divkapital
upravenyprirB=prirACD/prirastok*divkapital
upravenyprirA=prirBCD/prirastok*divkapital
```