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Markov Decision Processes in Finance

Master's Thesis

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Abstract

This thesis presents the theory applicable to the option pricing and shortfall risk minimization problem. The market is arbitrage-free without transaction costs and the underlying asset price process is assumed to possess a Markov chain structure. Under these assumptions, stochastic dynamic programming is exploited to price the European type option. By using the utility concept, the Fundamental Theorem of Asset Pricing is proved via portfolio optimization. Furthermore, it is shown how to use dynamic programming to control the risk related to the future payoff of the option. The approach extends to the case when there is restricted information on the underlying asset price evolution. The methods deal with both complete and incomplete markets.

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Chapter 1

Introduction

The purpose of this Master's Thesis is to present and develop applications of Markov Decision Processes to pricing problems and risk management. This first chapter discusses the motivation for the research, introduces the goals of the thesis, and describes the global contents.

1.1 Motivation for the research

The financial markets provide a huge range of financial instruments and for many years much effort has been going into the study how prices vary with time and how to valuate contracts fairly. Financial instruments can be categorized according to various criteria. One of them distinguishes financial instruments between cash instruments such as stocks, bonds, loans, deposits and derivative instruments (or derivatives for short), which are defined in terms of some underlying assets, rates or indexes that already exist on the market. Derivatives cannot be priced arbitrarily in relation to underlying asset prices, if one wants to avoid discrepancies between the individual prices. Then it is clear, that if the price of the underlying asset moves into the right direction, the derivative provides a profit. Otherwise, the investor loses money or the derivative becomes worthless. The diverse range of potential underlying assets and payoff alternatives leads to a huge variety of derivative contracts available to be traded on the market. Stock prices, bond prices, currency rates and interest rates fluctuate, and thereby they create risk. Due to a major growth in financial markets, risk management has become a very important area of study. Generally, main objectives of risk management are to assess the risk and then to develop strategies that minimize it. The latter can be obtained just by using financial derivatives.

An explosive growth in the variety of financial instruments traded over the last few years has increased the relevance of proper modeling and analyzing of financial markets to solve the problems such as asset pricing, risk management, portfolio optimization, forecasting, etc. Financial economics investigates structural reasons for certain asset prices and such an approach very often turns out to be insufficient. One needs more sophisticated methods in order to solve many problems and to make good decisions. Financial mathematics is just a branch of applied mathematics that comes to the financial investor's aid by the use of technical and advanced mathematical tools in both research and practice. In combination with increasing computing power, this area of science is getting more and more applicable and powerful, thereby gaining major recognition among companies and investors. Behavior of the financial markets and plenty of problems encountered there can be well formalized within mathematical frameworks; probability, statistics, numerical methods, stochastic calculus and stochastic differential equations provide strong methods for the analysis of various markets.

One of the most fundamental tasks in finance concerns assets and contract valuation. First of all, one can try to forecast future stock prices, interest rates, etc. in more or less precise ways. Asset price dynamics can be described with the aid of many different mathematical models that have been evolving through the years. However, there does not exist any privileged model reflecting real asset prices and there are several reasons for it. A mathematical model should correspond to real settings and lead to solutions of the considered problems, which is not always possible to achieve. Real markets are characterized by many restrictions, e.g., transaction costs, access to the financial markets is not always free, information about borrowing and lending opportunities may not be freely available, a single trader can have a significant impact on market prices, short selling can be allowed or not. Thus, one is very often forced to make additional assumptions about the market, but the mathematical model still has to deal with the considered problems. For example, discrete-time settings may be taken into account instead of continuous-time models for some simplifications. Moreover, in order to valuate derivatives by the use of mathematical methods one general assumption has to be set, namely that the market is arbitrage-free. That means no riskless profits can be made by trading in derivative contracts. Arbitrage-free pricing is a central topic of financial mathematics and if one aims at determining a fair price of a contract, then a proper choice of the underlying asset price dynamics is crucial. Derivative contracts also can be used for speculation and creating arbitrage opportunities. However, it is more interesting and desirable to focus on their usefulness in risk management. Since derivatives have strict connections with underlying assets, they can transfer risk and hence be used for reducing or totally canceling out the risk derived from investments. Such a strategy, designed to minimize the risk of financial loss from an adverse price change is called hedging.

The research for financial markets has therefore become an important area, fueled by the application of sophisticated mathematical methods. The underlying principle for financial markets is the Fundamental Theorem of Asset Pricing, which states that there exists a martingale measure if and only if the market is arbitrage-free. Under this no-arbitrage assumption, the problem how to optimally price some financial derivatives is solvable by the use of martingale measures, which are risk-neutral probability measures. The unique arbitrage-free price exists in a setting where the market is assumed to be complete. However, the additional assumption that the market is complete forms a severe restriction for practical applicability, so that one has to take refuge to incomplete markets to obtain more realistic models. Unfortunately, incomplete markets allow for an interval of arbitrage-free prices, thus no preference for independent pricing of a derivative is given. Hence, other criteria that would allow for the selection of only one arbitrage-free price have to be formalized. For example, the utility approach, that expresses an individual investor's attitude towards risk, leads to a unique solution.

All issues outlined above can be studied in particular within the Markov Decision Processes framework. The decision making in pricing or hedging problems can be defined and solved completely in a theoretical manner; moreover dynamic programming provides numerical results for specific models.

1.2 Goals and structure of the thesis

The main goal of this thesis is to create a general market model that enables a formulation of the utility-based option pricing and the shortfall risk minimization via dynamic portfolio optimization, and leads to solutions to the above issues; both in the case of the complete and incomplete information about the market. Within the scope of this thesis we plan to present an approach that combines and develops already existing techniques adapted to a Markovian setup.

The outline of this thesis is as follows. Chapter 2 is a general introduction to financial markets. It starts with mathematical modeling of the asset price and portfolios and later discusses the existing problems and well-known solutions in the simplest cases of the market. At the same time, it points out how difficulties may arise and prepares the basics that we will be relying on and building the theory upon. Chapter 3 establishes the model and provides solutions to option pricing and risk management by using the stochastic control approach, when all information about the market is given.

The presented methods take into account preferences of the investors. Chapter 4 shows the effectiveness of the proposed model. Specific examples, that fit into the general framework of Chapter 3 will be also elaborated on in the case of incomplete market information.

Chapter 2

Modeling of Financial Markets

2.1 Introduction

This chapter provides a brief overview of mathematical modeling of financial markets. As a starting point, a general model for asset price dynamics and a portfolio process are introduced, and are then followed by a discussion of financial derivatives and the formulation of the main problems. To deal with these problems, we need to use very important concepts – an arbitrage and a martingale measure. It is shown how these two concepts are interconnected and how we can exploit them in the option pricing problem. Finally, we familiarize with complete and incomplete markets.

2.2 Asset prices and portfolios

Let us consider the following model of a financial market, where investors can trade in securities and observe their prices only at the dates $0, 1, \ldots, N$. N is called the time horizon. In the model, we have K risky securities called *stocks*, and a risk-free one, called a *bond* (or a *saving account*).

2.2.1 Asset price dynamics

The bond price evolves according to a deterministic process defined by

$$B_n := (1 + r_1) \cdots (1 + r_n), \quad \text{with } n = 0, \ldots, N, \; B_0 = 1, \tag{2.1}$$

where $r_n \geq 0$ denotes the *interest rates* in a period $(n-1, n]$. For theoretical purposes the interest rates for borrowing and lending are assumed to be the same.

The stock prices at time t are modeled by a K–dimensional stochastic process

$$S_t = (S_t^1, \ldots, S_t^K), \ S_0 \in \mathbb{R}_+,$$

where all components of S_t are assumed to be positive and time t may be taken discrete or continuous. We assume that trading of securities takes place at discrete-time steps. Hence, it suffices to observe the dynamics of S_t in a discrete setting, i.e., stock prices are given by S_n for $n = 0, \ldots, N$. The dynamical behavior of the stock price process we introduced is very general; asset price dynamics can be described in various and more particular manners. The main point here is that almost all of them can be adjusted and formulated as a special case of this general model. For illustrative purposes, let us provide examples of stock price processes that are quite commonly used. They will be developed in more detail and studied in the next chapters by using Markov Decision Processes theory.

Examples

1) The multi-period binomial model is a discrete-time model. We assume here that $K = 1$ and $r_n = r$ for all n. The dynamics of the stock price are given by

$$S_{n+1} = S_n \cdot Z_n, \quad S_0 \in \mathbb{R}^+,$$

for $n = 0, \ldots, N-1$, where $Z_0, \ldots, Z_{N-1}$ are independent and identically distributed random variables such that

$$P(Z_n = u) = p_u,$$

$$P(Z_n = d) = p_d,$$

and

$$d \leq (1 + r) \leq u.$$

2) The stock price dynamics evolves according to a multi-dimensional continuous-time stochastic process S_t, satisfying the following stochastic differential equations:

$$dS_t^k = S_t^k(a_t^k dt + \sum_{j=1}^{d} \sigma_t^{kj} dW_t^j), \ S_0^k \in \mathbb{R}_+, \ k = 1, \ldots, K.$$

The stochastic processes a_t^k are called *appreciation rates* and they determine the main trends of the stock price changes. The d-dimensional stochastic processes $\sigma_t^k = (\sigma_t^{k1}, \ldots, \sigma_t^{kd})$, called *volatilities*, describe fluctuations of the stock price, $W_t = (W_t^1, \ldots, W_t^d)$ is a d-dimensional standard Brownian motion. For correctness of the model, the above-mentioned processes usually have to satisfy some regularity conditions, e.g., they need to be progressively measurable. However, here we will not go into the details. We can write the solution to the equations in the form of

$$S_t^k = e^{\int_0^t (a_s^k - \frac{1}{2}\sum_{j=1}^d (\sigma_s^{kj})^2) ds + \int_0^t \sum_{j=1}^d \sigma_s^{kj} dW_s},$$

in particular, the dynamics at discrete-times n are given by

$$S_{n+1}^k = S_n^k e^{\int_n^{n+1} (a_s^k - \frac{1}{2}\sum_{j=1}^d (\sigma_s^{kj})^2) ds + \int_n^{n+1} \sum_{j=1}^d \sigma_s^{kj} dW_s}.$$

This stock price process appears in the extended Black-Scholes model.

2.2.2 Portfolio process

Information about the observed stock prices, given at n, creates an increasing sequence of σ-fields $\mathcal{F}_n := \sigma(S_0, \ldots, S_n)$, $n = 0, \ldots, N$. On the basis of this information, at each time n, the investor decides about the composition of his portfolio, consisting of bonds and stocks. A *strategy* (*portfolio*) (ϕ, ψ) is defined as a pair of predictable stochastic processes ϕ_n and ψ_n. ϕ_{n+1} denotes the amount of money invested in bonds during time interval $(n, n+1]$, ψ_n is assumed to be a K–dimensional process where every kth component ψ_{n+1}^k denotes the amount invested in the kth risky asset during time interval $(n, n+1]$, $k = 1, \ldots, K$. If one considers negative values for ϕ or ψ this is identified with a loan or short selling of stocks, respectively. We assume the following facts about the market. First, the market allows short selling and fractional holdings, which means that (ϕ, ψ) can take on every value in $\mathbb{R}^2$. Moreover, for all assets the selling and buying prices are the same, and there are no transactions costs for trading. The last assumption states that investors can trade unlimited quantities on the market, in particular, they can get unlimited loans from the bank by short selling of bonds.

Definition 2.1 *The* ***value process*** *of the portfolio* (ϕ, ψ) *is defined by*

$$V_n = \phi_n B_n + \psi_n^\top \cdot S_n = \phi_n B_n + \sum_{k=1}^K \psi_n^k S_n^k, \; n = 0, \ldots, N.$$

The investor starts with an initial capital V_0 and later he may relocate money to bonds and stocks at every time n, but in such a way, that import or export of money is not allowed. That means that all changes in the portfolio value are only due to appreciation or depreciation of securities. This leads to the following definition.

Definition 2.2 *The strategy* (ϕ, ψ) *is defined to be* ***self-financing*** *if for* $n = 0, \dots, N-1$ *it satisfies*

$$\phi_n B_n + \psi_n^\top \cdot S_n = \phi_{n+1} B_n + \psi_{n+1}^\top \cdot S_n.$$

One may state that the *discounted stock price process* and the *discounted value process* are given by:

$$\begin{aligned} \tilde{S}_n &= \frac{S_n}{B_n} \ , \ n = 0, \dots, N, \\ \tilde{V}_n &= \frac{V_n}{B_n} \ , \ n = 0, \dots, N. \end{aligned} \tag{2.2}$$

Let us define the backward increment process by $\Delta X_n := X_n - X_{n-1}$. The above definitions and (2.2) lead to the formula

$$\tilde{V}_n = \tilde{V}_{n-1} + \psi_n^\top \cdot \Delta \tilde{S}_n. \tag{2.3}$$

We see that the discounted portfolio value at time n is conditional upon the strategy ψ. The same holds trivially for the portfolio value at n.

2.3 Option pricing and hedging – problem formulation

The financial markets provide a great variety of derivatives that are traded in huge volumes. Moreover, we can take into account many different risk factors which determine the payoff of such contracts. In this thesis, we mainly focus on specific types of derivatives, called options and in the model the value of such a financial instrument at time horizon N depends on the risk factor represented by the future stock prices.

Example
One of the most important examples is given by a European call option. It is an agreement signed at $n = 0$, which gives a buyer the right, but not the obligation, to buy a stock S at a predetermined price E called the *exercise price* at the expiration time N. The value of the European call option at maturity is easily determined, namely, if the price S_N of the stock is less than the exercise price E, then the contract is worthless and the buyer does not exercise his right. On the other hand, if the stock price is higher than the price E, the buyer will exercise the option and obtain a gain $S_N - E$ by selling it at the market price immediately. Thus, the value of the option at

the time N is given by a random variable $X = (S_N - E)^+$.

Now, we can formulate the main problems. If we take a fixed derivative, then first we want to know what a fair price is for the contract at all times till the expiration date. The second question concerns hedging. In general a seller of the option is exposed to a certain amount of financial risk at the date of expiration. Thus, what action should he take to hedge against the risk? These problems may be solved quite easily or pose a lesser or greater problem depending on the assumptions about the market and the construction of the payoff function. Let us now precise the mathematical framework.

Definition 2.3 *A* ***contingent claim*** *(financial derivative) with expiration time (maturity) N is any $\mathcal{F}_N$-measurable random variable X expressed in the form of*

$$X = h(S_N),$$

where the contract function h is some given real valued function.

A contingent claim represents the payoff of a contract at expiration time from a seller to a buyer. As one may suspect, thanks to investing in securities at time 0 and the choice of a suitable portfolio strategy we might get the portfolio with its terminal value equal to the value of a given contingent claim. Now we precise and explore this idea.

Definition 2.4 *A given contingent claim X is said to be* ***attainable*** *if there exists a self-financing portfolio (ϕ, ψ) such that*

$$V_N = X,$$

with probability 1.

In that case we say that portfolio (ϕ, ψ) is a *replicating* portfolio for a given contingent claim X.

2.4 The arbitrage-free market

An *arbitrage* means that on the market there exist opportunities to make profits with certainty that are strictly greater than profits obtained by investing in risk free assets. As an example, consider that at time 0 two portfolios are available with the same terminal values $V_T^1 = V_T^2$, and different initial values V_0^1 and V_0^2. The difference of both portfolios would have an initial value $V_0 = V_0^1 - V_0^2 \neq 0$ and $V_T = V_T^1 - V_T^2 = 0$. Hence, investing in a new portfolio defined as a suitably taken difference of both previous ones would lead to certain gains. Thus, an arbitrage has also the interpretation of the existence of mispricings on the market. The assumption about the arbitrage-free market is very important in the approach to a pricing problem, as it will be shown later. It is possible to put an arbitrage into a mathematical framework. There are many, more or less, equivalent variations to present the concept of an arbitrage. Now, for the sake of this thesis, we work with the following definition.

Definition 2.5 *The self-financing portfolio (ϕ, ψ) is an **arbitrage portfolio** if it satisfies the conditions*

$$V_0 = 0,$$

$$P(V_N \geq 0) = 1,$$

$$P(V_N > 0) > 0.$$

Definition 2.6 *The financial market is called to be **arbitrage-free** if there is no arbitrage portfolio on the market.*

These formal definitions seem to be reasonable. Indeed, according to the conditions, an arbitrage portfolio offers an opportunity to make money at horizon time N at zero initial expenditure. The next sections explain why the assumption about an arbitrage-free market is so relevant in order to get a pricing system that is consistent with the underlying asset price given by the market.

2.5 Martingale measures

Let Y be an adapted, integrable stochastic process on the given filtered probabilistic space $(\Omega, \mathcal{F}, \{\mathcal{F}_n\}, \mathbb{P})$. Then Y is defined to be a martingale if for all $m \leq n$: $E(Y_n|\mathcal{F}_m) = Y_m$. In case of the financial market, $\mathbb{P}$ is the physical probability that models the randomness of the stock price S. If one aims to price contingent claims in a fair sense, it is necessary to refer to *martingale measures.*

Definition 2.7 *A probabilistic measure $\tilde{\mathbb{P}}$ is defined to be a **martingale measure** if $\tilde{\mathbb{P}}$ is an equivalent measure to the given physical measure $\mathbb{P}$ and turns $\tilde{S}_n$ into a martingale, i.e.,*

$$E_{\tilde{\mathbb{P}}}(\tilde{S}^k_n|\mathcal{F}_{n-1}) = \tilde{S}^k_{n-1},$$

for $k = 1, \ldots, K$ and $n = 1, \ldots, N$.

We can define now

$$\mathcal{P} := \{\mathbb{P} : \mathbb{P} \text{ is a martingale measure}\}.$$

If $\mathcal{P}$ is non-empty, then every attainable contingent claim may be priced by the use of martingale measures. For a self-financing strategy (ϕ, ψ) that replicates a given contingent claim X, we can derive from the martingale property that for $\tilde{V}$:

$$V_n = B_n \tilde{V}_n = B_n E_{\tilde{\mathbb{P}}}(\tilde{V}_N | \mathcal{F}_n) = B_n E_{\tilde{\mathbb{P}}} \left(\frac{X}{B_N} | \mathcal{F}_n \right), \ \tilde{\mathbb{P}} \in \mathcal{P}. \tag{2.4}$$

In particular, the value of X at 0 equals

$$V_0 = E_{\tilde{\mathbb{P}}} \left(\frac{X}{B_N} \right), \ \tilde{\mathbb{P}} \in \mathcal{P}. \tag{2.5}$$

To fully justify the above derivation, it suffices to show why $\tilde{V}_n$ is a martingale. We have that (ϕ, ψ) is self-financing, predictable, and $\tilde{S}$ is a martingale, hence for $n = 0, \ldots, N–1$

$$\tilde{V}_n = \frac{\phi_n B_n + \psi_n^\top \cdot S_n}{B_n} = \frac{\phi_{n+1} B_n + \psi_{n+1}^\top \cdot S_n}{B_n} = \phi_{n+1} + \psi_{n+1}^\top \cdot \tilde{S}_n =$$
$$= \phi_{n+1} + \psi_{n+1}^\top \cdot E_{\tilde{\mathbb{P}}}(\tilde{S}_{n+1} | \mathcal{F}_n) = E_{\tilde{\mathbb{P}}}(\phi_{n+1} + \psi_{n+1}^\top \cdot \tilde{S}_{n+1} | \mathcal{F}_n) = E_{\tilde{\mathbb{P}}}(\tilde{V}_{n+1} | \mathcal{F}_n).$$

Thus, the price given by Formula (2.4) is a suitable candidate for a fair price of the contingent claim X at n in view of the following reasoning. First of all, the above method prices X in terms of the underlying a-priori given assets $S^1, \ldots, S^K$. Moreover, linearity of the conditional expectation implies that such a pricing system works well on the space of all contingent claims. However, in order to show that such a way of pricing is indeed reasonable, we would like to have no arbitrage opportunities offered on the market.

Fortunately, we can combine both concepts we introduced: an arbitrage and a martingale measure. The following theorem is a crucial result for financial mathematics; it clarifies why martingale measures play a dominant role in the valuation of contingent claims.

Theorem 2.1 (Fundamental Theorem of Asset Pricing) *There exists a martingale measure, i.e., $\mathcal{P}$ is non-empty if and only if the market is arbitrage-free.*

Proposition 2.1 (no-arbitrage condition) *The financial market is arbitrage-free if and only if for any process ψ for $n = 1, \ldots, N$,*

$$\psi_n^\top \cdot \Delta \tilde{S}_n \geq 0 \ \textit{almost surely} \Rightarrow \psi_n^\top \cdot \Delta \tilde{S}_n = 0 \ \textit{almost surely.}$$

Proof.
Let (ϕ, ψ) be a self-financing portfolio and $V_0 = 0$.
Furthermore, let $\psi_n^\top \cdot \Delta\tilde{S}_n \geq 0$ a.s. for $n = 1, \ldots, N$, and let us assume there exists an n such that $P(\psi_n^\top \cdot \Delta\tilde{S}_n > 0) > 0$. By using (2.3) we have

$$\tilde{V}_N = \sum_{n=1}^{N} \psi_n^\top \cdot \tilde{S}_n.$$

The above leads to a contradiction, because $V_N \geq 0$ a.s. and $P(V_N > 0) > 0$, which means that (ϕ, ψ) is an arbitrage portfolio.

□

A full proof of the Fundamental Theorem of Asset Pricing in the case when Ω is finite will be presented in the next chapter by the use of dynamic programming tools. However, the implication from the left to the right can be shown easily. Let $\tilde{\mathbb{P}}$ be a martingale measure. The martingale property for $\tilde{S}$ and $\psi_n^\top \cdot \Delta\tilde{S}_n \geq 0$ leads to $E_{\tilde{\mathbb{P}}}(\psi_n^\top \cdot \Delta\tilde{S}_n) = 0$. Hence, $\psi_n^\top \cdot \Delta\tilde{S}_n = 0$ $\tilde{\mathbb{P}}$–a.s. and simultaneously $\psi_n^\top \cdot \Delta\tilde{S}_n = 0$ $\mathbb{P}$–a.s., since $\tilde{\mathbb{P}}$ and $\mathbb{P}$ are equivalent measures.

2.6 Complete and incomplete markets

The pricing formula (2.4) looks very clear, but there is one problem we can encounter. If there exist several different martingale measures, then we can have several possible arbitrage-free prices for non-attainable contingent claims. To deal with that problem we can make a distinction between complete and incomplete markets.

The market is defined to be *complete* if every contingent claim is attainable, otherwise the market is said to be *incomplete*. It can be shown, that for complete markets an existing martingale measure is unique ($\mathcal{P}$ consists of one element $\tilde{\mathbb{P}}$). Hence, each contingent claim possesses a fair price given by (2.4). For all theoretical purposes, option pricing does not pose any problems in case of a complete market since the appropriate formula leads to the unique fair price. Of course, it is not always a trivial task to give a strict price V_0 for some contingent claims, mainly because of computational difficulties. Examples of complete markets are given by the Binomial model with one risky asset (discrete-time) or the Black-Scholes model in continuous-time.

If one takes into account incomplete markets that better correspond to real markets, then the no-arbitrage approach we outlined above can be still exploited in the pricing of all contingent claims. If it is possible to replicate a given contingent claim by some self-financing portfolio, then a unique fair price is obtained by (2.4), since for every $\mathbb{P} \in \mathcal{P}$ a contingent claim can be expressed as a martingale in terms of the replicating portfolio. Otherwise, if we consider non-attainable contingent claims, then there exist several "fair" prices defined by (2.4), since the incomplete market allows for several choices of martingale measures. Having the availability of the set of martingale measures, we can define new random variables[1]

$$\overline{V}_n := B_n \operatorname*{ess\,sup}_{\tilde{\mathbb{P}} \in \mathcal{P}} E_{\tilde{\mathbb{P}}} \left(\frac{X}{B_N} | \mathcal{F}_n \right),$$

$$\underline{V}_n := B_n \operatorname*{ess\,inf}_{\tilde{\mathbb{P}} \in \mathcal{P}} E_{\tilde{\mathbb{P}}} \left(\frac{X}{B_N} | \mathcal{F}_n \right).$$

It may be proved, that $\overline{V}_0$ is the minimum initial capital necessary to replicate a contingent claim. In the case of the seller of the option, $\overline{V}_0$ suffices to hedge the future payoffs with certainty. Such a risk-free action of the seller is called *superhedging*. On the other hand, $\underline{V}_0$ is the biggest risk-free price for the buyer. Prices outside an interval $[\underline{V}_0, \overline{V}_0]$ correspond to risk-free profits either for the buyer or the seller. Each price inside the interval $[\underline{V}_0, \overline{V}_0]$ is arbitrage-free, so it is a suitable candidate for a fair price. However such a price involves taking some risk on the buyer's or seller's part or both of them. Now, the question is what price should be taken as a fair price of the contingent claim? There is no unique answer; few approaches are possible. One is based on *utility functions* and takes into consideration the preferences of each investor. In another approach, the price is determined by the market – a suitable martingale measure is chosen with respect to observations of the prices for commonly traded options.

2.7 Conclusions

In this chapter we introduced the basics concerning mathematical modeling of financial markets in discrete-time trading settings. All fundamental concepts, problems, and ideas have been expressed in the form of definitions, formulas, and theorems. The risk neutral valuation formula is the main contribution of the chapter. The idea is that today's stock price is the discounted expected value of tomorrow's stock price if absence of arbitrage is assumed.

[1]For any set of random variables $\{X_\alpha : \alpha \in A\}$, an *essential supremum* is defined to be a random variable X (possibly with value ∞) such that $X \geq X_\alpha$ a.s. for every α and if $Y \geq X_\alpha$ a.s. for every α, then $Y \geq X$ a.s. This random variable is unique up to null sets and is denoted by $\operatorname{ess\,sup}_\alpha X_\alpha$. Similarly, we define an *essential infimum*.

However, it can be done only if we use theoretical probability measures instead of the physical one. Such a probability measure, which is risk neutral and equivalent to a given probability measure is called a martingale measure. We have explained and justified why that idea is applicable to realistic situations. Finally, we had a glance at how the pricing problem becomes more complicated if we consider a model of the incomplete market, which is a better approximation of reality.

Chapter 3

Theoretical Background

3.1 Introduction

In this chapter we introduce a Markovian model of the financial market. After a short description of the model we propose an elegant way to choose one preference price for each contingent claim in the case of an incomplete market. It is done due to the concept of utilities. This fair price will be shown to be consistent with the existing method that uses a martingale measure for pricing attainable contingent claims. In connection to this, the Fundamental Theorem of Asset Pricing will be proven; first, we will consider a one-period model, which will prepare us before going into multi-period models.[1] At the end, we will analyze different ways for hedging of future payoffs by the seller of the option. All these issues can be worked out by the use of Markov Decision Processes theory.

3.2 Model Specification

We have been studying the market model from the first chapter with additional conditions. All economical assumptions remain the same. Now, let us remind and precise all technical details.

Assumptions

- *Trading takes place at times* $n = 0, \ldots, N$,
- *The bond price process evolves according to (2.1),*

[1] The martingale measure construction presented in Section 3.4 is based on [4].

- *The randomness in the model is given by a fixed probability space* $(\Omega, \mathcal{F}, P)$, *where* Ω *is a finite sample space,* $\mathcal{F}$ *is a* σ*-field of subsets of* Ω*, and* P *is a probability measure defined on* $\mathcal{F}$*. Moreover, we assume that for all* $\omega \in \Omega$ *we have that* $P(\omega)$ *are strictly positive,*

- *Stock prices at time* n *are given by a* K*–dimensional stochastic process* S_n*. Furthermore, we assume that the evolution of every risky asset is modeled by a Markov chain, hence* S_n *follows a Markov chain on* $(\Omega, \mathcal{F}, P)$ *with a finite state space* $\mathfrak{S} = \mathfrak{S}_1 \cup \cdots \cup \mathfrak{S}_N \subset (\mathbb{R}^+)^K$*, where* $\mathfrak{S}_n$ *denotes a set of all possible values for* S_n*. Transition probabilities are given by* $p_{sq}(n) := P(S_{n+1} = q | S_n = s),\ s \in \mathfrak{S}_n, q \in \mathfrak{S}_{n+1}$*,*

- $\mathcal{F}_n := \sigma(S_0, \ldots, S_n),\ n = 0, \ldots, N$*,*

- *There are no transaction costs,*

- *The market is arbitrage-free.*

Proposition 3.1 *For* $k = 1, \ldots, K$*,*

(i) *if the Markov chain is homogeneous*[2] *in time, the evolution of the stock price process can be described in the following dynamical form*

$$S^k_{n+1} := \rho^k(S^k_n, Z^k_n),$$

where $\rho^k : \mathbb{R}^+ \times \mathbb{R} \mapsto \mathbb{R}^+$ *is a measurable function and* $Z^k_0, \ldots, Z^k_{N-1}$ *is a sequence of independent identically distributed random variables.*

(ii) *the time inhomogeneous Markov chain* S_n *can be described in a dynamical form as follows*

$$S^k_{n+1} := \rho^k_n(S^k_n, Z^k_n),$$

where $\rho^k_n : \mathbb{R}^+ \times \mathbb{R} \mapsto \mathbb{R}^+$ *is a measurable function and* $Z^k_0, \ldots, Z^k_{N-1}$ *is a sequence of independent random variables.*

The initial capital $V_0 = v$ and the strategy ψ determines the value of the portfolio. From now on, we will stick to the notation $V^\psi_n(v)$ instead of V_n. If we go back to the discounted portfolio value process, we can write (2.3) as follows

$$\tilde{V}^\psi_n(v) = \tilde{V}^\psi_{n-1}(v) + \psi_n^\top \cdot \Delta\tilde{S}_n. \tag{3.1}$$

[2]Here the homogeneity means that the description of the evolution of the Markov chain is independent of time.

3.3 Option pricing – utility approach

One of the methods that one can use to option pricing is based on an optimal solution to the portfolio optimization problem. The individual investor can be characterized by a utility function that is dependent on the portfolio value and the stock price. In portfolio theory, the utility function expresses the preferences for economic entities with respect to the perceived risk and the expected return.

For the following reasoning, the Markovian structure of the market and additional assumptions on Ω are not necessary. The utility approach still works in more general cases of financial markets, including the continuous-time setting. Now, let us present a formal definition.

Definition 3.1 *A function* $U : \mathbb{R} \times \mathfrak{S}_N \mapsto \mathbb{R}$ *such that* $w \mapsto U(w, s)$ *is strictly concave and continuously differentiable with derivative* $U'(w, s)$ *and* $U'(-\infty, s) > 0$*,* $U'(+\infty, s) \leq 0$ *is defined to be the* ***utility function****.*[3]

The investor aims at maximizing the expected utility of the terminal portfolio value. Moreover, if there exists an optimal strategy ψ^* such that the supremum is attained, we can write

$$E\left(U(\tilde{V}_N^{\psi^*}(v), S_N)\right) = \sup_{\psi} E\left(U(\tilde{V}_N^{\psi}(v), S_N)\right),$$

and the supremum equals the maximum. The right side of the equation is called the *maximal utility*. Now, we explain how we can benefit from the maximal utility when we consider the option pricing problem. Let us assume that the contingent claim X is available for trading with a fair purchase price π. Now we can suspect that the expected utility can increase or decrease if the investor, that owns the initial capital v, decides to change the structure of the portfolio at time 0 by investing into options or short-selling them. Trying to answer whether π is a fair price for the contingent claim, we may use the following justification. If π is a fair price for X, then such actions of the investor can not increase his maximal utility. We know that the investor follows the optimal strategy ψ^* and he decides to divert an amount $x \in \mathbb{R}$ to buy x/π shares of the option. Let us denote the discounted contingent claim X/B_N by $\tilde{X}$. The action of the investor can be described by the following equation

$$\tilde{V}_N^{\psi^*}(v - x) + \frac{x}{\pi} \cdot \tilde{X},$$

[3] The definition serves purposes in this and the next section. Usually, authors do not assume strict concavity, also assumptions on the signs of the derivative can differ in literature.

which equals

$$\tilde{V}_N^{\psi^*}(v) + x\left(\frac{1}{\pi}\tilde{X} - 1\right).$$

Thus the expected utility amounts to

$$u(\psi^*, x, \pi, v) = E\left(U(\tilde{V}_N^{\psi^*}(v) + x\left(\frac{1}{\pi}\tilde{X} - 1\right), S_N)\right),$$

in which the maximal utility is given by

$$\sup_{\psi} u(\psi, x, \pi, v) = u(\psi^*, 0, \pi, v).$$

A price π has a neutral effect on the maximal utility, i.e., $\frac{\partial}{\partial x}u(\psi^*, x, \pi, v)|_{x=0} = 0$. Let us explain the above in a more detailed way by using the indifference argument. Namely, the option price π makes the investor indifferent, in terms of the expected utility, between trading in the market with and without the option. This fact can be expressed by equate the partial derivatives to 0. Furthermore, $u(\psi^*, x, \pi, v)$ is strictly concave in x. Hence, the expected utility $u(\psi^*, x, \pi, v)$ is indeed maximal for $x = 0$. Thus, π is a fair price, because short-selling or purchasing the option do not imply higher profits of the investor in view of his utility. We can derive

$$\frac{\partial}{\partial x}u(\psi^*, x, \pi, v)\Big|_{x=0} = E\left(U'(\tilde{V}_N^{\psi^*}(v), S_N) \cdot \left(\frac{1}{\pi}\tilde{X} - 1\right)\right),$$

and hence we get a direct formula for π, given by

$$\pi = \frac{E\left(U'(\tilde{V}_N^{\psi^*}(v), S_N) \cdot \tilde{X}\right)}{E\left(U'(\tilde{V}_N^{\psi^*}(v), S_N)\right)} = E_{\mathbb{P}^U}(\tilde{X}).$$

In the next section we will show including all theoretical details how to construct the martingale measure $\mathbb{P}^U$, which will also be a proof of the Fundamental Theorem of Asset Pricing. Finally, we can remark that for all attainable contingent claims the utility approach to option pricing is consistent with Formula (2.4), obtained due to replicating a future payoff by the self-financing portfolio.

3.4 Martingale measure construction

3.4.1 The one-period model

We begin with the case $N = 1$. Let us define L as the smallest linear space in $\mathbb{R}^K$ such that $P(\Delta\tilde{S}_1 \in L) = 1$. Now, Proposition 2.1 leads immediately to the reformulation of the no-arbitrage condition

$$\text{For all } \psi_1 \in L \backslash \{0\} : P(\psi_1^\top \cdot \Delta\tilde{S}_1 < 0) > 0. \tag{3.2}$$

Let us remind that

$$\tilde{V}_1^\psi(v) = v + \psi_1^\top \cdot \Delta\tilde{S}_1.$$

We want to maximize the *expected utility* dependent on the terminal portfolio value and the stock price

$$f(v, \psi_1) := E\left(U(v + \psi_1^\top \cdot \Delta\tilde{S}_1, S_1)\right),$$

that results in a definition of the *maximal expected utility*

$$F(v) := \sup_{\psi_1 \in \mathbb{R}^K} f(v, \psi_1), \ v \in \mathbb{R}.$$

Lemma 3.1

(i) $(v, \psi_1) \mapsto f(v, \psi_1)$ *is continuous,*

(ii) $\psi_1 \mapsto f(v, \psi_1)$ *attains the maximum on* $\mathbb{R}^K$ *for each* $v \in \mathbb{R}$ *and the maximum point* $\psi^* := \psi^*(v)$ *is unique,*

(iii) $v \mapsto f(v, \psi_1)$ *is strictly concave for fixed* ψ_1*,*

(iv) $F(v)$ *is strictly concave in* v*.*

Proof.
(i) This part is straightforward.
(ii) Let $n \in \mathbb{N}$. At first,

$$\begin{aligned}
&\frac{1}{n}(f(v, n\psi_1) - f(v, 0)) = \\
&\qquad = E\left(D(n, \psi_1^\top \cdot \Delta\tilde{S}_1, v, S_1) \cdot \mathbf{1}_{\{\psi_1^\top \cdot \Delta\tilde{S}_1 > 0\}}\right) + \\
&\qquad\qquad + E\left(D(n, \psi_1^\top \cdot \Delta\tilde{S}_1, v, S_1) \cdot \mathbf{1}_{\{\psi_1^\top \cdot \Delta\tilde{S}_1 < 0\}}\right),
\end{aligned}$$

in which

$$D(n, w, v, s) := \frac{1}{n} \cdot (U(v + nw, s) - U(v, s)).$$

Since $U(\cdot, s)$ is concave, we can apply Lemma A1 (see Appendix). Hence $D(n, w, v, s)$ is decreasing in n. Then we can denote

$$D(\infty, w, v, s) := \lim_{n \to \infty} D(n, w, v, s).$$

Thanks to the derivative's properties of the utility function we get

$$D(\infty, w, v, s) \leq 0 \text{ for } w > 0,$$

$$D(\infty, w, v, s) < 0 \text{ for } w < 0.$$

The use of the Monotone Convergence Theorem (Theorem A2 in Appendix) leads to

$$\lim_{n\to\infty} \frac{1}{n}(f(v, n\psi_1) - f(v, 0)) =$$
$$= E\left(D(\infty, \psi_1^\top \cdot \Delta\tilde{S}_1, v, S_1) \cdot \mathbf{1}_{\{\psi_1^\top \cdot \Delta\tilde{S}_1 > 0\}}\right) +$$
$$+ E\left(D(\infty, \psi_1^\top \cdot \Delta\tilde{S}_1, v, S_1) \cdot \mathbf{1}_{\{\psi_1^\top \cdot \Delta\tilde{S}_1 < 0\}}\right).$$

Now, if Θ denotes the orthogonal projection on L, we know that

$$\psi_1^\top \cdot \Delta\tilde{S}_1 = (\Theta\psi_1)^\top \cdot \Delta\tilde{S}_1.$$

It becomes apparent that we can restrict our attention to L. By (3.2) it is clear that for $\psi_1 \in L\backslash\{0\}$

$$\lim_{n\to\infty} \frac{1}{n}(f(v, n\psi_1) - f(v, 0)) < 0 \text{ a.s.}$$

The above implies that

$$\lim_{n\to\infty} \frac{1}{n} f(v, n\psi_1) = -\infty,$$

for all $\psi_1 \in L\backslash\{0\}$. By a combination of the preceding with (i) we get the existence of the maximum.
For the proof of uniqueness see the proof for (iv).
(iii) This part is obvious.
(iv) Let $v, w \in \mathbb{R}$, and $\lambda \in (0, 1)$. Further, let $\psi_v, \psi_w \in L$ such that

$$V(v) = f(v, \psi_v),$$

$$V(w) = f(w, \psi_w).$$

We can consider $v > w$ and $v = w, \psi_v \neq \psi_w$, that implies

$$P(v + \psi_v^\top \cdot \Delta\tilde{S}_1 \neq w + \psi_w^\top \cdot \Delta\tilde{S}_1) > 0,$$

otherwise we have

$$(\psi_v - \psi_w)^\top \cdot \Delta\tilde{S}_1 = v - w,$$

and in consequence a contradiction of the no-arbitrage condition or a contradiction of the condition $\psi_v - \psi_w \neq 0$ for the cases that we consider, respectively. The strict concavity of $U(\cdot, s)$ gives us

$$\begin{aligned}\lambda_1 F(v) + \lambda_2 F(w) &= \lambda_1 f(v, \psi_v) + \lambda_2 f(w, \psi_w) \\ &< f(\lambda_1 v + \lambda_2 w, \lambda_1 \psi_v + \lambda_2 \psi_w) \leq F(\lambda_1 v + \lambda_2 w),\end{aligned}$$

with $\lambda_1 \in (0, 1)$ and $\lambda_1 + \lambda_2 = 1$. If $v = w$, then we get a contradiction in the form of $V(v) < V(v)$. Hence we deduce that in this case it has to be that $\psi_v = \psi_w$, and the proof is complete.

□

Lemma 3.2

(i) *$v \mapsto f(v, \psi_1)$ is differentiable with derivative*

$$f'(v, \psi_1) = E\left(U'(v + \psi_1 \cdot \Delta\tilde{S}_1, S_1)\right).$$

(ii) *$\psi_1 \mapsto f(v, \psi_1)$ is partially differentiable for $v > 0$ with partial derivatives*

$$\partial_k f(v, \psi_1) = E\left(U'(v + \psi_1 \cdot \Delta\tilde{S}_1, S_1) \cdot \Delta\tilde{S}_1^k\right).$$

The proof for both cases is obvious since Ω is finite.

Theorem 3.1 *$F(v)$ is differentiable with derivative $F'(v) = f'(v, \psi_1^*)$, where $\psi_1^* = \psi_1^*(v)$ is the unique maximum point of the function $\psi_1 \mapsto f(v, \psi_1)$*

Proof.
The existence of the unique ψ_1^* is given by Lemma 3.1(ii) and we can write

$$F(v + h) - F(v) \geq f(v + h, \psi_1^*) - f(v, \psi_1^*),$$

hence

$$f'_+(v, \psi_1^*) \leq F'_+(v) \leq F'_-(v) \leq f'_-(v, \psi_1^*),$$

where $F'_\pm(v)$ and $f'_\pm(v, \psi_1^*)$ denote the right and left derivatives, respectively. An application of Lemma 3.2(i) finishes the proof.

□

Theorem 3.2 *For each $v \in \mathbb{R}$ and for $k = 1, \ldots, K$,*

$$E\left(U'(v + \psi_1^{*\top} \cdot \Delta\tilde{S}_1, S_1) \cdot \Delta\tilde{S}_1^k\right) = 0,$$

with ψ_1^* as in the preceding theorem.

Proof.
By Lemma 3.2(ii) the function $\psi_1 \mapsto f(v, \psi_1)$ is partially differentiable. It is well known that all partial derivatives equal 0 if ψ_1^* is the maximum point.

□

Corollary 3.1 *If $U'(v, s)$ is strictly positive for all x and s, or generally speaking, if $U'(v + \psi_1^{*\top} \cdot \Delta\tilde{S}_1, S_1) > 0$ on Ω, then for every constant $c \neq 0$ the probability measure $\tilde{\mathbb{P}}$ defined by $d\tilde{\mathbb{P}} = c \cdot U'(v + \psi_1^{*\top} \cdot \Delta\tilde{S}_1, S_1)d\mathbb{P}$ is a martingale measure.*

3.4.2 The multi-period model

Now we study a general case for $N \in \mathbb{N}$. To serve our purposes we introduce the notion of a *local arbitrage* in the following definition.

Definition 3.2 *The no-arbitrage condition holds locally if and only if for every $n = 1, \ldots, N$,*

$$P(\psi_n^\top \cdot \Delta\tilde{S}_n \geq 0 | S_{n-1} = s) = 1 \;\Rightarrow\; P(\psi_n^\top \cdot \Delta\tilde{S}_n = 0 | S_{n-1} = s) = 1, \; \psi_n \in \mathbb{R}^K.$$

It can be shown that the no-arbitrage condition and the local no-arbitrage condition are equivalent.

The discounted value of the portfolio at time n can be expressed again due to Formula (2.3) as

$$\tilde{V}_N^\psi(v) = \tilde{V}_n^\psi(v) + \sum_{m=n+1}^{N} \psi_m^\top \cdot \Delta\tilde{S}_m.$$

Now let us define the expected utility and the maximal expected utility of the terminal portfolio value given the stock price s and the discounted portfolio value w at time n:

$$\begin{aligned} f_n(w, s, \psi) &:= E\left(U(w + \sum_{m=n+1}^{N} \psi_m^\top \cdot \Delta\tilde{S}_m, S_N) | S_n = s \right), \\ F_n(w, s) &:= \sup_\psi f_n(w, s, \psi). \end{aligned} \tag{3.3}$$

Let us notice, that if s is replaced by a random variable S_n, then $f_n(w, S_n, \psi)$ becomes a random variable, and in the definition of F_n we should put ess sup instead of sup. Of course, in the case when S_n is given as a fixed s, ess sup

reduces to an ordinary sup.
It is obvious that for all ψ,

$$f_N(w,s,\psi) = F_N(w,s) = U(w,s).$$

The situation is described by a Markov decision model with a finite horizon N and the terminal reward given by the utility function. One can remark that f_n and F_n have the following interpretations – as a conditional expected reward and a maximal conditional expected reward. In the following proposition we formulate the general results for Markov Decision Processes.

Proposition 3.2 *If for $n = 0, \ldots, N{-}1$, we define a reward operator*

$$\Lambda_n^{\psi_{n+1}} g(w,s) := E\left(g(w + \psi_{n+1}^\top \cdot \Delta\tilde{S}_{n+1}, S_{n+1}) | S_n = s\right),$$

where g can be taken as

$$(w,s) \mapsto f_{n+1}(w,s,\psi)$$

or

$$(w,s) \mapsto F_{n+1}(w,s),$$

then

$$f_n(w,s,\psi) = \Lambda_n^{\psi_{n+1}} f_{n+1}(w,s,\psi) \tag{3.4}$$

and

$$F_n(w,s) = \sup_{\psi_{n+1}} \Lambda_n^{\psi_{n+1}} F_{n+1}(w,s). \tag{3.5}$$

Proof.
Let us take fixed n. We start with a proof of (3.4):

$$\Lambda_n^{\psi_{n+1}} f_{n+1}(w,S_n,\psi) = E\left(f_{n+1}(w + \psi_{n+1}^\top \cdot \Delta\tilde{S}_{n+1}, S_{n+1}, \psi) | S_n\right) =$$

$$= E\left(E\left(U(w + \psi_{n+1}^\top \cdot \Delta\tilde{S}_{n+1} + \sum_{m=n+2}^{N} \psi_m^\top \cdot \Delta\tilde{S}_m, S_N) | S_{n+1}\right) | S_n\right) =$$

$$= E\left(U(w + \sum_{m=n+1}^{N} \psi_m^\top \cdot \Delta\tilde{S}_m, S_N) | S_n\right) = f_n(w,S_n,\psi).$$

Taking $S_n = s$ we obtain the first assertion.
We continue with a proof of (3.5).

$$\sup_{\psi_{n+1}} \Lambda_n^{\psi_{n+1}} F_{n+1}(w,s) =$$

$$= \sup_{\psi_{n+1}} E\left(F_{n+1}(w + \psi_{n+1}^\top \cdot \Delta\tilde{S}_{n+1}, S_{n+1})|S_n = s\right) =$$

$$= \sup_{\psi_{n+1}} E\left(\operatorname{ess\,sup}_{\psi} f_{n+1}(w + \psi_{n+1}^\top \cdot \Delta\tilde{S}_{n+1}, S_{n+1}, \psi)|S_n = s\right) =$$

$$^{(\star)} = \sup_{\psi} \Lambda_n^{\psi_{n+1}} f_{n+1}(w, s, \psi) =$$

$$= \sup_{\psi} f_n(w, S_n, \psi) = F_n(w, s)$$

A full justification of equality $(\star)$ involves more technical computations and detailed approach. We sketch only the intuition. First of all, you have to realize that $\psi = (\psi_1, \ldots, \psi_N)$, i.e., the policy ψ consists of decision rules ψ_n that tell you what actions to take at each moment n. Therefore, an optimal ψ can be obtained by maximizing each decision rule, or by maximizing some decision rules and then maximizing with respect to the complete policy. Thus, $\sup_\psi f_n = \sup_{\psi_{n+1}} \Lambda_n^{\psi_{n+1}} \sup_\psi f_{n+1}$. In fact, in the proof we do the other way around. f_{n+1} depends directly on ψ_{n+1}, and then continues optimally. Therefore, in that case, you can also do $\sup_\psi f_n = \sup_{\psi_{n+1}} \Lambda_n^{\psi_{n+1}} \sup_\psi f_{n+1}$, because when $g := \Lambda_n^{\psi_{n+1}} f_{n+1}$, then $\sup_\psi g$ only depends on ψ_{n+2}, ψ_{n+3}, etc. □

In the Markov Decision Processes theory, Formula (3.4) is called the *fundamental equation* and Formula (3.5) is known as the *optimality equation*. The reward operator can be expressed by the transition probabilities as follows

$$\Lambda_n^{\psi_{n+1}} g(w, s) = \sum_{q \in \mathfrak{S}_{n+1}} p_{sq}(n) \cdot g(w + \psi_{n+1}^\top \cdot (\tilde{q} - \tilde{s}), q).$$

As a consequence, $\Lambda_n^{\psi_{n+1}} g(w, s)$ can be expressed by an ordinary expectation and for fixed s we can apply the results of the previous section.

Lemma 3.3 *Let g be as above, then for every $n = 0, \ldots, N{-}1$,*

(i) $(w, \psi_{n+1}) \mapsto \Lambda_n^{\psi_{n+1}} g(w, s)$ *is continuous,*

(ii) $\psi_{n+1} \mapsto \Lambda_n^{\psi_{n+1}} g(w, s)$ *attains the maximum on $\mathbb{R}^K$ for each $w \in \mathbb{R}$ and the maximum point $\psi_{n+1}^* := \psi_{n+1}^*(w, s)$ is unique,*

(iii) $G(w, s) := \max_{\psi_{n+1}} \Lambda_n^{\psi_{n+1}} g(w, s)$ *is strictly concave in w,*

(iv) *Furthermore, $w \mapsto F_n(w, s)$ is differentiable with derivative $F_n'(w, s)$ for $n = 0, \ldots, N$, strictly concave, and $F_N'(-\infty, s) > 0, F_N'(\infty, s) \leq 0$.*

Proof.

• First we prove the theorem for $g = f_n$.

(i) Continuity follows from the definition of f_n.

(ii) $f_n(\cdot, s, \psi)$ is strictly concave, differentiable (see the proof of Lemma 3.2(i)), with $f_n'(-\infty, s, \psi) > 0$ and $f_n'(\infty, s, \psi) \leq 0$. We use (3.4), and from now on we can follow the proof of Lemma 3.1(ii) and (iv) with appropriate modifications.

(iii) This part is obvious.

(iv) Differentiability is given by analogy to Theorem 3.1 We will show that $F_n(\cdot, s)$ is strictly concave for each n using backward induction. We know that $F_N = U$, hence we get strict concavity and $F_n'(-\infty, s) > 0, F_n'(\infty, s) \leq 0$. Now by using (3.5) we have that every F_n is strictly concave and $F_n'(-\infty, s) > 0, F_n'(\infty, s) \leq 0$.

• Let $g = F_n$

(i) It follows straightforwardly from the definition.

(ii) Thanks to (iv), this part is proved as Lemma 3.1 (ii) and (iv).

(iii) We can exploit the proof of Lemma 3.1 (iv).

□

Let us summarize the main conclusions resulted from the theorem.

Corollary 3.2 *There exists a unique policy $\psi^* = (\psi_n^*) := (\psi_n^*(w, s))$ such that for $n = 0, \ldots, N$–1:*

$$\Lambda_n^{\psi_{n+1}^*} F_{n+1}(w, s) = \max_{\psi_{n+1}} \Lambda_n^{\psi_{n+1}} F_{n+1}(w, s) = F_n(w, s),$$
$$F_n(w, s) = f_n(w, s, \psi^*).$$

Lemma 3.4 *Let $v \in \mathbb{R}$ and ψ^* be the optimal policy of Corollary 3.2. Then*

$$F_0'(v) = E\left(U'(\tilde{V}_N^{\psi^*}(v), S_N)\right).$$

Proof.

We want to prove by induction that

$$F_0'(v) = E\left(F_n'(\tilde{V}_n^{\psi^*}(v), S_n)\right).$$

Then, in the case $n = N$, we will obtain the result.

• $n = 1$

We have

$$F_0(v) = \max_{\psi_1} E\left(F_1(v + \psi_1^\top \cdot \Delta\tilde{S}_1, S_1)\right),$$

and the maximum is attained in $\psi_1^* := \psi_1^*(v, s_0)$. By Lemma 3.2 (i) and Theorem 3.1 we get

$$F_0'(v) = f_0'(v, \psi_1^*) = E\left(F_1'(v + \psi_1^{*\top} \cdot \Delta\tilde{S}_1, S_1)\right) = E\left(F_1'(\tilde{V}_1^{\psi^*}(v), S_1)\right).$$

Thus, the first induction step is shown to be valid. Note, that for $N = 1$ the result of the theorem holds. By induction with respect to n, we continue the proof for $N > 1$.

• $n - 1 \Rightarrow n$

Let us take fixed $s \in \mathfrak{S}_n$, then

$$F_{n-1}(w, s) = \max_{\psi_n} E\left(F_n(w + \psi_n^\top \cdot \Delta\tilde{S}_n, S_n)|S_{n-1} = s\right),$$

with the maximum point $\psi_{n-1}^* := \psi_{n-1}^*(v, s)$. As above we get

$$F_{n-1}'(w, s) = E\left(F_n'(w + \psi_n^{*\top} \cdot \Delta\tilde{S}_n, S_n)|S_{n-1} = s\right).$$

By (3.1), we have $\tilde{V}_n^{\psi^*}(v) = \tilde{V}_{n-1}^{\psi^*}(v) + \psi_n^{*\top} \cdot \Delta\tilde{S}_n$, and

$$\begin{aligned} F_{n-1}'(\tilde{V}_{n-1}^{\psi^*}(v), S_{n-1}) &= \\ &= E\left(F_n'(\tilde{V}_{n-1}^{\psi^*}(v) + \psi_n^{*\top} \cdot \Delta\tilde{S}_n, S_n)|S_{n-1}\right) = \\ &= E\left(F_n'(\tilde{V}_n^{\psi^*}(v), S_n)|S_{n-1}\right). \end{aligned} \tag{3.6}$$

Now, by assumption for $n - 1$, we obtain

$$\begin{aligned} F_0'(v) = E\left(F_n'(\tilde{V}_{n-1}^{\psi^*}(v), S_{n-1}\right) &= \\ E\left(E\left(F_n'(\tilde{V}_n^{\psi^*}(v), S_n)|S_{n-1}\right)\right) &= E\left(F_n'(\tilde{V}_n^{\psi^*}(v), S_n)\right). \end{aligned}$$

□

Theorem 3.3 *Let U' be positive, or more generally, $U'(\tilde{V}_N^{\psi^*}(v), S_N)$ be positive, in which ψ^* is the optimal policy of Corollary 3.2 and $v \in \mathbb{R}$. Let the process Z_n be defined as follows:*

$$Z_n := F_n'(\tilde{V}_n^{\psi^*}(v), S_n), \ n = 0, \ldots, N.$$

In particular,

$$Z_0 = V_0'(v),$$
$$Z_N = U'(\tilde{V}_N^{\psi^*}(v), S_N).$$

Then

(i) *a martingale measure* $\mathbb{P}^U$ *is obtained on* Ω *by*

$$\mathbb{P}^U(\{\omega\}) = \frac{Z_N(\omega)}{Z_0}\,\mathbb{P}(\{\omega\}),$$

(ii) Z_n/Z_0 *is a martingale under* $\mathbb{P}$ *and is called the density process of* $d\mathbb{P}^U/d\mathbb{P}$*,*

(iii) *the following holds*

$$E_{\mathbb{P}^U}\left(\rho(I_1,\ldots,S_N)|\mathcal{F}_{n-1}\right) = \frac{1}{Z_{n-1}} E\left(\rho(I_1,\ldots,S_N)\cdot Z_n|\mathcal{F}_{n-1}\right),$$

for any function ρ*.*

Proof.
(ii) See the proof of Lemma 3.3, Formula (3.6).
(iii) It can be obtained as a result of the following proposition, namely:
($\star$) Let P and Q be probabilistic measures on $(\Omega,\mathcal{F})$, such that there exists a density $dQ/dP = Z > 0$. Let $\mathcal{G}\subset\mathcal{F}$ and X be a Q–integrable random variable. Then,

$$E_Q(X|\mathcal{G}) = \frac{E_P(XZ|\mathcal{G})}{E_P(Z|\mathcal{G})}.$$

For a proof of the proposition we need to show, that for some $A\in\mathcal{G}$ the following holds:

$$\int_A XZdP = \int_A E_Q(X|\mathcal{G})E_P(Z|\mathcal{G})dP.$$

We can derive

$$\int_A XZdP = \int_A XdQ = \int_A E_Q(X|\mathcal{G})dQ = \int_A E_Q(X|\mathcal{G})ZdP =$$
$$= \int_A E_P(E_Q(X|\mathcal{G})Z|\mathcal{G})dP = \int_A E_Q(X|\mathcal{G})E_P(Z|\mathcal{G})dP.$$

Now, the assertion of the theorem is obtained by proposition ($\star$) applied to $P := \mathbb{P}$, $Q := \mathbb{P}^U$, and $Z = Z_n/Z_0$.
(i) Lemma 3.4 and the fact that U' is positive lead to the conclusion that $\mathbb{P}^U$ is a probability measure equivalent to $\mathbb{P}$. We get as in Theorem 3.2 that for $k = 1,\ldots,K$,

$$E\left(F_n'(w+\psi_n^{*\top}\cdot\Delta\tilde{S}_n, S_n)\cdot\Delta\tilde{S}_n^k|S_{n-1}=s\right) = 0, \qquad (3.7)$$

since ψ^* is the unique maximum point of the function

$$\psi_n \mapsto E\left(F_n'(w + \psi_n^\top \cdot \Delta\tilde{S}_n^k, S_n) \cdot \Delta\tilde{S}_n^k | S_{n-1} = s\right).$$

From (3.7) we have as in the proof of (3.6), that

$$E\left(Z_n \cdot \Delta\tilde{S}_n^k | \mathcal{F}_{n-1}\right) = 0, \; k = 1, \ldots, K,$$

and finally as a result of (iii) we obtain

$$E_{\mathbb{P}^U}\left(\Delta\tilde{S}_n^k | \mathcal{F}_{n-1}\right) = 0, \; k = 1, \ldots, K.$$

□

The theorem presented above proves The Fundamental Theorem of Asset Pricing. We assumed that the market was arbitrage-free and step by step we showed how to produce a martingale measure.

3.5 Hedging

Hedging is a strategy designed to reduce or cancel out the risk related to an investment. The seller of the option tries to minimize exposure to an unwanted risk appearing when the value of the payoff at expiration time N is positive. Then, he is obliged to pay money to the buyer. Of course, in the case of a complete market there exists only one fair price for a given contingent claim and the situation is apparent; the seller is capable of covering the future payoff completely by investment in the replicating portfolio using money received for the selling of the contract. The problem arises when we go into incomplete markets. *Superhedging*, defined to be a portfolio strategy such that it generates a payoff at least as high as that of the given contingent claim, eliminates the risk totally. However, very often it requires too much initial capital. If one decides to invest in hedging a portfolio with less money than necessary for superhedging, then the future financial commitment given by a contingent claim may not be covered completely. In such a situation we talk about the *shortfall risk*.

3.5.1 Superhedging

In order to formulate a mathematical background, we start with a given contingent claim X. We want to derive the initial value v^* defined as follows

$$v^* := \inf\{v \in \mathbb{R} : \text{ there exists strategy } \psi \text{ such that } V_N^\psi(v) \geq X \text{ a.s.}\}.$$

Thus v^* is the smallest amount of money that allows for superhedging X by investment in some self-financing portfolio. We stated that v^* equals $\overline{V}_0$ (see the previous chapter for a definition). Let us recall the notation $\tilde{X}$ for a discounted contingent claim and present the following theorem.

Theorem 3.4

$$v^* = \sup_{\tilde{\mathbb{P}} \in \mathcal{P}} E_{\tilde{\mathbb{P}}}(\tilde{X}).$$

Proof.
Here we carry out the proof only for the case $N = 1$.
The problem can be expressed as follows,

$$v^* = \inf\{v \in \mathbb{R} : \ \exists \psi_1 \in \mathbb{R}^K : \ v + \psi_1^\top \cdot \Delta \tilde{S}_1 \geq \tilde{X} \text{ a.s.}\}.$$

Since Ω is finite, we can use linear programming.
Let $\sharp\Omega = p$ and

$$x := \begin{bmatrix} v \\ \psi_1^1 \\ \vdots \\ \psi_1^K \end{bmatrix}, \ c := \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \in \mathbb{R}^{K+1},$$

$$A := \begin{bmatrix} 1 & \Delta\tilde{S}_1^1(\omega_1) & \cdots & \Delta\tilde{S}_1^K(\omega_1) \\ \vdots & \vdots & \ddots & \vdots \\ 1 & \Delta\tilde{S}_1^1(\omega_p) & \cdots & \Delta\tilde{S}_1^K(\omega_p) \end{bmatrix}, \ b := \begin{bmatrix} \tilde{X}(\omega_1) \\ \vdots \\ \tilde{X}(\omega_p) \end{bmatrix}.$$

The primal problem can now be formulated as

$$\begin{aligned} &\text{minimize } c^\top x \\ &\text{subject to } Ax \geq b. \end{aligned}$$

The equivalent dual problem is

$$\begin{aligned} &\text{maximize } b^\top y \\ &\text{subject to } A^\top y \geq c, \ y \geq 0. \end{aligned}$$

Now, taking $y := y(\omega)$ we can identify

$$\mathcal{Y} := \{y : A^\top y = c, y > 0, \omega \in \Omega\}$$

with a set of martingale measures. Obviously, $\{x : Ax \geq b\}$ is not empty. Let $\bar{\mathcal{Y}}$ be the closure of $\mathcal{Y}$. Then, $\mathcal{Y}$ and $\bar{\mathcal{Y}}$ are not empty since the no-arbitrage condition holds and we can apply the Fundamental Theorem 2.1. Now, by the use of the duality theorem of linear programming and the fact that $\mathcal{Y}$ is dense in $\bar{\mathcal{Y}}$ we obtain as a basis solution

$$v^* = \min_{Ax \geq b} c^\top x = \max_{y \in \bar{\mathcal{Y}}} \sum_{i=1}^{p} y(\omega_i)\tilde{X}(\omega_i) = \sup_{y \in \mathcal{Y}} \sum_{i=1}^{p} y(\omega_i)\tilde{X}(\omega_i) = \sup_{y \in \mathcal{Y}} E_y(\tilde{X})$$

□

3.5.2 Shortfall risk minimization

The *shortfall risk* and the *shortfall probability* are defined by the following expressions, respectively,

$$E\left((X - V_N^\psi(v))^+\right),$$
$$P\left((X - V_N^\psi(v)) > 0\right).$$

Analogously, we can consider the discounted shortfall risk.

Definition 3.3 *The* ***loss function*** *is defined to be a convex increasing continuously differentiable function* $l : \mathbb{R}^+ \mapsto \mathbb{R}^+$ *with* $l(0) = 0$.

The loss function describes the investor's attitude towards the shortfall. For a given contingent claim X such that $X \geq 0$ a.s., and initial capital v we want to minimize the loss associated with the discounted shortfall risk. In general, we aim to find

$$\inf_\psi E\left(l\big((\tilde{X} - \tilde{V}_N^\psi(v))^+\big)\right). \tag{3.8}$$

Let us briefly discuss a few remarks on the shortfall risk minimization approach. First note, that if l is regarded as $l(x) = x$ or $l(x) = 1_{\{x>0\}}$, then problem (3.8) corresponds to the minimization of the discounted shortfall risk or the shortfall probability. Furthermore, superhedging of the contingent claim fits also to this framework. We can write

$$v^* = \inf\left\{v : \inf_\psi E\left(l\big((\tilde{X} - \tilde{V}_N^\psi(v))^+\big)\right) = 0\right\},$$

since the condition

$$\inf_\psi E\left(l\big((\tilde{X} - \tilde{V}_N^\psi(v))^+\big)\right) = 0,$$

implies that there exists at least one strategy ψ such that $V_N^\psi(v) \geq X$. Finally, let us mention that the minimization of the loss function can be treated as a particular case of the utility maximization problem.

3.5.3 Dynamic Programming

Problem (3.8) can be solved by using dynamic programming. For the discounted value of the portfolio w and the stock price s given at every time n we have the following backward algorithm:

$$
\begin{aligned}
F_N(w,s) &:= l\big((\tilde{h}(s)-w)^+\big),\\
F_n(w,s) &:= \inf_{\psi_{n+1}} \Lambda_n^{\psi_{n+1}} F_{n+1}(w,s),
\end{aligned}
\tag{3.9}
$$

with

$$
\Lambda_n^{\psi_{n+1}} F_{n+1}(w,s) := E\left(F_{n+1}(w+\psi_{n+1}^{\top}\cdot \Delta\tilde{S}_{n+1}, S_{n+1})|S_n=s\right),
$$

and

$$
\tilde{h}(s) := \frac{h(s)}{B_N},\ s\in\mathfrak{S}_N.
$$

To justify the validity of (3.9), let us refer to the utility maximization. Some part of the theory presented in Section 3.4 may be adapted to the loss minimization problem. If we change the problem formulation and some definitions, in consequence it will turn out that a modified Proposition 3.2 works in the case of (3.8). Similarly as in the previous section, the reward operator can be expressed by

$$
\Lambda_n^{\psi_{n+1}} F_{n+1}(w,s) = \sum_{q\in\mathfrak{S}_{n+1}} p_{sq}(n)\cdot F_{n+1}(w+\psi_{n+1}^{\top}\cdot(\tilde{q}-\tilde{s}),q),\ s\in\mathfrak{S}_n.
$$

The last step of the dynamic programming algorithm leads to the solution of the minimization problem, namely

$$
\inf_{\psi} E\left(l\big((\tilde{X}-\tilde{V}_N^{\psi}(v))^+\big)\right) = F_0(v,s_0).
$$

3.6 Conclusions

This chapter reached the theoretical solutions for the option pricing and hedging problems. It was shown, how dynamic programming can be exploited at the specific setup of the financial market. Let us notice, that in the case of the utility function such as it was defined, the maximum point can be easily determined (with using a derivative), while studying the shortfall risk minimization is more complicated. To clarify the point, there does not exist a unique and explicit way that says how to determine the infimum in the iteration steps. Depending on the regarded stock price process and a loss function, every specific model needs a distinct treatment.

Chapter 4

Models with restricted information

4.1 Introduction

Although the main ideas of the previous chapter are correct, there is a technical problem for applicability to real life situations. We imposed a Markovian structure of the market. So far, we assumed that all information is given; but we can take into account Markovian models with *incomplete* or *partial information*. Some parameters may be hidden, e.g., the transition probabilities are not given. At first view, it seems that we are not able to carry out computations using the introduced algorithm. However, it is still possible to improve the presented methods in such a way, that they deal with incomplete information. In this chapter we look at some specific cases of market models. First, based on the multinomial model example, we build up the theory working on the principle that the stock price process is a homogeneous Markov chain. Later, we carry on with an example of a continuous-time market model. In fact, that model can be transformed and considered as a discrete-time one. The chapter ends with solutions to the pricing and hedging problems for the latter model, both with complete and incomplete information; we extend the methods obtained under the assumption, that S_n possesses a Markovian structure.

4.2 Examples of Markovian market models

4.2.1 Multinomial model

First, let us assume that $\Omega := \{\omega_1, \dots, \omega_M\}$. Moreover, we set $r_n = r$ for every n. The evolution of the stock price process is described by

$$S^k_{n+1} = S^k_n Z^k_n,\ k = 1, \dots, K,$$

in which $Z^k_1, \dots, Z^k_n$ is a sequence of independent and identically distributed random variables taking values on $\{z^1_k, \dots, z^M_k\}$ for $\omega_1, \dots, \omega_M$, respectively. The assumption

$$\text{for } k = 1, \dots, K : \min\{z^1_k, \dots, z^M_k\} \leq (1 + r) \leq \max\{z^1_k, \dots, z^M_k\},$$

makes the market arbitrage-free. If we denote

$$Z_n := (Z^1_n, \dots, Z^K_n),$$

and

$$z^m := (z^m_1, \dots, z^m_K),$$

then we write

$$p_m := P(\omega_m) = P(Z_n = z^m).$$

Now, the transition probabilities can be obtained as follows

$$p_{sq_m}(n) = P(S_{n+1} = q_m | S_n = s) = p_m,$$

with

$$q_m := (s^1 z^m_1, \dots, s^K z^m_K) \in \mathfrak{S}_{n+1},\ s \in \mathfrak{S}_n.$$

The transition probabilities do not depend on n. Thus, the multinomial model is an example in which the Markov chain S_n is homogeneous (see also Proposition 3.1). If the p_m are known, then dynamic programming works correctly. Otherwise, we need to improve the method. The Bayesian estimation can be used for our purposes.

Before going into the details we briefly sketch the intuition concerning unknown transition probabilities. In real life, some economic factors affect the stock prices and very often they may be not observable. This situation can impinge on transition probabilities. Later, we will explore this idea more precisely.

Bayesian estimation

To formalize the framework we begin with the following definitions.

Definition 4.1 *A random vector $(X_1, \ldots, X_M)$ is **multinomially distributed** if the probability function is given by*

$$P(X_1 = x_1, \ldots, X_M = x_M) = \begin{cases} \frac{n!}{x_1! \cdots x_M!} p_1^{x^1} \cdots p_M^{x^M}, & \text{when } \sum_{m=1}^{M} x_m = n, \\ 0, & \text{otherwise,} \end{cases}$$

for non-negative integers $x_1, \ldots, x_K$ and a probability vector $(p_1, \ldots, p_M)$.

We can give an interpretation of the multinomial distribution. Imagine there are n independent trials, each trial results in one of the fixed finite number M of possible outcomes with probabilities $p_1, \ldots, p_M$. We use a random variable X_m to indicate the number of times outcome number m was observed over the n trials.

Definition 4.2 *A **Dirichlet distribution** of order M is given by a probability density function f defined for a vector $p = (p_1, \ldots, p_M)$, $\sum_{m=1}^{M} p_m = 1$ such that*

$$f(p) \propto \prod_{m=1}^{M} p_m^{\alpha_m - 1},$$

where $\alpha = (\alpha_1, \ldots, \alpha_M)$ is a parameter vector with $\alpha_m > 0$.

Corollary 4.1 *Let a random vector $(X_1, \ldots, X_M)$ possess a Dirichlet distribution with parameter α. Then, we have*

$$E(X_m) = \frac{\alpha_m}{|\,\alpha|},$$

with $|\,\alpha| := \sum_{m=1}^{M} \alpha_m$.

Proposition 4.1 *Consider a multinomially distributed random vector $\beta|p = (\beta_1, \ldots, \beta_M)$, where β_m is the number of occurrences of z_m in a sample of n points from the discrete distribution of Z on $\{z_1, \ldots, z_M\}$ defined by p, such that the a-priori distribution of p is given as a Dirichlet distribution of order M with parameter $\alpha = (\alpha_1, \ldots, \alpha_m)$. Then the a-posteriori distribution $p|\beta$ for $\beta = (\beta_1, \ldots, \beta_M)$ has a Dirichlet distribution of order M with parameter $\alpha + \beta = (\alpha_1 + \beta_1, \ldots, \alpha_M + \beta_M)$.*

Proof
Let us denote
- $f(p)$ – the prior density of p,
- $f(p|\beta)$ – the posterior density after n observations,
- γ – the probability function of $\beta|p$.
From the Bayes Theorem we obtain

$$f(p|\beta) = \frac{\gamma(\beta|p)f(p)}{\int_{\mathbb{R}^M} \gamma(\beta|p)f(p)dp}$$

$$\propto \prod_{m=1}^{M} p_m^{\beta_m} \prod_{m=1}^{M} p_m^{\alpha_m - 1} = \prod_{m=1}^{M} p_m^{\alpha_m + \beta_m - 1}.$$

□

We showed that the Dirichlet distribution is a conjugate to the multinomial distribution. This result guarantees, that if we only know the prior distribution of p, in the next steps we can update the parameters using the information that becomes successively available. In particular, let the prior density be given by

$$f_0(p) \propto \prod_{m=1}^{M} p_m^{\alpha_0^m - 1},$$

where $\alpha_0 = (\alpha_0^1, \ldots, \alpha_0^M)$ is a parameter. Then the posterior density function at time n is as follows

$$f_n(p) \propto \prod_{m=1}^{M} p_m^{\alpha_n^m - 1},$$

with

$$\alpha_n^m = \alpha_0^m + \beta_n^m \text{ with } \beta_n^m = \sum_{j=0}^{n-1} \mathbf{1}_{\{Z_j = z^m\}}.$$

Now, in the dynamic programming algorithm we replace an unknown p_m by $\frac{\alpha_n^m}{|\alpha_n|}$ at nth step.

Corollary 4.2 *Taking into consideration Proposition 3.1, we note as the obvious conclusion, that the Bayesian estimation presented above deals with every homogeneous Markov chain S_n with incomplete information given in the form of unknown transition probabilities.*

4.2.2 Continuous stock price process driven by a homogeneous Markov chain

We assume that the interest rate is constant and equals r. The stock price dynamics evolve according to continuous-time processes described by the set of equations as follows

$$dS_t^k = S_t^k(a_k(X_t)dt + \sum_{j=1}^{d} \sigma_{kj}(X_t)dW_t^j),\ S_0^k \in \mathbb{R}_+,\ k = 1, \dots, K,$$

where:
– X_t is a homogeneous and finite space Markov chain with time step 1, the state space $\{x_1, \dots x_J\}$ and the given transition probability matrix $P = \{p_{ij}\}_{i,j=1,\dots,J}$,
– $a_k(\cdot) : \mathbb{R} \mapsto \mathbb{R}$, $\sigma_k(\cdot) : \mathbb{R} \mapsto \mathbb{R}^d$ for $k = 1, \dots, K$, are proper functions representing appreciation rates and volatilities, respectively,
– $W_t = (W_t^1, \dots, W_t^d)$ is a d-dimensional standard Brownian motion[1] independent of X_t,
– X_t may be not observable.
Thus, the dynamics of S_t at moment n at which we can observe stock prices can be written as

$$S_{n+1}^k = S_n^k e^{a_k(X_n) - \frac{1}{2}\sum_{j=1}^d \sigma_{kj}^2(X_n) + \sum_{j=1}^d \sigma_{kj}(X_n)(W_{n+1}^j - W_n^j)}.$$

The above result can be derived by the multi-dimensional Ito's formula. Let us rewrite the above

$$S_{n+1}^k = S_n^k Z_n^k, \tag{4.1}$$

where, conditionally on X_n, Z_n^k are log-normally distributed, which means:

$$\ln Z_n^k \sim \mathcal{N}\left(a_k(X_n) - \frac{1}{2}\sum_{j=1}^d \sigma_{kj}^2(X_n), \sum_{j=1}^d \sigma_{kj}^2(X_n)\right) =: \mathcal{N}(\mu_k(X_n), \Sigma_k(X_n)).$$

We remark, that $Z_1^k, \dots, Z_N^k$ is a sequence of independent random variables, if only $X_1, \dots, X_n$ are independent. It is an immediate result that follows from the definition of the multi-dimensional Brownian motion, the fact that increments of Brownian motion are independent, and Theorem A3 (see Appendix). Furthermore, one may notice obviously, that S_n is a Markov process then.

[1] The definition says that W_i are independent one-dimensional Brownian motions.

Reduction to a discrete-time model

Let us present, how to adjust the above model to the framework introduced in the previous chapter. In the first place, the procedure requires transferring the continuous-time model into a discrete counterpart. To discretize the process Z_n^k, we can take a finite partition of $\mathbb{R}^+$ and for each interval choose a representative element. Let us expand on that idea. For $k = 1, \ldots, K$, we consider $\{A_m\}_{m=1,\ldots,M}$ such that

$$\bigcup_m A_m = \mathbb{R}^+, \text{ and } A_m \cap A_{m'} = \emptyset \text{ for } m \neq m'.$$

Now, for every m, one should choose[2]

$$z_m^k \in A_m.$$

Discretized random variables Z_n^k take the values in the sets $\{z_1^k, \ldots, z_M^k\}$, for $k = 1, \ldots, K$. If we declare

$$Z_n := (Z_n^1, \ldots, Z_n^K),$$
$$z^m := (z_1^m, \ldots, z_K^m),$$

then we can define

$$p_m(X_n) := P(Z_n = z^m | X_n) = P(\ln Z_n = \ln z_m | X_n) = \int_{\mathcal{A}_m} f_n(z) dz, \quad (4.2)$$

with

$$\mathcal{A}_m = \underbrace{A_m \times \ldots \times A_m}_{M},$$

and the joint density function f_n, which is conditional upon X_n. The following proposition clarifies all details.

Proposition 4.2 *For every $n = 1, \ldots, N$, the random vector Z_n is, conditionally on X_n, K-variate log-normally distributed, i.e.,*

$$\ln Z_n = (\ln Z_n^1, \ldots, \ln Z_n^K) \sim \mathcal{N}_K(\mu(X_n), \Sigma(X_n)),$$

with

$$\mu(X_n) := \begin{bmatrix} \mu_1(X_n) \\ \vdots \\ \mu_K(X_n) \end{bmatrix}, \ \Sigma(X_n) := \begin{bmatrix} \Sigma_{1,1}(X_n) & \cdots & \Sigma_{1,K}(X_n) \\ \vdots & \ddots & \vdots \\ \Sigma_{K,1}(X_n) & \cdots & \Sigma_{K,K}(X_n) \end{bmatrix},$$

where $\Sigma_{k,l}(X_n) = \sum_{j=1}^d \sigma_{kj}\sigma_{lj}(X_n)$. Moreover, the foregoing distribution determines the density function f_n uniquely.

[2]This selection should take into consideration a validity of the no-arbitrage condition.

Proof.
With using the introduced notation we can write

$$\ln Z_n^k = \mu_k(X_n) + \sum_{j=1}^{d} \sigma_{kj}(X_n)\Delta W_{n+1}^j.$$

Since ΔW_{n+1}^j are independent for $j = 1, \ldots, d$, we know that the joint probability distribution of $(\Delta W_n^1, \ldots, \Delta W_n^d)$ is a d-variate normal one. Thus, we can apply Lemma A4 (see Appendix) and the following holds

$$a^{\top} \cdot \Delta W_{n+1}^j \ \text{ is normally distributed for every } a \in \mathbb{R}^d. \tag{4.3}$$

Now, let us take $b \in \mathbb{R}^K$. Again, according to Lemma A4 it suffices to show that

$$b^{\top} \cdot \ln Z_n \ \text{ is normally distributed,}$$

Continuing the proof,

$$b^{\top} \cdot \ln Z_n = \sum_{k=1}^{K} b_k \ln Z_n^k =$$

$$= \sum_{k=1}^{K} b_k \cdot \left(\mu_k(X_n) + \sum_{j=1}^{d} \sigma_{kj}(X_n)\Delta W_{n+1}^j \right),$$

and now (4.3) implies that $b^{\top} \cdot \ln Z_n$ is normally distributed. Finally, let us determine the covariance matrix $\Sigma(X_n)$.

$$\Sigma_{k,l}(X_n) = \mathrm{Cov}(\ln Z_n^k, \ln Z_n^l) = E(\ln Z_n^k \ln Z_n^l) - E(\ln Z_n^k)E(\ln Z_n^l).$$

We derive $E(\ln Z_n^k \ln Z_n^l)$:

$$E\left(\left(\mu_k(X_n) + \sum_{j=1}^{d} \sigma_{kj}(X_n)\Delta W_{n+1}^j\right)\left(\mu_l(X_n) + \sum_{j=1}^{d} \sigma_{lj}(X_n)\Delta W_{n+1}^j\right) \right) =$$

$$= \mu_k(X_n)\mu_l(X_n) + \sum_{j=1}^{d} \sigma_{kj}\sigma_{lj}(X_n) E\big((\Delta W_{n+1}^j)^2\big) =$$

$$= \mu_k(X_n)\mu_l(X_n) + \sum_{j=1}^{d} \sigma_{kj}\sigma_{lj}(X_n).$$

Hence, the covariance equals $\sum_{j=1}^{d} \sigma_{kj}\sigma_{lj}(X_n)$.

□

The proposition provides, that the probabilities (4.2) are defined correctly[3]. Furthermore, we point out, that S_n does not remain a Markov chain.

[3]Usually, marginal distributions do not determine a joint distribution.

Dynamic programming

Process S_n is conditionally distributed, its values depend on X_n. If we define a new process $I_n := (S_n, X_n)$, then it turns out that I_n becomes a homogeneous Markov chain. This fact results from the following computations:

First, we denote
$$\frac{s_{n+1}}{s_n} := \left(\frac{s^1_{n+1}}{s^1_n}, \ldots, \frac{s^K_{n+1}}{s^K_n}\right),$$
and assume that
$$\frac{s_{n+1}}{s_n} = z^m.$$
Then,
$$P(I_{n+1} = i_{n+1} | I_n = i_n, \ldots, I_0 = i_0) =$$
$$= \frac{P(I_{n+1} = i_{n+1}, I_n = i_n, \ldots, I_0 = i_0)}{P(I_n = i_n, \ldots, I_0 = i_0)} =$$
$$= \frac{P\Big((S_{n+1} = s_{n+1}, X_{n+1} = x_{n+1}), (S_n = s_n, X_n = x_n), \ldots, (S_0 = s_0, X_0 = x_0)\Big)}{P\Big((S_n = s_n, X_n = x_n), \ldots, (S_0 = s_0, X_0 = x_0)\Big)} =$$
$$= \frac{P(S_n Z_n = s_{n+1}, S_n = s_n, \ldots, S_0 = s_0, X_{n+1} = x_{n+1}, X_n = x_n, \ldots, X_0 = x_0)}{P(S_n = s_n, \ldots, S_0 = s_0, X_n = x_n, \ldots, X_0 = x_0)} =$$
$$= \frac{P(Z_n = \frac{s_{n+1}}{s_n}, S_n = s_n, \ldots, S_0 = s_0 | X_{n+1} = x_{n+1}, X_n = x_n, \ldots, X_0 = x_0)}{P(S_n = s_n, \ldots, S_0 = s_0 | X_n = x_n, \ldots, X_0 = x_0)}$$
$$\cdot \frac{P(X_{n+1} = x_{n+1}, X_n = x_n, \ldots, X_0 = x_0)}{P(X_n = x_n, \ldots, X_0 = x_0)} = \ldots$$
Let us notice, that if one knows values of $X_1, \ldots, X_{n+1}$, then it implies an independence of random variables $Z_1, \ldots, Z_N$. In combination with Formula (4.1), we can easily conclude that Z_{n+1} is independent of $S_0, \ldots, S_n$ given $X_{n+1} = x_{n+1}, \ldots, X_1 = x_1$. Hence, we carry on with the computations as follows
$$\ldots = \frac{P(S_n = s_n, \ldots, S_0 = s_0 | X_{n+1} = x_{n+1}, X_n = x_n, \ldots, X_0 = x_0)}{P(S_n = s_n, \ldots, S_0 = s_0 | X_n = x_n, \ldots, X_0 = x_0)}$$
$$\cdot P(Z_n = z^m | X_{n+1} = x_{n+1}, X_n = x_n, \ldots, X_0 = x_0)$$
$$\cdot P(X_{n+1} = x_{n+1} | X_n = x_n, \ldots, X_0 = x_0) =$$
$$= \frac{P(S_n = s_n, \ldots, S_0 = s_0 | X_n = x_n, \ldots, X_0 = x_0)}{P(S_n = s_n, \ldots, S_0 = s_0 | X_n = x_n, \ldots, X_0 = x_0)}$$

$$\cdot P(Z_n = z^m | X_{n+1} = x_{n+1}) P(X_{n+1} = x_{n+1} | X_n = x_n) = \dots$$

Let $x_{n+1} = x_j, x_n = x_i$. Then, the above equality can be written as

$$\dots = p_m(x_j) p_{ij}.$$

By a similar argument we have that

$$P(I_{n+1} = i_{n+1} | I_n = i_n) = p_m(x_j) p_{ij}.$$

The transition probabilities $p_m(x_j)p_{ij}$ do not depend on n, hence we get that I_n is a homogeneous Markov chain.

Now, let us refer to dynamic programming. At n we have additional information about the market, namely $X_n = x$. If we redefine the utility functions in (3.3) and the reward operator as follows

$$\begin{aligned}
f_n(w, (s, x), \psi) &:= E\left(U(w + \sum_{m=n+1}^{N} \psi_m^\top \cdot \Delta \tilde{S}_m, S_N) | S_n = s, X_n = x \right), \\
F_n(w, (s, x)) &:= \sup_{\psi} f_n(w, (s, x), \psi), \\
\Lambda_n^{\psi_{n+1}} g(w, (s, x)) &:= E\left(g(w + \psi_{n+1}^\top \cdot \Delta \tilde{S}_{n+1}, (S_{n+1}, X_{n+1})) | S_n = s, X_n = x \right),
\end{aligned}$$

and take σ-fields $\mathcal{F}_n := \sigma(I_0, \dots, I_n)$, $n = 0, \dots, N$, then all results of Chapter 3 hold. Since the utility function and the loss function depend only on S_N, we can express the reward operator by using transition probabilities in the following way

$$\Lambda_{N-1}^{\psi_N} F_N(w, s) = \sum_{m=1}^{M} p_m(x) F_N(w + \psi_{n+1}^\top \cdot (\tilde{q}_m - \tilde{s}), (q_m, x_j)),$$

with

$$q_m := (s^1 z_1^m, \dots, s^K z_K^m) \in \mathfrak{S}_N, \ s \in \mathfrak{S}_{N-1},$$

and for $n = 0, \dots, N-2$, as

$$\Lambda_n^{\psi_{n+1}} F_{n+1}(w, s) = \sum_{j=1}^{J} \sum_{m=1}^{M} p_{xj} p_m(x) F_{n+1}(w + \psi_{n+1}^\top \cdot (\tilde{q}_m - \tilde{s}), (q_m, x_j)), \quad (4.4)$$

with

$$p_{xj} := P(X_{n+1} = x_j | X_n = x),$$

and

$$q_m := (s^1 z_1^m, \dots, s^K z_K^m) \in \mathfrak{S}_{n+1}, \ s \in \mathfrak{S}_n.$$

Bayesian dynamic programming

If the process X_n is not observable, then we can estimate its value from the past and current observations of stock prices. One may repeat the procedure given by the dynamic programming algorithm, however it is necessary to replace some expressions in the reward operator (4.4) as follows

- We do not observe X_n given by x, thus instead of the values $p_m(x)$ we take $E(p_m(X_n)|\mathcal{F}_n^S)$, with $\mathcal{F}_n^S := \sigma(S_0, \ldots, S_n)$. Let us express the above expectation into a different form, namely

$$E(p_m(X_n)|\mathcal{F}_n^S) = E(\sum_{i=1}^{J} p_m(x_i) 1_{\{X_n = x_i\}}|\mathcal{F}_n^S) =$$

$$= \sum_{i=1}^{J} p_m(x_i) E(1_{\{X_n = x_i\}}|\mathcal{F}_n^S) = \sum_{i=1}^{J} p_m(x_i) P(X_n = x_i|\mathcal{F}_n^S).$$

  If one denotes

$$\Pi_n^i := P(X_n = x_i|\mathcal{F}_n^S),$$

  then

$$E(p_m(X_n)|\mathcal{F}_n^S) = \sum_{i=1}^{J} p_m(x_i) \Pi_n^i.$$

- X_n is replaced by a random vector $\Pi_n := (\Pi_n^1, \ldots, \Pi_n^J)$. One can remark, that Π_n forms probability vectors. To determine such vectors, it suffices to specify $J-1$ of its components.

- Instead of the transition probabilities p_{ij} for the process X_n, we take into account transition probabilities $p_{\pi_{n-1}\pi_n}$ for Π_n, where π_{n-1}, π_n are given values of Π_{n-1} and Π_n, respectively. The following proposition and corollary clarify how to get a suitable formula expressing $p_{\pi_{n-1}\pi_n}$.

Proposition 4.3 (Recursive Bayes' formula) *Let us define*

$$\pi_n^j(m) := P(X_n = x_j|\Pi_{n-1} = \pi_{n-1}, Z_{n-1} = z^m),$$

for $m = 1, \ldots, M,\ j = 1, \ldots, J$. Then

$$\pi_n^j(m) = \frac{\sum_{i=1}^{J} \pi_{n-1}^i p_{ij} p_m(x_i)}{\sum_{i=1}^{J} \pi_{n-1}^i p_m(x_i)}.$$

Proof.

$$\pi_n^j(m) = P(X_n = x_j|\Pi_{n-1} = \pi_{n-1}, Z_{n-1} = z^m) =$$

$$= \frac{P(X_n = x_j, \Pi_{n-1} = \pi_{n-1}, Z_{n-1} = z^m)}{P(\Pi_{n-1} = \pi_{n-1}, Z_{n-1} = z^m)} =$$

$$= \frac{\sum_{i=1}^J \pi_{n-1}^i P(X_n = x_j, X_{n-1} = x_i, Z_{n-1} = z^m)}{\sum_{i,l=1}^J \pi_{n-1}^i P(X_n = x_l, X_{n-1} = x_i, Z_{n-1} = z^m)} =$$

$$= \frac{\sum_{i=1}^J \pi_{n-1}^i P(X_n = x_j, Z_{n-1} = z^m|X_{n-1} = x_i)}{\sum_{i,l=1}^J \pi_{n-1}^i P(X_n = x_l, Z_{n-1} = z^m|X_{n-1} = x_i)} =$$

$$= \frac{\sum_{i=1}^J \pi_{n-1}^i P(X_n = x_j|X_{n-1} = x_i)P(Z_{n-1} = z^m|X_{n-1} = x_i)}{\sum_{i,l=1}^J \pi_{n-1}^i P(X_n = x_l|X_{n-1} = x_i)P(Z_{n-1} = z^m|X_{n-1} = x_i)} =$$

$$= \frac{\sum_{i=1}^J \pi_{n-1}^i p_{ij} p_m(x_i)}{\sum_{i,l=1}^J \pi_{n-1}^i p_{il} p_m(x_i)} = \frac{\sum_{i=1}^J \pi_{n-1}^i p_{ij} p_m(x_i)}{\sum_{i=1}^J \pi_{n-1}^i p_m(x_i)}.$$

□

Corollary 4.3 *The only possible transitions at time n for the process Π_n are from π_{n-1} to $\pi_n(m) = (\pi_n^1(m), \ldots, \pi_n^J(m))$, $m = 1, \ldots, M$, with transition probabilities equal*

$$\sum_{i=1}^J \pi_{n-1}^i p_m(x_i).$$

Proof. We are at the moment n, and we want to specify possible values of Π_n given $\Pi_{n-1} = \pi_{n-1}$ and the proper transition probabilities . Let us note, that all history up to n is known, except the process X_n, which is not observable. Values of Π_n are determined by observations of the stock price process S_n, hence, by the value of Z_{n-1} (since $Z_{n-1} = S_n/S_{n-1}$). Now, realize that Π_n takes on M possible values, as we stated in the assertion of the corollary, with the transition probabilities as follows

$$P(Z_{n-1} = z^m|\Pi_{n-1} = \pi_{n-1}) = \sum_{i=1}^J \pi_{n-1}^i p_m(x_i).$$

□

4.3 Conclusions

The above examples emphasize very wide applications of the model that has been introduced in Chapter 3, and techniques exploiting dynamic programming to the portfolio optimization. First of all, the discrete-time models have their own value and right of existence, in particular, in connection with a huge computational computer power. The latter example shows, that the discrete-time models can be viewed as approximations to the continuous-time models. Moreover, if information concerning the stock price evolution is incomplete, in the meaning of hidden economic factors, then the Bayesian estimation makes dynamic programming still valid in the case when we consider market models described by a homogeneous Markov chain.

Chapter 5

Conclusions

Financial markets can be modeled by using two alternatives; either by using continuous-time models or discrete-time models. In this thesis we have considered the discrete-time setting. Moreover, we have assumed the underlying asset price process to take only a finite number of values. Such a family of discrete-time models is called tree models. The tree models are very powerful and play an important role in the sense that they are easily implementable. Of course, for larger time horizons the number of possibilities increases exponentially, leading to limits in the applicability of dynamic programming. In general, the problem is called the *curse of dimensionality*. Calculations become very time consuming or even impossible. However, there are some natural procedures for recombination and simplification of a tree. As a consequence, for more complex cases there exist approximation methods that allow to carry out computations at reasonable computer workloads. The need of research for tree models is also justified by the fact, that they approximate continuous-time models. For example, with suitable re-scaling of time and choices of the parameters the binomial model converges to the basic Black-Scholes model.

The discrete-time models describing the complete market are the most basic and easily solvable models. Under the no-arbitrage assumption, option pricing and hedging are reduced to one problem, namely, one needs to determine the unique martingale measure. This is done by finding the unique solution to the equations given by (2.4). Then, the fair price ensures simultaneously money necessary to replicate the future payoffs with certainty. If one tries to find more general and more realistic limiting models, the market usually turns out to be incomplete. From a variety of approaches that deal with the problem of option pricing and hedging in the case of incomplete markets, it seems to us that the utility based approach is the most reasonable. It produces the most optimal policies; intuitively, under this approach we take into consideration the investor's attitude toward risk related to the contract, while other alternative methods are mainly preference-free. Due to the Markovian structure of the financial market we can exploit dynamic programming to specify the optimal (in the sense of the above explanation) policy.

It has been recognized that there are needs for some models to better capture the price movements of the underlying securities. Indeed, in reality many factors influence the stock price behavior. It becomes desirable to modify the general model, where the stock price process is assumed to be Markovian. We can improve this model by incorporating market trends with other economic factors. Especially, for a longer horizon such an improvement seems to be more suitable; it is quite obvious that economic factors are more sensitive if we lengthen the duration time of the contract. Let us remark, that if changes of economic factors follow a Markov chain, then the general model, that we have introduced, can be extended and adjusted to this setting, still working and leading to the solution (the previous chapter has provided one example). For some reasons economic factors may be not observable. Fortunately, there exist various methods and techniques, which deal with this restriction. We have discussed one of them, which is based on a Bayesian estimation of the parameters and benefits when the underlying Markov chain is homogeneous.

Finally, let us point out, that there are several different directions in which our work could be continued and extended. One can base pricing and hedging on our model, when the financial market is considered with both fixed and proportional transaction costs. Continuing, it would be very interesting to see how other mathematical techniques work with hidden information about the market, in particular, in the case of an inhomogeneous underlying Markov chain. Next, in this thesis we have provided only theoretical solutions. For the specific models one can present step by step how to implement the backward algorithm, as a final result obtaining explicit solutions. Overall, there are no additional theoretical issues; just determining optimal policies by maximization or minimization the utility function or the loss function, respectively. In literature some particular models have already been solved[1], however, it is still desirable to examine other models with all details. This issue is very important for practical interest, especially if we aim for numerical solutions. Moreover, for practical purposes and applicability one can study particular discrete-time models that converge to continuous-time ones, and later assess the accuracy of the obtained results. Next, extensions may

[1] The shortfall risk minimization for the binomial case is treated in [10], while [11] discusses the trinomial model.

concern the curse of dimensionality, which appears for larger time horizons. Various approximation methods are still improved and developed. One might gather all of them, compare with respect to precision and advantages followed from computational simplifications, and later discuss applicability to specific market models.

Appendix

Lemma A1 Let $f : (a, b) \mapsto \mathbb{R}$ is concave, with $-\infty \leq a < b \leq \infty$. Let $c \in (a, b)$, then the functions:

$$\{x \in (a,b) : x < c\} \ni x \mapsto \frac{f(c) - f(x)}{c - x} \in \mathbb{R},$$

$$\{x \in (a,b) : x > c\} \ni x \mapsto \frac{f(x) - f(c)}{x - c} \in \mathbb{R},$$

are decreasing.

Theorem A2 Let E be a measure space and $\{f_n\}$ be a monotonically increasing sequence of non-negative measurable functions defined on E. Then the following holds

$$\int_E \lim_{n\to\infty} f_n \, d\mu = \lim_{n\to\infty} \int_E f_n \, d\mu.$$

Theorem A3 Let us assume that random variables

$$X_{1,1}, \ldots, X_{1,k_1}, X_{2,1}, \ldots, X_{2,k_2}, \ldots\ldots, X_{n,1}, \ldots, X_{n,k_n},$$

are independent. Then, the random variables

$$Y_j = \varphi_j(X_{j,1}, \ldots, X_{j,k_j}), \; j = 1, \ldots, n,$$

where φ_j are measurable functions such that Y_j are well defined, are independent.

Lemma A4 The random vector $X = (X_1, \ldots, X_d)$ is $\mathcal{N}_d(\mu, \Sigma)$-distributed if and only if $a^\top X$ is $\mathcal{N}_1(a^\top \mu, a^\top \Sigma a)$-distributed for every $a \in \mathbb{R}^d$.

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