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Master Thesis

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Quantitative Credit Risk Modeling: Ratings under Stochastic Time

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Abstract

In presented work we extensively studied Markov chains, especially homogeneous continuous–time Markov chains. We described methods how to estimate the generator in the case of partially observed data from Inamura (2006). Further we add minor extension to these methods. Then we moved to the stochastic time and modeled the continuous–time Markov chain under the stochastic time. In the case when the stochastic time evolution is modeled via Lévy subordination we derived conditions under which the original system, which is now subordinated, follows again the continuous–time Markov chain, but with different generator. This result appears useful in the situation when we have some model of the time evolution. We also gave an overview about current credit risk models. In the end of the thesis we suggested some directions of further research.

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1
Introduction

The credit risk became a big issue in the last few decades, but the need for credit risk measurement is here since people have started borrowing any goods that they needed (not just money). In our context the subject of debt is money and the lender is the bank, but the principle is same in any situation when the subject of debt is whatever else than money.

When the bank borrows some amount of money to a debtor, the debtor is obliged to return the full amount plus some compensation for borrowing the money. Of course, there always exists a danger that the debtor will fail to fulfill this obligation. Reasons for the failure can be financial distress of the debtor, fraud or something else. In those case the lender loses some fraction of his money or all of it. How big fraction he lose depend on the debtor's financial situation, willingness of returning money, bankruptcy costs, the obligation of debtor to other counterparty and on many other circumstances. The lender should be aware of the risk. If he acts rationally, he will lend money only in the case when the profit from lending the money is higher than the expected loss from the provided loan plus some risk premium for bearing the risk. Therefore he needs to estimate the probability distribution of the credit loss for the entire portfolio of loans.

After estimation of the credit loss distribution the bank would like to hedge against the risk. Here comes into play the rich variety of credit risk derivatives which transfer the credit risk to third counterparty. Since these credit derivatives provide some protection against the credit risk, the bank is obliged to pay for it. Hence the profit from providing loans is smaller but without a credit risk.

There are two points of view on credit risk models that are closely connected. The first point of view is estimating the expected loss under a real world probability measure that is useful for a bank asset–liability management. The second one is estimating the loss under a risk–neutral probability measure,[1] that is useful for a risk neutral pricing of credit derivatives.

The first approach to credit risk modelling is important for bank manager who should be interested in the following questions. What is the expected loss from a loan portfolio in next five years? What is the probability of default of a particular debtor? What is the correlation between particular loans? What is the VaR (CVaR)[2] of the loan portfolio? How the bank's loan portfolio dependents on the evolution of the economy? We will try to answer some of these questions in the presented work. We will refer to the this approach as to the portfolio models (the second approach are pricing models).

A different point of view is adapted by the trader who wants to enter into a contract with the bank and sell a protection against the credit risk. His main issue is, what price he should charge for the protection. Of course, the bank, as a protection buyer, is also interested in fair valuation of the derivative. Many models for the credit derivatives pricing are known. These models can be divided into two groups — structural models and reduced–form models. There also exist some hybrid models that try to integrate both, the structural and the reduced–form approach. Pricing models see the debt as a defaultable zero–coupon bond or as some structure

[1] For a definition of a risk–neutral measure and conditions for existence see Delbaen and Schachermayer (2006).

[2] Var – Value at Risk, CVaR - Conditional Value at Risk, for definition see Chapter 3.

build from it. Hence the main issue is how to price a defaultable zero–coupon bond. Roots of structural models go down to the work of Black and Scholes (1973) and Merton (1974), later extended by Black and Cox (1976) and by many others. Reduced–form models were introduced mainly by Jarrow and Turnbull (1995), Jarrow et al. (1997), and Lando (1998) and were extended by many others. One of the most widespread approach integrating both approaches together is the incomplete information approach started with Duffie and Lando (2001). A nice short introduction to pricing models is given in Giesecke (2004). All these pricing models try to explain spreads of defaultable zero–coupon bonds. For more details about credit risk derivative pricing we refer to Bielecki and Rutkowski (2002), Duffie and Singleton (2003), Schönbucher (2003), and Lando (2004). The pricing modeling and portfolio modeling can not be separated apart, but we are going to study mainly portfolio modeling issues and the pricing of credit derivatives will be mentioned just marginally. For portfolio models there is much less available literature. One of the most recent books is for example Bluhm et al. (2002).

There is a lot of problematic issues in the credit risk modeling. First of all, one needs to define a default. For many purposes a different definition is used. One possibility is the random time when a firm's value will drop bellow some default barrier; then we are dealing with a hitting time theory. A different possibility is a time when obliger is delayed in fulfilling his obligations for some given amount of time. Even if we have some definition of a default one needs to estimate the exposition at default[3] and the recovery at default.[4] For these estimations banks have very little data history. The lack of data is one the biggest problem in the credit risk modeling.

This work has the following structure. In the second chapter we give a necessary mathematical theory about tools used in the credit risk modeling, especially Markov chains. We study the behavior of the continuous–time Markov chain under a stochastic time, where the stochastic time evolution is modelled via a Lévy subordinator. We derived some maximum likelihood estimators in two cases, either if stochastic time is part of the model, or is given by parameters. The maximum likelihood estimator can be used only in the case of continuously observed data which is often not the case in the credit risk. Hence we study few methods which can be used in the situation of partial data and we extend one of them into stochastic time. In the third chapter we give an overview about common used models in the portfolio management and in the last forth chapter we do a real data study where we show the performance of our method. Introducing stochastic time evolution into the Markov chain modeling gives us a 35 % better fit to data. We think it is a very nice improvement. In the end of forth chapter we give an overview about future directions of our research.

[3]The height of the debt at time of default.

[4]The fraction of the debt which will be covered at default.

2
Preliminary Theory

In this chapter we recall known results from the theory of stochastic processes which we will need later. Nevertheless, some elementary probability concepts will be used implicitly and can be found in every introduction to probability such as Feller (1968), Karlin and Taylor (1975), Karlin and Taylor (1981), Shiryaev (1995), Kallenberg (1997), Chung (2001), and many others.

2.1 Stochastic Processes

Let us assume a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathsf{P})$. A *stochastic process* $(X_t, t \in T)$ is a family of random variables where index set $T \subset \mathbb{R}_+ := [0, \infty)$. In the case when $T = \mathbb{N}_0$, we rather talk about a random sequence denoted by $\{X_n\}_{n\in\mathbb{N}_0}$. An adapted process X_t with $\mathsf{E}\,|X_t| < \infty$, for every $t \in T$, is called

- *submartingale* if for every $s < t$ holds

$$\mathsf{E}\,[X_t|\mathcal{F}_s] \geq X_s,$$

- *supermartingale* if for every $s < t$ holds

$$\mathsf{E}\,[X_t|\mathcal{F}_s] \leq X_s,$$

- *martingale* if for every $s < t$ holds

$$\mathsf{E}\,[X_t|\mathcal{F}_s] = X_s.$$

Let N_t be a nonnegative right–continuous submartingale with $N_0 = 0$. Then from the Doob–Mayer decomposition (see Karatzas and Shreve (1991, Theorem 1.4.10) or Kallenberg (1997, Theorem 22.5)) there exists a unique right–continuous martingale M_t and a right–continuous nondecreasing predictable process A_t such that

$$N_t = M_t + A_t, \qquad t \geq 0,$$

with $A_0 = 0$. The process A_t is called the *compensator* of the process N_t. If A_t is differentiable it can be written in form

$$A_t = \int_0^t \lambda_s \,\mathrm{d}s,$$

where λ_s is a nonnegative right continuous predictable process. This form will be useful later.

If N_t has independent increments, then A_t is a deterministic function equal to expected value of N_t

$$A_t = \mathsf{E}\,N_t, \qquad t \geq 0. \tag{2.1}$$

2.1.1 Counting Processes

Let $\{T_n\}_{n\in\mathbb{N}}$ be an increasing sequence of random times with values in $(0,\infty)$ and

$$\mathsf{P}[T_n \xrightarrow{n\to\infty} \infty] = 1.$$

Then we can define an *associated counting process* N_t as

$$N_t = \sum_{n=1}^{\infty} \mathbf{1}_{\{T_n \le t\}}.$$

The counting process N_t is process that counts some events which have arrived up to time t. It starts at 0 and is nondecreasing with jumps of size 1 (the possibility of two events arrived at the same time is excluded). From that follows that it can take values only from $\mathbb{N}_0$.

Random times between event arrivals are called an *inter–arrival times* and are denoted by

$$\begin{aligned} \tau_1 &= T_1, \\ \tau_i &= T_i - T_{i-1}, \qquad i > 1, \end{aligned}$$

If inter–arrival times $\tau_1, \tau_2, \dots$ are independent identically distributed random variables with an exponential distribution with parameter λ, then N_t is called a *Poisson process* with intensity λ. Since inter–arrival times $\tau_1, \tau_2, \dots$ are independent exponentially distributed, N_t has independent stationary increments with the Poisson distribution

$$\mathsf{P}[N_t - N_s = k] = \mathsf{P}[N_{t-s} = k] = \frac{\big(\lambda(t-s)\big)^k}{k!} \mathrm{e}^{-\lambda(t-s)}, \qquad s < t.$$

Since a Poisson process has independent increments and the mean value is

$$\mathsf{E}\, N_t = \lambda t,$$

we conclude from (2.1) that the compensator of the process N_t is

$$A_t = \lambda t.$$

In a matter of fact it is not hard to prove that the only counting process with stationary independent increments is the Poisson process.

In general, the intensity parameter λ does not need to be constant. In the case when intensity is a time dependent function, we denote it by λ_t and speak about an *inhomogeneous Poisson process*. Increments of inhomogeneous Poisson process are not stationary anymore and their Poisson distribution is time dependent in following way

$$\mathsf{P}[N_t - N_s = k] = \frac{\Big(\int_s^t \lambda_u \,\mathrm{d}u\Big)^k}{k!} \mathrm{e}^{-\int_s^t \lambda_u \,\mathrm{d}u}, \qquad s < t.$$

We can go even further and allow λ_t to be a stochastic nonnegative process such that if we condition on a particular $\omega \in \Omega$, the counting process N_t with intensity $\lambda_t(\omega)$ becomes an inhomogeneous Poisson process. The process is called a *Cox process* or a *doubly stochastic Poisson process*. Then for $s < t$ holds

$$\mathsf{P}[N_t - N_s = k] = \mathsf{E}\left[\frac{\Big(\int_s^t \lambda_u \,\mathrm{d}u\Big)^k}{k!} \mathrm{e}^{-\int_s^t \lambda_u \,\mathrm{d}u}\right].$$

For more technical details about the Cox process see Grandell (1976) or Kallenberg (1997, Chapter 10).

2.1.2 Survival Analysis

Let us assume a nonnegative random variable $\tau : \Omega \to \mathbb{R}$ which has an interpretation of a random time before some particular event will happen. For convenience we can assume that $\mathsf{P}[\tau > 0] = 1$. In context of a survival analysis τ is often called a *survival time*.

Let us denote a cumulative distribution function of τ by $F(t)$. Then we can define a *survival function* $S(t)$ as

$$S(t) = 1 - F(t) = \mathsf{P}[\tau > t].$$

The survival function shows what is the probability that the survival time τ will be bigger than t. Furthermore, we assume that there exists a density function $f(t)$ that is a derivative of $F(t)$. Then let us define a *hazard rate* λ_t as

$$\lambda_t = \lim_{h \to 0^+} \frac{\mathsf{P}[t \leq T < t + h | T > t]}{h} = \lim_{h \to 0^+} \frac{F(t+h) - F(t)}{h} \frac{1}{1 - F(t)} = \frac{f(t)}{1 - F(t)}.$$

The hazard rate λ_t shows the conditional intensity with which the random event will occur in an infinitesimal time increment after t, conditioned that the random event has not occurred yet. For λ_t hold following identities

$$\lambda_t = \frac{f(t)}{S(t)} = -\frac{S'(t)}{S(t)} = -\frac{\partial \log S(t)}{\partial t}. \tag{2.2}$$

If we know the hazard rate, we can easily recover the cumulative distribution function and the survival function from (2.2) as

$$S(t) = \exp\left(-\int_0^t \lambda_s \, \mathrm{d}s\right).$$

It is clear that if and only if the hazard rate is constant, the random time T has an exponential distribution with intensity λ.

In the survival analysis there is often used more general Weibull distribution with the cumulative distribution function

$$F(t) = 1 - \mathrm{e}^{-(ct)^k}.$$

The exponential distribution is a special type of the Weibull distribution for $k = 1$. The hazard rate for the Weibull distribution is

$$\lambda_t = ck(ct)^{k-1}.$$

Let us define the counting processes N_t associated with the random time τ as

$$N_t = \mathbf{1}_{\{\tau \leq t\}}.$$

It is obvious, that the counting process N_t is a right–continuous nonnegative submartingale with $N_0 = 0$, hence there exists a nondecreasing compensator A_t of the process N_t. Let us assume that λ_t is the hazard rate of the random time τ, then

$$A_t = \int_0^{t \wedge \tau} \lambda_s \, \mathrm{d}s = \int_0^t \lambda_s \mathbf{1}_{\{s \leq \tau\}} \, \mathrm{d}s.$$

The compensator A_t describes a cumulative conditional likelihood of default. It is obvious that the counting process N_t is a Poisson process (resp. time inhomogeneous Poisson process, resp. Cox process) stopped at random time τ if the hazard rate λ_t is constant (resp. deterministic, resp. random). Stochastic dynamics of the counting process is then

$$\mathrm{d}N_t = (1 - N_t)\lambda_t \,\mathrm{d}t + \mathrm{d}M_t,$$

where M_t is a martingale.

2.2 Markov Processes

A stochastic process X_t is Markov if and only if for every $0 \leq s < t$ holds

$$\mathsf{P}[X_t \leq x|\mathcal{F}_s] = \mathsf{P}[X_t \leq x|\sigma(X_s)], \tag{2.3}$$

where $\mathcal{F}_s$ is a σ-algebra generated by the process X_t up to time s and $\sigma(X_s)$ is a σ-algebra generated by X_s. If the process X_t satisfies (2.3) we say that X_t satisfies the *Markov property*. The Markov property does not say anything different then that the conditional probability distribution of the process X_t conditioned on the whole past till time s is same as the conditional probability distribution of the process X_t conditioned by knowledge of the value of the process at time s. The special case of a Markov process is a *Markov chain* which is a Markov process with a countably large state space E.

Markov chains are very useful processes in finance and a lot of results are well-known. We recall basic definitions and some results which we will need later. For a deeper Markov chain theory see Karlin and Taylor (1975), Karlin and Taylor (1981), Resnick (1992), Kijima (1997), or Stroock (2005).

2.2.1 Discrete–time Markov Chains

We call a random sequence $\{X_n\}_{n\in\mathbb{N}_0}$ with values in a countable state space E a *discrete–time Markov chain* if and only if it satisfies the Markov Property, i.e.,

$$\mathsf{P}[X_{n+m} = j|X_n = i, X_{n-1} = i_{n-1}, \ldots, X_0 = i_0] = \mathsf{P}[X_{n+m} = j|X_n = i],$$

for all $n, m \in \mathbb{N}$ and for all $i, j, i_0, \ldots, i_{n-1} \in E$ such that

$$\mathsf{P}[X_n = i, X_{n-1} = i_{n-1}, \ldots, X_0 = i_0] > 0.$$

Let us denote the transition probabilities

$$\mathsf{P}[X_{n+m} = j|X_n = i_n] = p_{ij}(n, n + m).$$

If the state space E is finite, we call X_n a finite discrete–time Markov chain. If E is countably infinite it can be labeled by natural numbers. Hence we can work with $E = \mathbb{N}$. In the credit risk management we are interested mainly in finite Markov chains, therefore in the following we will assume $E = \{1, 2, \ldots, K\}$.

We call X_n a *homogeneous Markov chain* if transition probabilities $p_{ij}(n, n + m)$ do not depend on n but only on m, i.e.,

$$\mathsf{P}[X_{n+m} = j|X_n = i] = \mathsf{P}[X_m = j|X_0 = i] = p_{ij}^{(m)}, \tag{2.4}$$

for all $i, j \in E$, $n \in \mathbb{N}_0$ and $m \in \mathbb{N}$. If (2.4) does not hold we speak about an *inhomogeneous Markov chain*. In the following we assume X_n to be homogeneous. The matrix

$$\mathbf{P}^{(m)} = \left(p_{ij}^{(m)}\right)_{i,j=1}^{K},$$

is called the m-step transition probability matrix of the chain. It is easy to verify the so–called Chapman–Kolmogorov equations

$$\mathbf{P}^{(n)}\mathbf{P}^{(m)} = \mathbf{P}^{(n+m)}. \tag{2.5}$$

Equation (2.5) gives us a way to compute the n-step transition probability matrix $\mathbf{P}^{(n)}$ from the one–step transition probability matrix $\mathbf{P}$ as

$$\mathbf{P}^{(n)} = \mathbf{P}^{n}.$$

We call the probability distribution of X_0, the *initial distribution* of the process and denote it by

$$p_i = \mathsf{P}[X_0 = i].$$

Then it follows that

$$\mathsf{P}[X_0 = i_0, X_1 = i_1, \ldots, X_{n-1} = i_{n-1}, X_n = i_n] = p_{i_0} p_{i_0 i_1} \ldots p_{i_{n-1} i_n}. \tag{2.6}$$

Maximum Likelihood Estimation

We see that the one–step transition probability matrix is a building stone for a discrete–time Markov chain modeling. However, in many practical situations we have observations $x_0, x_1, \ldots, x_m$ of a process X_n and we would like to estimate the transition probability matrix $\mathbf{P}$ from these observations. Let us denote by n_{ij} the frequency of transitions from i to j that occurred in observation $x_0, x_1, \ldots, x_m$ and define

$$n_{i.} = \sum_{j=1}^{K} n_{ij}.$$

Since $\mathbf{P}$ is a probability matrix, its row–sums must be 1 and hence we can write

$$p_{iK} = 1 - \sum_{k=1}^{K-1} p_{ik}. \tag{2.7}$$

Using (2.6) and (2.7) the log–likelihood function is given by

$$\begin{aligned} l(\mathbf{P}) &= \log \mathsf{P}[X_0 = i_0, \ldots, X_m = i_m] \\ &= \log\left(p_{i_0} \prod_{i,j} p_{ij}^{n_{ij}}\right) \\ &= \log(p_{i_0}) + \sum_{i=1}^{K}\left(\sum_{j=1}^{K-1} n_{ij}\log(p_{ij}) + n_{iK}\log\left(1 - \sum_{k=1}^{K-1} p_{ik}\right)\right). \end{aligned}$$

After differentiation of $l(\mathbf{P})$ and setting the derivatives to 0, we get the following equations

$$\frac{\partial}{\partial p_{ij}} l(\mathbf{P}) = \frac{n_{ij}}{p_{ij}} - \frac{n_{iK}}{1 - \sum_{k=1}^{K-1} p_{ik}} = 0, \tag{2.8}$$

for $i = 1, 2, \ldots, K$ and $j = 1, 2, \ldots, K-1$. By solving (2.8) together with (2.7) we get the maximum likelihood estimator of the one–step transition probability matrix

$$\hat{\mathbf{P}} = \left(\frac{n_{ij}}{n_{i.}} \right)_{i,j=1}^{K}. \tag{2.9}$$

The estimator of the transition probability matrix $\hat{\mathbf{P}}$ is often called the *cohort estimator* or the estimation by the *cohort method*.

2.2.2 Continuous–time Markov Chains

A *continuous–time Markov chain* can be seen as a generalization of a discrete–time Markov chain. In the discrete–time setting the Markov chain can change the state only in some natural multiples of time unit. If we replace these discrete–times by random exponentially distributed variables we get a continuous–time Markov chain, where change of the state can happen any time.

Let $T = [0, \infty)$ and let the stochastic process $(X_t, t \in T)$ with values in a countable state space E satisfies the Markov property, i.e., for every $s \geq 0$ and $t \geq 0$,

$$\mathsf{P}[X_{t+s} = i | X_u, 0 \leq u \leq s] = \mathsf{P}[X_{t+s} = i | X_s],$$

then we say that X_t is a continuous–time Markov chain. Let us denote transition probabilities by

$$\mathsf{P}[X_{t+s} = i | X_s] = p_{ij}(s, s+t).$$

Similar as in the case of a discrete–time Markov chain we talk about a homogeneous continuous–time Markov chain if transition probabilities $p_{ij}(s, s+t)$ do not depend on s, but only on t. In that case we write $p_{ij}(t)$ for short. In the following we assume a finite homogeneous chain ($E = \{1, \ldots, K\}$) if not stated otherwise.

One can see, that we need a whole family of transition probability matrices

$$\mathbf{P}(t) = \left(p_{ij}(t) \right)_{i,j=1}^{K},$$

that depend on time t. It is convenient to define

$$\mathbf{P}(0) = \mathbf{I},$$

where $\mathbf{I}$ is the identity matrix. Let us denote the probability distribution of X_t at time t by

$$\boldsymbol{p}(t) = \left(p_i(t) \right)_{i=1}^{K},$$

where $p_i(t) = \mathsf{P}[X_t = i]$. Then $\boldsymbol{p}(0)$ is the initial distribution of the process and for every $0 < t_1 < \ldots < t_n$ and $i_0, i_1, \ldots, i_n \in E$ we have

$$\mathsf{P}[X_0 = i_0, X_{t_1} = i_1, \ldots, X_{t_n} = i_n] = p_{i_0}(0)p_{i_0 i_1}(t_1)p_{i_1 i_2}(t_2 - t_1)\ldots p_{i_{n-1} i_n}(t_n - t_{n-1}).$$

Also for a continuous–time Markov chain is easy to verify the Chapman–Kolomogorov equations

$$\mathbf{P}(s)\mathbf{P}(t) = \mathbf{P}(s + t).$$

The transition probability matrix $\mathbf{P}(t)$ is differentiable with respect to t (see Kijima (1997, Theorem 4.4)) and since $\mathbf{P}(0) = \mathbf{I}$, we can compute the matrix $\mathbf{Q}$ as the right–side derivative of $\mathbf{P}(t)$ at point 0, i.e.,

$$\mathbf{Q} = \lim_{t \to 0+} \frac{\mathbf{P}(t) - \mathbf{I}}{t}.$$

The matrix $\mathbf{Q} = (q_{ij})_{i,j=1}^{K}$ is called the *infinitesimal generator*, or *generator* for short, of a finite continuous–time Markov chain. The generator matrix $\mathbf{Q}$ satisfies for every $i \in E$

$$\begin{aligned} q_{ij} &\geq 0, \qquad i \neq j, \\ q_{ii} &\leq 0, \\ q_i = -q_{ii} &= \sum_{j \neq i} q_{ij}, \end{aligned}$$

where q_{ij} are called *intensities* and q_i is often called the *total intensity*. If we have the generator $\mathbf{Q}$ we can get the transition probability matrix $\mathbf{P}(t)$ as the unique solution of the backward Kolmogorov differential equation

$$\mathrm{d}\mathbf{P}(t) = \mathbf{Q}\mathbf{P}(t)\,\mathrm{d}t, \qquad t \geq 0,$$

or the forward Kolmogorov differential equation

$$\mathrm{d}\mathbf{P}(t) = \mathbf{P}(t)\mathbf{Q}\,\mathrm{d}t, \qquad t \geq 0,$$

with the initial condition $\mathbf{P}(0) = \mathbf{I}$. The unique solution is given by

$$\mathbf{P}(t) = \exp(\mathbf{t}\mathbf{Q}) = \sum_{n=0}^{\infty} \frac{t^n \mathbf{Q}^n}{n!}, \qquad t \geq 0.$$

Let us denote a random time, when the process X_t changes the state for the i-th time, by T_i. We get an increasing sequence of stopping times $T_1 < T_2 < \ldots$. Times between jumps (changes of state) are random variables with exponential distribution (see Kijima (1997, Theorem 4.6)). The intensity of the exponential distribution depends on the state in which the process X_t is. If the process X_t is in the state i, the random *sojourn time* before the next jump will occur is an exponential random variable τ_i with intensity q_i,

$$\mathsf{P}[\tau_i \leq t] = 1 - \mathrm{e}^{-q_i t}, \qquad t \geq 0. \tag{2.10}$$

In the case of the zero intensity q_i the sojourn time is ∞ and the state i is absorbing, i.e., once the process X_t enters the state i it will stay there forever.[5]

When the process X_t is in state i and a jump occurred, the probability that the next state will be state j is (for proof see Kijima (1997, Theorem 4.7))

$$\mathsf{P}[X_t = j | X_{t-} = i, X_t \neq i] = \frac{q_{ij}}{q_i}. \tag{2.11}$$

Probabilities (2.11) are transition probabilities of the embedded discrete–time Markov chain.

The continuous–time Markov chain can be defined through the embedded Markov chain and specifying intensities q_i for the distribution of sojourn times. This definition will be handy for extending Markov chains to semi–Markov chains in Section 2.2.3.

Maximum Likelihood Estimation

Same as in the case of a discrete–time Markov chain we are interested in the maximum likelihood estimator of the generator matrix $\mathbf{Q}$ from observed data. If we have a continuous observation of the process X_t up to the present time T and we know, that exactly m jumps occurred, we can denote the jump times of the process by $J_1, J_2, \ldots, J_m$. It means that the process X_t has started in some initial state $i_0 \in E$, at time J_1 it has changed the state i_0 to state i_1 and so on until at time J_m it has jumped to state i_m, where it resides. Assuming we are given by initial distribution we use (2.10) and (2.11) and get the maximum likelihood equation in form

$$\begin{aligned}
L(\mathbf{Q}) &= \lambda_{i_0} \exp(-\lambda_{i_0} J_1) \\
&\quad \times \prod_{k=1}^{m-1} \frac{\lambda_{i_{k-1} i_k}}{\lambda_{i_{k-1}}} \lambda_{i_k} \exp\big(- \lambda_{i_k} (J_k - J_{k-1})\big) \\
&\quad \times \frac{\lambda_{i_{m-1} i_m}}{\lambda_{i_{m-1}}} \exp\big(- \lambda_{i_m} (T^* - J_m)\big) \\
&= \exp(-q_{i_0} J_1) \prod_{k=1}^{m-1} q_{i_{k-1} i_k} \exp\big(- q_{i_k} (J_k - J_{k-1})\big) q_{i_{m-1} i_m} \exp\big(- q_{i_m} (T - J_m)\big) \\
&= \prod_{i=1}^{K} \exp\big(- q_i R_i(T)\big) \prod_{\substack{j=1 \\ j \neq i}}^{K} q_{ij}^{n_{ij}(T)} \\
&= \prod_{i=1}^{K} \exp\Big(- \sum_{k \neq i} q_{ik} R_i(T)\Big) \prod_{\substack{j=1 \\ j \neq i}}^{K} q_{ij}^{n_{ij}(T)} \\
&= \prod_{i=1}^{K} \prod_{\substack{j=1 \\ j \neq i}}^{K} q_{ij}^{n_{ij}(T)} \exp\big(- q_{ij} R_i(T)\big),
\end{aligned}$$

where $n_{ij}(T)$ is the total number of transitions from state i to state j up to time T, $R_i(T) = \int_0^T \mathbf{1}_{\{x(t)=i\}} \, \mathrm{d}t$ is the total time spend by the process in state i up to time T and term $\exp\big($ –

[5] If the state i is absorbing, i-th row of the generator matrix is just zero vector.

$q_{i_m}(T - J_m)\big)$ is the probability that the last sojourn time is bigger than $T - J_m$. The log–likelihood function is given by

$$l(\mathbf{Q}) = \log L(\mathbf{Q}) = \sum_{i=1}^{K} \sum_{\substack{j=1 \\ j \neq i}}^{K} \big(n_{ij}(T) \log q_{ij} - R_i(T) q_{ij}\big).$$

Let us differentiate $l(\mathbf{Q})$ with respect to q_{ij} and set the derivative equal to 0. We get

$$\frac{\partial}{\partial q_{ij}} l(\mathbf{Q}) = \frac{1}{q_{ij}} n_{ij}(T) - R_i = 0, \qquad i \neq j.$$

The maximum likelihood estimator of the generator matrix $\mathbf{Q}$ is then

$$\begin{aligned} \hat{q}_{ij} &= \frac{n_{ij}(T)}{R_i(T)}, \qquad i \neq j, \\ \hat{q}_{ii} &= -\sum_{\substack{j=1 \\ j \neq i}}^{K} \hat{q}_{ij}. \end{aligned} \tag{2.12}$$

If we observe data continuously, the estimator (2.12) is useful. But if we observe the state of the process just in discrete equidistant times there is a problem to estimate the generator $\mathbf{Q}$. The way to deal with it, is in Section 2.5.

Inhomogeneous Continuous–time Markov Chains

In the case of an inhomogeneous continuous–time Markov chain the generator $\mathbf{Q}$ is not constant anymore and it depends on time. For $0 \leq s < t$ we have

$$\begin{aligned} \mathbf{Q}(t) &= \lim_{h \to 0+} \frac{\mathbf{P}(t, t+h) - \mathbf{I}}{h}, \\ \mathbf{P}(s,t) &= \exp\left(\int_s^t \mathbf{Q}(u)\, \mathrm{d}u \right) = \sum_{n=0}^{\infty} \frac{\left(\int_s^t \mathbf{Q}(u)\, \mathrm{d}u \right)^n}{n!}, \end{aligned}$$

where the integral from a matrix is assumed to be componentwise, i.e.,

$$\int_s^t \mathbf{Q}(u)\, \mathrm{d}u = \begin{pmatrix} \int_s^t q_{1,1}(u)\, \mathrm{d}u & \int_s^t q_{1,2}(u)\, \mathrm{d}u & \cdots & \int_s^t q_{1,K}(u)\, \mathrm{d}u \\ \int_s^t q_{2,1}(u)\, \mathrm{d}u & \int_s^t q_{2,2}(u)\, \mathrm{d}u & \cdots & \int_s^t q_{2,K}(u)\, \mathrm{d}u \\ \vdots & \vdots & \ddots & \vdots \\ \int_s^t q_{K,1}(u)\, \mathrm{d}u & \int_s^t q_{K,2}(u)\, \mathrm{d}u & \cdots & \int_s^t q_{K,K}(u)\, \mathrm{d}u \end{pmatrix}.$$

For every $t \geq 0$ and $i \in E$ it holds

$$\begin{aligned} q_{ij}(t) &\geq 0, \qquad j \neq i, \\ q_{ii}(t) &\leq 0, \\ q_i(t) &= -q_{ii}(t) = \sum_{j \neq i} q_{ij}(t). \end{aligned}$$

2.2.3 Semi–Markov Chains

The continuous–time Markov chain is an extension of the discrete–time Markov chain when sojourn times are not a natural multiples of time unit anymore, but they are replaced with an exponential random variable with a constant intensity. If we go further and replace the exponential distribution by any general distribution we will get a *semi–Markov chain* and if we allow a dependency of the distribution on time t we get an inhomogeneous semi–Markov chain. In this section we describe basic properties of finite inhomogeneous semi–Markov chains. Properties for homogeneous semi–Markov chains can be derived by relaxing the dependency on time t. Semi–Markov chains are often used in the renewal theory and the reliability theory. For the general theory of semi–Markov chains we refer to Janssen and Manca (2006).

Similar as in the case of a finite continuous–time Markov chain we have an embedded discrete–time Markov chain $\{J_n\}_{n\in\mathbb{N}_0}$ with the state space $\{1,\ldots,K\}$ and an increasing sequence of stopping times $\{T_n\}_{n\in\mathbb{N}}$. A semi–Markov chain X_t is fully described by (J_n, T_n) in following way:

$$\begin{aligned} X_t &= J_0, \qquad t \in [0, T_1), \\ X_t &= J_1, \qquad t \in [T_1, T_2), \\ &\vdots \\ X_t &= J_n, \qquad t \in [T_n, T_{n+1}), \\ &\vdots \end{aligned}$$

Let us define a kernel associated to the semi–Markov chain X_t as

$$\mathbf{H}(s,t) = \Big(h_{ij}(s,t)\Big)_{i,j=1}^{K},$$

where

$$h_{ij}(s,t) = \mathsf{P}[J_{n+1} = j, T_{n+1} \le t | J_n = i, T_n = s].$$

Next, let us define

$$S_i(s,t) = \sum_{j=1}^{K} h_{ij}(s,t) = \mathsf{P}[T_{n+1} \le t | J_n = i, T_n = s],$$

that is, the probability distribution of the sojourn time in state i conditioned that the last jump occurred at time s. Finally, we can compute the transition probabilities $\mathbf{P}(s,t) = (p_{ij}(s,t))_{i,j}^{K}$ of the process X_t from evolution equations

$$p_{ij}(s,t) = \delta_{ij}\big(1 - S_i(s,t)\big) + \sum_{k=1}^{K} \int_s^t p_{kj}(u,t) h_{ik}(s,\, \mathrm{d}u), \tag{2.13}$$

where $\delta_{ij} = \mathbf{1}_{\{i=j\}}$ is Kronecker symbol. The general integral equations of form (2.13) are called Volterra integral equations. The equations (2.13) are extended Kolmogorov differential equations and can not be solved without any further specification of sojourn time distribution and inhomogeneity structure. If we assume some distribution, we can use maximum likelihood estimation. If we do not want to assume any particular distribution we can use a nonparametric estimation for which see Lucas et al. (2006). Work with a nonparametric estimation is harder than with a parametric estimation.

2.3 Lévy Processes

Classical references for Lévy processes theory are Bertoin (1996) and Sato (1999). An overview of most common Lévy processes with references to original papers can be found in Schoutens (2003, Chapter 5).

Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathsf{P})$ be the probability space equipped by filtration $\mathcal{F}_t, t \geq 0$ that satisfies the "usual hypothesis", i.e., is right–continuous and for every $t \geq 0$ σ-algebra $\mathcal{F}_t$ contains all null sets of σ-algebra $\mathcal{F}$. We will describe a one–dimensional Lévy processes properties. For multidimensional properties see literature above.

A stochastic cadlag[6] process X_t adapted to filtration $\mathcal{F}_t$ with values in $\mathbb{R}$ is a Lévy process if and only if it has following properties:

(L1) $X_0 = 0$ a.s.

(L2) For every sequence of times $t_0 < t_1 < \ldots < t_n$ the random variables $X_{t_0}, X_{t_1} - X_{t_0}, \ldots, X_{t_n} - X_{t_{n-1}}$ are independent.

(L3) The process X_t has stationary increments, i.e., for every $t > 0$ and $s > 0$

$$\mathcal{L}(X_{t+s} - X_t) = \mathcal{L}(X_s).$$

(L4) The process X_t is continuous in probability, i.e., for every $\varepsilon > 0$ holds

$$\lim_{h\to 0} \mathsf{P}[|X_{t+h} - X_t| > \varepsilon] = 0.$$

The cadlag property is not necessary in the definition of the Lévy process, but as we will see later, the Lévy process is a semimartingale[7] and every semimartingale admits the unique modification that is cadlag. Therefore we have included the cadlag property in the definition.

We say that the distribution μ is *infinitely divisible* if for every $n \in \mathbb{N}$ there exist $Y_1 + \ldots + Y_n$ i.i.d. random variables such that has the probability distribution μ. If μ_n is the probability distribution of random variables $Y_1, \ldots, Y_n$ then μ is the n-th convolution of μ_n.

There exists a close connection between infinitely divisible distributions and Lévy processes. In fact for every Lévy process X_t exists an infinitely divisible distribution μ such that $\mathcal{L}(X_1) = \mu$. Conversely for any infinitely divisible distribution μ exists a Lévy process with $\mathcal{L}(X_1) = \mu$. Indeed, for every $n \in \mathbb{N}$ we can write X_1 as

$$X_1 = X_{\frac{1}{n}} + \left(X_{\frac{2}{n}} - X_{\frac{1}{n}}\right) + \ldots + \left(X_{\frac{n}{n}} - X_{\frac{n-1}{n}}\right),$$

and since X_t has independent stationary increments it follows that

$$\begin{aligned}\mathcal{L}(X_1) &= \mathcal{L}\left(X_{\frac{1}{n}} + \left(X_{\frac{2}{n}} - X_{\frac{1}{n}}\right) + \ldots + \left(X_{\frac{n}{n}} - X_{\frac{n-1}{n}}\right)\right) \\ &= \mathcal{L}\left(X_{\frac{1}{n}}\right) * \ldots * \mathcal{L}\left(X_{\frac{1}{n}}\right),\end{aligned}$$

hence the distribution of X_1 is the n-th convolution of the distribution of $X_{\frac{1}{n}}$ and therefore the distribution of X_1 is infinitely divisible. The converse implication can be proved by the

[6] The cadlag process is a process which has every sample path right–continuous and admits left limit.

[7] We say that a process X_t is a semimartingale if and only if

$$X_t = X_0 + M_t + C_t,$$

where M_t is a local martingale and C_t is a cadlag adapted process of a locally bounded variation.

construction of the process using the Kolmogorov theorem and verifying that the constructed process is a Lévy process. Details of the construction can be found in Sato (1999, Theorem 7.10).

Let us define the characteristic function of X_t as

$$\phi_{X_t}(z) = \mathsf{E}\left[\mathrm{e}^{izX_t}\right], \qquad z \in \mathbb{R}.$$

For the characteristic function $\phi_{X_{t+s}}(z)$, where $t, s > 0$ it holds

$$\begin{aligned}
\phi_{X_{t+s}}(z) &= \mathsf{E}\left[\mathrm{e}^{izX_{t+s}}\right] \\
&= \mathsf{E}\left[\mathrm{e}^{iz(X_{t+s}-X_s+X_s)}\right] \\
&= \mathsf{E}\left[\mathrm{e}^{iz(X_{t+s}-X_s)}\right]\mathsf{E}\left[\mathrm{e}^{izX_s}\right] \\
&= \mathsf{E}\left[\mathrm{e}^{izX_t}\right]\mathsf{E}\left[\mathrm{e}^{izX_s}\right] \\
&= \phi_{X_t}(z)\phi_{X_s}(z), \qquad ,
\end{aligned} \tag{2.14}$$

since Lévy process has independent stationary increments. Now let $t \in \mathbb{Q}^+$, then there exists $m, n \in \mathbb{N}$ such that $t = \frac{m}{n}$ and from (2.14) follows

$$\phi_{X_m}(z) = \left(\phi_{X_{\frac{m}{n}}}(z)\right)^n,$$

and

$$\begin{aligned}
\phi_{X_t}(z) &= \phi_{X_{\frac{m}{n}}}(z) \\
&= \left(\phi_{X_m}(z)\right)^{\frac{1}{n}} \\
&= \left(\phi_{X_1}(z)\right)^{\frac{m}{n}}.
\end{aligned}$$

Then from the continuity in probability it follows that for every $t \geq 0$

$$\phi_{X_t}(z) = \left(\phi_{X_1}(z)\right)^t = \mathrm{e}^{t\psi(z)},$$

where the exponential form is implied by the property (2.14) and ψ is called the *characteristic exponent*.

2.3.1 Examples of Lévy Processes

We mention just few examples of Lévy processes. For more see (Schoutens, 2003, Chapter 5).

Brownian motion

A *standard Brownian motion* B_t is a Lévy process such that

1. $B_t \sim N(0, t)$ for each $t \geq 0$.
2. B_t has continuous sample path.

From property 1 follows that the characteristic function of the Brownian motion is

$$\phi_{B_t}(z) = \frac{1}{\sqrt{2\pi}} \int_{\infty}^{\infty} \mathrm{e}^{izx}\mathrm{e}^{-\frac{x^2}{2}}\,\mathrm{d}x = \exp\left(-\frac{1}{2}tz^2\right).$$

The Brownian motion is a very useful construction used in many different areas for modeling various phenomenas. Let $b \in \mathbb{R}$ and $\sigma > 0$ then the process

$$X_t = bt + \sigma B_t,$$

is called the *Brownian motion with drift*. The process X_t is a Gaussian Lévy process with $X_t \sim N(bt, \sigma^2 t)$. Its characteristic function is given by

$$\phi_{X_t}(z) = \exp\left(ibtz - \frac{1}{2} t\sigma^2 z^2 \right).$$

Poisson Process

We have already defined a Poisson process N_t in Section 2.1.1. The Poisson process is a Lévy process such that N_t has a Poisson distribution with parameter λt, i.e.,

$$\mathsf{P}[N_t = k] = \frac{(\lambda t)^k}{k!} \mathrm{e}^{-\lambda t}, \qquad \text{for } k = 0, 1, \ldots.$$

Therefore the characteristic function of the Poisson process is

$$\phi_{N_t}(z) = \sum_{k=0}^{\infty} \mathrm{e}^{izk} \frac{(\lambda t)^k}{k!} \mathrm{e}^{-\lambda t} = \exp\left(\lambda t (\mathrm{e}^{iz} - 1) \right).$$

Recall that the Poisson process is the only counting process with independent stationary increments.

From Section 2.1.1 we know that the Poisson process has the compensator λt. Therefore we can define the *compensated Poisson process* $\tilde{N}_t$ as

$$\tilde{N}_t = N_t - \lambda t,$$

which is a martingale.

Compounded Poisson process

A Poisson process is a process with positive jumps of size 1. If we generalise a Poisson process and allow a random size of jumps we will get the *compounded Poisson process*. More precisely let N_t be the Poisson process and $Y_1, Y_2, \ldots$ are i.i.d. random variables with probability distribution F independent with the Poisson process N_t. Then the process

$$X_t = \sum_{i=1}^{N_t} Y_i, \qquad t \geq 0,$$

is the compounded Poisson process.[8] The compounded Poisson process jumps at same times as the original Poisson process N_t, but has random size of jumps. The characteristic function

[8] We assume $\sum_1^0 \cdot = 0$.

of the compounded Poisson process is

$$\begin{aligned}\phi_{X_t}(z) &= \mathsf{E}\,\mathrm{e}^{izX_t} \\ &= \mathsf{E}\left[\mathsf{E}\left[\mathrm{e}^{iz\sum_{i=1}^{N_t}Y_i}|N_t\right]\right] \\ &= \mathsf{E}\left[\left(\mathsf{E}\left[\mathrm{e}^{izY_1}\right]\right)^{N_t}\right] \\ &= \sum_{k=0}^{\infty}\left(\mathsf{E}\left[\mathrm{e}^{izY_1}\right]\right)^k \frac{(\lambda t)^k}{k!}\mathrm{e}^{-\lambda t} \\ &= \exp\left(t\int(\mathrm{e}^{izy}-1)\lambda\,\mathrm{d}F(y)\right).\end{aligned}$$

α-stable distribution

An important subclass of infinitely divisible distributions are so called *stable distributions.*

We say that distribution μ is stable if and only if for all $n > 1$ there exist constants $c_n > 0$ and $d_n \in \mathbb{R}$ such that

$$\mathcal{L}(X_1 + X_2 + \ldots + X_n) = \mathcal{L}(c_n X + d_n),$$

where $X, X_1, X_2, \ldots, X_n$ are i.i.d. random variables with distribution μ. Distribution μ is called *strictly stable* if and only if $d_n = 0$ for all n. It can be proved that the only possibility of choosing c_n is

$$c_n = n^{\frac{1}{\alpha}},$$

where $\alpha \in (0, 2]$ is called the *index of stability.* If X has the index of stability α we say that X is α-stable.

The random variable X is stable if and only if

$$X = aY + b,$$

where $a > 0$, $b \in \mathbb{R}$ and Y is random variable with the characteristic function

$$\mathsf{E}\,\mathrm{e}^{izY} = \begin{cases} \exp\left(-|z|^\alpha\left(1 - i\beta(\operatorname{sgn} z)\tan\frac{\pi\alpha}{2}\right)\right), & \alpha \neq 1, \\ \exp\left(-|z|\left(1 - i\beta(\operatorname{sgn} z)(-\frac{\pi}{2}\log|z|)\right)\right), & \alpha = 1, \end{cases}$$

where $-1 \leq \beta \leq 1$. For proof see Sato (1999, Theorem 14.15). If $b = 0$ and $\beta = 0$ then X is symmetric around zero and characteristic function of $X = aZ$ is simply

$$\mathsf{E}\,\mathrm{e}^{izaY} = \exp(-a^\alpha|z|^\alpha).$$

Some special cases of α-stable distributions are:

Normal distribution $N(\mu, \sigma^2)$

$$\begin{aligned} \alpha &= 2, & a &= \frac{\sigma}{\sqrt{2}}, \\ \beta &= 0, & b &= \mu, \end{aligned}$$

Cauchy distribution with parameters γ and δ

$$\alpha = 1, \qquad a = \gamma,$$
$$\beta = 0, \qquad b = \delta,$$

Lévy distribution with parameters γ and δ

$$\alpha = \frac{1}{2}, \qquad a = \gamma,$$
$$\beta = 1, \qquad b = \delta,$$

One of interesting thing is that the stable distribution has finite second moment only for $\alpha = 2$ and finite first moment for $1 < \alpha \leq 2$.

2.3.2 Lévy–Ito Decomposition and Lévy–Khintchin Formula

First, we will mention few properties of infinitely divisible distributions which are also properties of Lévy processes since their straightforward connection. It is obvious that if μ and ν are independent infinitely divisible distributions then also convolution $\mu * \nu$ is infinitely divisible. If μ_k is a sequence of infinity divisible distributions and $\mu_k \to \mu$ then also μ is infinitely divisible.

It can be even proved that every infinitely divisible distribution μ is a limit of the sequence of compounded Poisson processes with the distribution μ_n, even if μ is not a compounded Poisson process.

If every infinitely divisible distribution can be written as a limit of the sequence of compounded Poisson distributions one can ask, if there exists some general form for the characteristic function of a Lévy process. The answer is the Lévy–Khintchin formula. For better understanding of the formula we will present few ideas which are behind it, but we will not prove it. The Lévy–Khintchin formula can be proved directly, see Sato (1999, Theorem 8.1), or firstly proving the Lévy–Ito decomposition and then as an easy corollary, see Cont and Tankov (2003, Theorem 3.1).

Let us define the *jump process* ΔX_t associated to the Lévy process X_t as

$$\Delta X = X_t - X_{t-}.$$

Since the Lévy process X_t is continuous in probability it follows that for fixed t, $\Delta X_t = 0$ a.s.. Some difficulties in Lévy processes modeling come from the fact that it is possible for any $t > 0$ have

$$\sum_{0 \leq s \leq t} \Delta |X_s| = \infty, \qquad \text{a.s.},$$

but we always have

$$\sum_{0 \leq s \leq t} \Delta |X_s|^2 < \infty, \qquad \text{a.s.}$$

Let $A \subset \mathbb{R}$ be any Borel set such that 0 does not belong to A, $0 \notin A$. Then we can define for $\omega \in \Omega$ a counting measure

$$N_t(A)(\omega) = \sum_{0 \leq s \leq t} \mathbf{1}_{\{\Delta X_s(\omega) \in A\}},$$

which assign to the set A a number of jumps with size in A which has happened before time t. Since for any fixed $\omega \in \Omega$ and $t \geq 0$ $N_t(A)(\omega)$ is a counting measure on $\mathbb{R}$, $N_t(A)$ is a counting process.

It is easy to verify that $N_t(A) - N_s(A)$ is independent of $\mathcal{F}_s$. From stationarity of Lévy process X_t also follows that

$$\mathcal{L}\big(N_t(A) - N_s(A)\big) = \mathcal{L}\big(N_{t-s}(A)\big).$$

Let A be bounded from 0, i.e., $0 \notin \bar{A}$, then $N_t(A) < \infty$ a.s.. From the fact that the counting process $N_t(A)$ has independent stationary increments follows that $N_t(A)$ is the Poisson process with intensity

$$\nu(A) = \mathsf{E}\, N_1(A),$$

where $\nu(A)$ is a σ-finite measure called the *Lévy measure*. Further, let us define the *compensated Poisson measure* $\tilde{N}_t(A)$ as

$$\tilde{N}_t(A) = N_t(A) - t\nu(A),$$

which is martingale valued, i.e., for $s < t$

$$\mathsf{E}\,[\tilde{N}_t(A)|\mathcal{F}_s] = \tilde{N}_s(A).$$

If A is not bounded from 0 then $\mathsf{E}\, N_t(A)$ and also $\mathsf{E}\,\tilde{N}_t(A)$ can be possibly infinite for any $t > 0$.

Now we would like to study the jump part of the Lévy process X_t, which should be intuitively equal to the sum of all jumps up to time t

$$\sum_{0 \leq s \leq t} \Delta X_s.$$

We must be careful here, since the Lévy process can have infinitely many small jumps in an arbitrary small time interval. That is reason why we divide jumps of the Lévy process X_t into two groups — jumps which are in absolute value smaller than some arbitrary constant $c > 0$ and the rest that are bigger or equal to c. It is irrelevant which c we choose since there might be infinitely many jumps smaller than any constant $c > 0$. The common choice for constant c is 1. Then

$$\int_{|x| \geq 1} x N_t(\,\mathrm{d}x) = \sum_{0 \leq s \leq t} \Delta X_s \mathbf{1}_{\{|\Delta X_s| \geq 1\}},$$

is sum of jumps of size bigger or equal to 1. This sum is a finite random variable since there is just finitely many jumps bigger then 1, but it does not need to have finite moments. On the other side

$$X_t - \int_{|x| \geq 1} x N_t(\,\mathrm{d}x), \qquad t \geq 0,$$

is a Lévy process with all moments finite. It can be shown that integral

$$\int_{\varepsilon_n \leq |x| < 1} x \tilde{N}_t(\,\mathrm{d}x),$$

converges almost surly and uniformly as $\varepsilon_n \downarrow 0$ (see Sato (1999, Lemma 20.6)). Hence the sum of compensated small jumps

$$\int_{|x|<1} x\tilde{N}_t(\,\mathrm{d}x),$$

is a martingale. Finally it can be proved that

$$B_t = X_t - bt - \int_{|x|<1} x\tilde{N}_t(\,\mathrm{d}x) - \int_{|x|\geq 1} xN_t(\,\mathrm{d}x), \qquad t \geq 0,$$

with

$$b = \mathsf{E}\left(X_1 - \int_{|x|\geq 1} xN_1(\,\mathrm{d}x)\right),$$

is a martingale with continuous paths starting at 0 and a variance $t\sigma^2$, where $\sigma > 0$. From the Lévy theorem follows that the process B_t is a Brownian motion.

We get the Lévy–Ito decomposition which says that if X_t is a Lévy process then there exists a Brownian motion B_t with drift $b \in \mathbb{R}$, variance $\sigma > 0$ and a random Poisson measure N_t such that

$$X_t = bt + B_t + \int_{|x|<1} x\tilde{N}_t(\,\mathrm{d}x) + \int_{|x|\geq 1} xN_t(\,\mathrm{d}x), \qquad t \geq 0,$$

where all terms are independent.

Recall that a process X_t is a semimartingale if and only if

$$X_t = X_0 + M_t + C_t,$$

where M_t is a local martingale and C_t is an adapted process of finite variation. We see that a Lévy process is a semimartingale where

$$M_t = B_t + \int_{|x|<1} x\tilde{N}_t(\,\mathrm{d}x),$$
$$C_t = bt + \int_{|x|\geq 1} xN_t(\,\mathrm{d}x).$$

Now if we compute the characteristic function of a general Lévy process using the Lévy–Ito decomposition we will get the Lévy–Khintchine formula

$$\begin{aligned}\mathsf{E}\,\mathrm{e}^{izX_t} &= \mathsf{E}\,\mathrm{e}^{t\psi(z)} \\ &= \exp\left(t\left(ibz - \frac{1}{2}\sigma^2 z^2 + \int_{\mathbb{R}} \left(\mathrm{e}^{izx} - 1 - izx\mathbf{1}_{\{|x|<1\}}\right)\nu(\,\mathrm{d}x)\right)\right), \qquad z \in \mathbb{R},\end{aligned}$$

where the Lévy measure ν satisfies following two properties

$$\nu(\{0\}) = 0,$$
$$\int_{\mathbb{R}} (|x|^2 \wedge 1)\,\nu(\,\mathrm{d}x) < \infty.$$

The triplet (b, σ, ν) is called the characteristic triplet of the infinitely divisible distribution of X_1. This triplet uniquely define the distribution of X_1. The characteristic triplet of the distribution of X_t is then $(tb, t\sigma, t\nu)$.

The jump part of the Lévy process X_t is of finite variation if and only if the Lévy measure ν satisfies

$$\int_{|x|<1} |x| \, \nu(\mathrm{d}x) < \infty.$$

In that case we can rewrite the characteristic exponent in the Lévy–Khintchin formula in form

$$\psi(z) = ib'z - \frac{1}{2}\sigma^2 z^2 + \int_{\mathbb{R}} \left(\mathrm{e}^{izx} - 1\right) \nu(\mathrm{d}x),$$

where

$$b' = b - \int_{|x|<1} izx \ \nu(\mathrm{d}x).$$

And we see that it is a convolution of a Brownian motion with drift b' and a compounded Poisson process with unit intensity and jumps size distribution ν.

2.3.3 Subordinators

A subordinator is a real one–dimensional Lévy process which is almost surely increasing and can be interpret as a random time evolution. It is clear that if a subordinator T_t is almost surely increasing then the Brownian part of the process has to be zero, drift b has to be nonnegative and the process can have only positive jumps. A subordinator has finite variation and hence the characteristic exponent in the Lévy–Khintchin formula takes the form

$$\psi(z) = ibz + \int_0^\infty \left(\mathrm{e}^{izx} - 1\right) \nu(\mathrm{d}x), \qquad z \in \mathbb{R},$$

where

$$\begin{aligned} b &\geq 0, \\ \nu\big((-\infty, 0)\big) &= 0, \\ \int_0^\infty (1 \wedge x)\nu(\mathrm{d}x) &< \infty. \end{aligned}$$

In the case of T_t is a subordinator, it is more convenient to work instead of the characteristic exponent ψ with the Laplace exponent η, which is defined as

$$\eta(u) = -\psi(iu) = bu + \int_0^\infty \left(1 - \mathrm{e}^{-ux}\right) \nu(\mathrm{d}x), \qquad u \in \mathbb{C}, \Re(u) \geq 0,$$

Then it holds

$$\mathsf{E}\, \mathrm{e}^{-uT_t} = \mathrm{e}^{-t\eta(u)}.$$

Sometimes it is useful to work with the Laplace exponent $\eta(u)$ for u such that $\Re(u) < 0$. According to Sato (1999, Theorem 25.17), $\mathsf{E}\,\mathrm{e}^{-uT_t} < \infty$ if and only if

$$\int_1^\infty \mathrm{e}^{-ux}\nu(\,\mathrm{d}x) < \infty.$$

The Laplace exponent is defined on the interval $(\bar{u}, \infty)$ or $[\bar{u}, \infty)$. The left endpoint $\bar{u}$ may or may not belong into the interval on which the Laplace exponent is defined, but every time $\bar{u} \leq 0$.

Note that since a subordinator T_t has only positive jumps $\nu((-\infty, 0)) = 0$, and nonnegative drift b it is clear that

$$T_t \geq bt.$$

The line bt is a lower bound for the process T_t. We need to consider if we should allow all values from interval $[0, \infty)$ for T_t, that is do not have any lower bound, or there is some lower bound which can be equipped by term bt. In most cases there is not such a bound and we should set the drift term equal to zero $b = 0$.

Some authors mean by the term subordinator a more general process T'_t which is a subordinator T_t killed with rate λ. More precisely, let T_t be a subordinator and τ is a random variable with exponential distribution with intensity λ independent of T_t. Then the process

$$T'_t = \begin{cases} T_t, & t < \tau, \\ \infty, & t \geq \tau \end{cases}$$

is a subordinator killed with rate λ. As a special case when $\lambda = 0$ we get a classical subordinator T_t. We will not need killed subordinators in the presented work but we mention it just to be clear what we mean by the term subordinator. For more about (killed) subordinators we refer to Bertoin (1999).

Examples of Subordinators

Compounded Poisson process One of the simplest subordinator is a Poisson process or a compounded Poisson process. A compounded Poisson process is a subordinator if the probability distribution of jumps is nonnegative. If λ is an intensity of a Poisson process and F is the nonnegative probability distribution of jumps then the Lévy measure of subordinator T_t is

$$\nu(\,\mathrm{d}x) = \lambda F(\,\mathrm{d}x).$$

α-stable subordinator An α-stable subordinator T_t is a subordinator with the Laplace exponent

$$\eta(u) = u^\alpha,$$

where $0 < \alpha \leq 1$. If we want to get a Lévy measure of the subordinator we need to

notice that

$$\begin{aligned}
u^\alpha &= \frac{u^\alpha}{\Gamma(1-\alpha)} \int_0^\infty \mathrm{e}^{-s} s^{-\alpha} \,\mathrm{d}s \\
&= \frac{\alpha}{\Gamma(1-\alpha)} \int_0^\infty u\mathrm{e}^{-uy} \frac{1}{\alpha} y^{-\alpha} \,\mathrm{d}y && (s = uy) \\
&= \frac{\alpha}{\Gamma(1-\alpha)} \int_0^\infty u\mathrm{e}^{-uy} \int_y^\infty x^{-\alpha-1} \,\mathrm{d}x \,\mathrm{d}y \\
&= \frac{\alpha}{\Gamma(1-\alpha)} \int_0^\infty \int_0^x u\mathrm{e}^{-uy} \,\mathrm{d}y x^{-\alpha-1} \,\mathrm{d}x \\
&= \frac{\alpha}{\Gamma(1-\alpha)} \int_0^\infty (1-\mathrm{e}^{-ux}) x^{-\alpha-1} \,\mathrm{d}x.
\end{aligned}$$

We see that the Lévy measure ν is

$$\nu(\,\mathrm{d}x) = \frac{\alpha x^{-\alpha-1}}{\Gamma(1-\alpha)} \,\mathrm{d}x.$$

The case when $\alpha = 1$, is degenerate and in that case we get deterministic $T_t = t$. Note that an α-stable subordinator is a special case of an α-stable process for choice of parameters $\beta = 1$, $b = 0$ and $a^\alpha = \cos(\alpha\pi/2)$.

Gamma subordinator Let T_t be a Lévy process with increments which are gamma distributed. We say T_t is a *Gamma process* with the density function

$$f_{T_t}(x) = \frac{a^{bt}}{\Gamma(bt)} e^{-ax} x^{bt-1}, \qquad x > 0,$$

where $a, b > 0$. It follows

$$\begin{aligned}
\mathsf{E}\,\mathrm{e}^{-uT_t} &= \int_0^\infty \mathrm{e}^{-ux} \frac{a^{bt}}{\Gamma(bt)} e^{-ax} x^{bt-1} \,\mathrm{d}x \\
&= \frac{a^{bt}}{(a+u)^{bt}} \int_0^\infty \frac{(a+u)^{bt}}{\Gamma(bt)} e^{-(a+u)x} x^{bt-1} \,\mathrm{d}x \\
&= \left(\frac{a}{a+u}\right)^{bt} \\
&= \exp\left(bt \log\left(\frac{a}{a+u}\right)\right) \\
&= \exp\left(bt \int_0^\infty \frac{\mathrm{e}^{-(a+u)x} - \mathrm{e}^{-ax}}{x} \,\mathrm{d}x\right) \\
&= \exp\left(-t \int_0^\infty b(1-\mathrm{e}^{-ux}) x^{-1} \mathrm{e}^{-ax} \,\mathrm{d}x\right),
\end{aligned}$$

where we used the Frullani's integral.[9] We see that the Lévy measure is

$$\nu(\,\mathrm{d}x) = bx^{-1}\mathrm{e}^{-ax} \,\mathrm{d}x.$$

[9] $\int_0^\infty \frac{f(ax)-f(bx)}{x} \,\mathrm{d}x = \big(f(0) - f(\infty)\big) \log \frac{b}{a}$.

2.3.4 Time–Inhomogeneous Lévy Processes

Lévy processes has stationary independent increments, which is very useful for computations. We can extend the class of Lévy processes further to the time–inhomogeneous Lévy processes setting, where the most of nice properties of Lévy processes remains, but we get a more general class of stochastic processes which still have independent increments, but increments are no longer stationary. Time–inhomogeneous Lévy processes are also called processes with independent increments and absolutely continuous characteristics. For more about processes with independent increments and their characteristics see Jacod and Shiryaev (2003, Chapter 2).

Let T^* be some finite time horizon. A stochastic cadlag process X_t adapted to filtration $\mathcal{F}_t$ with values in $\mathbb{R}$ is an inhomogeneous Lévy process if and only if it has following properties:

(L1) $X_0 = 0$ a.s.

(L2) For every sequence of times $t_0 < t_1 < \ldots < t_n$ random variables $X_{t_0}, X_{t_1} - X_{t_0}, \ldots, X_{t_n} - X_{t_{n-1}}$ are independent.

(L3) For all $0 \leq t \leq T^*$, X_t has time dependent characteristic exponent

$$\psi_t(z) = \int_0^t \left(ib_s z - \frac{1}{2}\sigma_s^2 z^2 + \int_{\mathbb{R}} \left(\mathrm{e}^{izx} - 1 - iux\mathbf{1}_{\{|x|<1\}} \right) \nu_s(\,\mathrm{d}x) \right) \mathrm{d}s, \qquad z \in \mathbb{R},$$

where $b_s \in \mathbb{R}$, $\sigma_s > 0$ and ν_s is a measure on $\mathbb{R}$ that integrate $(|x|^2 \wedge 1)$ and satisfying $\nu\{0\} = 0$. Furthermore, we assume that

$$\int_0^{T^*} \left(|b_s| + \sigma_s^2 + \int_{\mathbb{R}} \left(|x|^2 \wedge 1 \right) \nu_s(\,\mathrm{d}x) \right) \mathrm{d}s < \infty.$$

Random variable X_t has the infinitely divisible distribution with the characteristic triplet (b_t, σ_t, ν_t). If the characteristic triplet depends on time linearly, i.e., $(b_t, A_t, \nu_t) = (tb, tA, t\nu)$, the process X_t is a time–homogeneous Lévy process.

From the definition we see that for the characteristic function $\phi_{X_t}(z)$ holds

$$\phi_{X_t}(z) = \mathrm{e}^{\psi_t(z)},$$

and only in the case of a time–homogeneous Lévy processes holds

$$\phi_{X_t}(z) = \mathrm{e}^{\psi_t(z)} = \mathrm{e}^{t\psi(z)}.$$

Similar, in the case of a Lévy time–inhomogeneous subordinator, the Laplace exponent η_t is now time dependent.

Time–inhomogeneous Lévy processes are not very common used processes. Some examples can be found in Cont and Tankov (2003, Chapter 14). Sometimes the time interval $(0, T^*)$ is divided into subpartions on which the process X_t is modeled via a time–homogeneous Lévy processes with different parameters on each subpartition. On the whole interval $(0, T^*)$ is then the process X_t a time–inhomogeneous process. More about time–inhomogeneous Lévy processes can be found for example in Kluge (2005).

2.4 Time–changed Continuous–time Markov Chain

In this section we recall few known results from matrix algebra, which we will need later. Then we will try model time evolution with a stochastic process and show how a continuous–time Markov chain evolves under the stochastic time.

2.4.1 Matrix Exponential

Let us begin with known fact that for any given $n \times n$ matrix $\mathbf{A}$, over an algebraically closed field,[10] there exists invertible matrix $\mathbf{B}$ such that

$$\mathbf{A} = \mathbf{B}\mathbf{D}\mathbf{B}^{-1}, \tag{2.15}$$

where $\mathbf{D}$ is a block diagonal matrix given by

$$\mathbf{D} = \begin{pmatrix} \mathbf{J}_{n_1}(\lambda_1) & & & \mathbf{0} \\ & \mathbf{J}_{n_2}(\lambda_2) & & \\ & & \ddots & \\ \mathbf{0} & & & \mathbf{J}_{n_k}(\lambda_k) \end{pmatrix},$$

and the block $\mathbf{J}_{n_i}(\lambda_i)$ is $n_i \times n_i$ square matrix given in form

$$\mathbf{J}_{n_i}(\lambda_i) = \begin{pmatrix} \lambda_i & 1 & \dots & 0 \\ 0 & \lambda_i & \ddots & \vdots \\ \vdots & & \ddots & 1 \\ 0 & \dots & 0 & \lambda_i \end{pmatrix},$$

such that $n_1 + n_2 + \ldots + n_k = n$. Values $\lambda_1, \ldots, \lambda_k$ are eigenvalues of the matrix $\mathbf{A}$. Form (2.15) is called the *Jordan canonical representation* and $\mathbf{J}_{n_i}(\lambda_i)$ a *Jordan block* belonging to eigenvalue λ_i. In the case $n = k$, that is, the block $\mathbf{J}_{n_i}(\lambda_i)$ is 1-dimensional for every i, we say, that the matrix $\mathbf{A}$ is *diagonalizable*.

Let $\mathbf{J}_n(\lambda)$ be an arbitrary Jordan block. It can be decomposed into

$$\mathbf{J}_n(\lambda) = \lambda\mathbf{I} + \mathbf{N},$$

where $\mathbf{I}$ is the identity matrix and $\mathbf{N}$ is a nilpotent matrix[11] in form

$$\mathbf{N} = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ & & & \ddots & \\ \vdots & \vdots & & \ddots & 1 \\ 0 & 0 & & & 0 \end{pmatrix}.$$

[10] $\mathbb{R}$ is not closed, but $\mathbb{C}$ is.

[11] We say that the matrix $\mathbf{N}$ is nilpotent of order $a \in \mathbb{N}$ if $\mathbf{N}^k \neq \mathbf{0}$ for $k < a$ and $\mathbf{N}^a = \mathbf{0}$.

By easy multiplication we see that

$$\mathbf{N}^2 = \begin{pmatrix} 0 & 0 & 1 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 \\ & & & \ddots & \vdots \\ & & & & 1 \\ \vdots & \vdots & & \ddots & 0 \\ 0 & 0 & & & 0 \end{pmatrix}$$

$$\vdots$$

$$\mathbf{N}^{n-1} = \begin{pmatrix} 0 & 0 & 0 & \dots & 1 \\ 0 & 0 & 0 & \dots & 0 \\ & & & \ddots & \\ \vdots & \vdots & & \ddots & 0 \\ 0 & 0 & & & 0 \end{pmatrix},$$

i.e., with higher power of the matrix $\mathbf{N}$ the line of ones shifts one element to the right. Hence the order of nilpotency of the matrix $\mathbf{N}$ is $n-1$.

It is easy to verify that matrices $\lambda\mathbf{I}$ and $\mathbf{N}$ commute. Then it holds

$$\begin{aligned} \mathrm{e}^{\lambda\mathbf{I}+\mathbf{N}} &= \sum_{i=0}^{\infty} \frac{(\lambda\mathbf{I}+\mathbf{N})^i}{i!} \\ &= \sum_{i=0}^{\infty} \frac{\sum_{j=0}^{i} \binom{i}{j} (\lambda\mathbf{I})^j \mathbf{N}^{i-j}}{i!} \\ &= \sum_{i=0}^{\infty} \sum_{j=0}^{i} \frac{\lambda^j \mathbf{N}^{i-j}}{j!(i-j)!} \\ &= \sum_{j=0}^{\infty} \frac{\lambda^j}{j!} \sum_{i=j}^{\infty} \frac{\mathbf{N}^{i-j}}{(i-j)!} \\ &= \mathrm{e}^{\lambda}\mathrm{e}^{\mathbf{N}}. \end{aligned}$$

Since the matrix $\mathbf{N}$ is nilpotent of order $n-1$, the Taylor expansion of $\mathrm{e}^{\mathbf{N}}$ is a finite sum consisting of n terms. The first term of the expansion is identity matrix and every other term shifts ones on the diagonal one item to the right as described above. We see that

$$\mathrm{e}^{\mathbf{J}_n(\lambda)} = \mathrm{e}^{\lambda\mathbf{I}+\mathbf{N}} = \mathrm{e}^{\lambda} \begin{pmatrix} 1 & 1 & \frac{1}{2} & \dots & \frac{1}{(n-1)!} \\ 0 & 1 & 1 & \dots & \frac{1}{(n-2)!} \\ 0 & 0 & 1 & \dots & \frac{1}{(n-3)!} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \end{pmatrix}. \tag{2.16}$$

Now we can compute the exponential of an arbitrary matrix $\mathbf{A}$. Notice that

$$\begin{aligned} \mathrm{e}^{\mathbf{A}} &= \sum_{i=0}^{\infty} \frac{\left(\mathbf{B}\mathbf{D}\mathbf{B}^{-1}\right)^i}{i!} \\ &= \mathbf{B} \sum_{i=0}^{\infty} \frac{\mathbf{D}^i}{i!} \mathbf{B}^{-1} \\ &= \mathbf{B}\mathrm{e}^{\mathbf{D}}\mathbf{B}^{-1}. \end{aligned}$$

We need to compute the exponential of the block diagonal matrix $\mathbf{D}$. Since the exponent of a block diagonal matrix is a block diagonal matrix with exponentials of original blocks on the diagonal, we have

$$\begin{aligned} \mathrm{e}^{\mathbf{D}} &= \exp\left[\begin{pmatrix} \mathbf{J}_{n_1}(\lambda_1) & & & \mathbf{0} \\ & \mathbf{J}_{n_2}(\lambda_2) & & \\ & & \ddots & \\ \mathbf{0} & & & \mathbf{J}_{n_k}(\lambda_k) \end{pmatrix}\right] \\ &= \begin{pmatrix} \mathrm{e}^{\mathbf{J}_{n_1}(\lambda_1)} & & & \mathbf{0} \\ & \mathrm{e}^{\mathbf{J}_{n_2}(\lambda_2)} & & \\ & & \ddots & \\ \mathbf{0} & & & \mathrm{e}^{\mathbf{J}_{n_k}(\lambda_k)} \end{pmatrix}, \end{aligned}$$

where exponentials of Jordan blocks are given by (2.16).

2.4.2 Continuous–time Markov Chain under Stochastic Time

Recall that transition probabilities of a homogeneous continuous–time Markov chain are given by

$$\mathbf{P}(t) = \exp\left(t\mathbf{Q}\right).$$

Let us try to move to the stochastic time, where the time evolution is modeled through a time–inhomogeneous Lévy subordinator T_t.

Conditioned on the realization T_t the transition probability matrix is

$$\mathbf{P}_{T_t}(t) = \exp\left(T_t\mathbf{Q}\right).$$

The unconditioned transition probability matrix is then

$$\mathbf{P}(t) = \mathsf{E}\left[\exp\left(T_t\mathbf{Q}\right)\right].$$

Let $\mathbf{Q} = \mathbf{B}\mathbf{D}\mathbf{B}^{-1}$ be the Jordan decomposition of the matrix $\mathbf{Q}$, then we have from previous subsection

$$\begin{aligned} \mathbf{P}(t) &= \mathsf{E}\left[\exp\left(T_t\mathbf{Q}\right)\right] \\ &= \mathbf{B}\mathsf{E}\left[\exp\left(T_t\mathbf{D}\right)\right]\mathbf{B}^{-1}. \end{aligned}$$

We need to compute the expectation

$$\mathsf{E}\left[\exp\left(T_t\mathbf{D}\right)\right] = \mathsf{E}\left[\begin{pmatrix} \exp\left(T_t\mathbf{J}_{n_1}(\lambda_1)\right) & 0 & \cdots & 0 \\ 0 & \exp\left(T_t\mathbf{J}_{n_2}(\lambda_2)\right) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \exp\left(T_t\mathbf{J}_{n_k}(\lambda_k)\right) \end{pmatrix}\right].$$

Now, we need to investigate what have changed in the computation of the exponential of a Jordan block by moving to the stochastic time in comparison with (2.16),

$$\mathrm{e}^{T_t\mathbf{J}_n(\lambda)} = \mathrm{e}^{T_t\lambda\mathbf{I}+T_t\mathbf{N}} = \mathrm{e}^{T_t\lambda}\begin{pmatrix} 1 & T_t & \frac{T_t^2}{2} & \dots & \frac{T_t^{n-1}}{(n-1)!} \\ 0 & 1 & T_t & \dots & \frac{T_t^{n-2}}{(n-2)!} \\ 0 & 0 & 1 & \dots & \frac{T_t^{n-3}}{(n-3)!} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \end{pmatrix}.$$

Note that powers of T_t in the exponential matrix $\mathrm{e}^{T_t\mathbf{N}}$ came from the Taylor series. We see that we need to compute

$$\mathsf{E}\left[T_t^i\mathrm{e}^{T_t\lambda}\right], \qquad i = 0, \dots, n-1,$$

if expectations exist. For $i = 0$ the expectation exists since λ is nonpositive[12] and we get the Laplace transform. For $i > 0$ we need to now particular distribution of Lévy process.

In the special case when the Lévy subordinator T_t is time–homogeneous and the matrix $\mathbf{A}$ is diagonalizable (matrix $\mathbf{D}$ is diagonal), Jordan blocks are one–dimensional and equal to

$$\begin{aligned}\mathsf{E}\left[\mathrm{e}^{T_t\mathbf{J}_n(\lambda)}\right] &= \mathsf{E}\left[\mathrm{e}^{T_t\lambda}\right] \\ &= \mathrm{e}^{t\eta(\lambda)}.\end{aligned}$$

where $\eta(u)$ is the Laplace exponent of the subordinator T_t. Let $\mathbf{E}$ be

$$\mathbf{E} = \begin{pmatrix} \eta(D_{11}) & 0 & \cdots & 0 \\ 0 & \eta(D_{22}) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \eta(D_{KK}) \end{pmatrix},$$

then it holds

$$\mathsf{E}\left[\exp\left(T_t\mathbf{D}\right)\right] = \exp\left(t\mathbf{E}\right),$$

and transition probabilities are then given by

$$\begin{aligned}\mathbf{P}(t) &= \mathbf{B}\exp\left(t\mathbf{E}\right)\mathbf{B}^{-1} \\ &= \exp\left(t\mathbf{B}\mathbf{E}\mathbf{B}^{-1}\right) \\ &= \exp(t\tilde{\mathbf{Q}}),\end{aligned}$$

[12] The eigenvalue of the generator matrix $\mathbf{Q}$ with the biggest real part, is only one and equals to 0. The rest of eigenvalues have negative real part. Proof can be found for example in Dupač and Dupačová (1980, Page 43).

where $\tilde{\mathbf{Q}} = \mathbf{BEB}^{-1}$. We see that we get a homogeneous continuous–time Markov chain with a different generator matrix. However, it is only the special case, if we have a time inhomogeneous subordinator or at least one Jordan block with dimension higher than 1 we can get a time–inhomogeneous continuous–time Markov chain (the generator $\mathbf{Q}_t$ will depend on time t.

2.4.3 Maximum Likelihood Estimation

Firstly we will derive the maximum likelihood of a time–changed exponentially distributed random variable and later on we will extend it to the generator matrix estimator.

Time–changed Exponential Maximum Likelihood

Let τ be a sojourn time, such that its time–changed probability distribution (changed with Lévy subordinator) has the exponential distribution with parameter λ. That is for fixed realization T_t

$$\mathsf{P}[\tau \leq T_t] = 1 - \mathrm{e}^{-T_t\lambda}.$$

The probability distribution of the time–unchanged τ, which we denote by $\bar{\tau}$, is then

$$\begin{aligned}\mathsf{P}[\bar{\tau} \leq t] &= \mathsf{E}\left[\mathbf{1}_{\{\bar{\tau}\leq t\}}\right] \\ &= \mathsf{E}\left[\mathsf{E}\left[\mathbf{1}_{\{\tau\leq T_t\}}|T_t\right]\right] \\ &= 1 - \mathsf{E}\left[\mathrm{e}^{-T_t\lambda}\right] \\ &= 1 - \mathrm{e}^{-\eta_t(\lambda)}, \end{aligned} \tag{2.17}$$

where $\eta_t(\lambda)$ is the Laplace exponent of the Lévy subordinator. By differentiating (2.17) we get the density function of $\bar{\tau}$

$$f_{\bar{\tau}}(t) = \mathrm{e}^{-\eta_t(\lambda)}\eta_t'(\lambda),$$

where $\eta_t'(\lambda)$ is the derivative of the Laplace exponent with respect to time t, that is

$$\eta_t'(\lambda) = b_t u + \int_0^\infty \left(1 - \mathrm{e}^{-ux}\right)\nu_t(\,\mathrm{d}x)$$

Let $t_1, \ldots, t_n$ be independent realizations of the sojourn time $\bar{\tau}$. Further let $\boldsymbol{\alpha}$ be the p-dimensional vector of parameters of the probability distribution of the subordinator with $p + 1 < n$. The maximum likelihood is then

$$L(\lambda, \boldsymbol{\alpha}) = \prod_{i=1}^{n} \mathrm{e}^{-\eta_{t_i}(\lambda;\boldsymbol{\alpha})}\eta_{t_i}'(\lambda;\boldsymbol{\alpha}),$$

and the log–likelihood is

$$l(\lambda, \boldsymbol{\alpha}) = \sum_{i=1}^{n} \log\left(\eta_{t_i}'(\lambda;\boldsymbol{\alpha})\right) - \sum_{i=1}^{n} \eta_{t_i}(\lambda;\boldsymbol{\alpha}).$$

Let us differentiate the log–likelihood with respect to λ and components of $\boldsymbol{\alpha}$, and set derivatives equal to 0. We get $p + 1$ equations.

Note that in the conditioned distribution function of the exponential distribution we have term

$$\mathrm{e}^{-T_t\lambda},$$

where we can take any $c > 0$ and do the following

$$\mathrm{e}^{-T_t\lambda} = \mathrm{e}^{-cT_t\frac{\lambda}{c}} = \mathrm{e}^{-\tilde{T}_t\tilde{\lambda}}. \tag{2.18}$$

The positive multiple of the subordinator T_t is also the subordinator and the positive multiple of the intensity λ is also the intensity. We see that the solution of log–likelihood derivative equations will not be unique. Therefore we need to add some additional condition on an arbitrary parameter.

As an example let us compute the estimator if we use the α-stable subordinator. The Laplace exponent of the α-stable subordinator is

$$\eta_t(\lambda;\alpha) = t\lambda^\alpha.$$

The log–likelihood is

$$\begin{aligned} l(\lambda,\alpha) &= \sum_{i=1}^{n}\log\left(\eta'_{t_i}(\lambda;\alpha)\right) - \sum_{i=1}^{n}\eta_{t_i}(\lambda;\alpha) \\ &= n\alpha\log(\lambda) - \lambda^\alpha\sum_{i=1}^{n}t_i. \end{aligned}$$

By differentiating with respect to λ and setting equal to 0 we get

$$\begin{aligned} 0 = \frac{\partial l(\lambda,\alpha)}{\partial\lambda} &= \frac{n\alpha}{\lambda} - \alpha\lambda^{\alpha-1}\sum_{i=1}^{n}t_i \\ 0 &= n - \lambda^\alpha\sum_{i=1}^{n}t_i \\ \lambda^\alpha &= \frac{n}{\sum_{i=1}^{n}t_i}. \end{aligned}$$

By differentiating the log–likelihood with respect to α we get exactly same equation. As we remarked above, the solution is not unique. Note that $\alpha \in (0,1]$ and $\alpha = 1$ is the degenerate case when the time evolution is deterministic. Hence let us choose α equal to an arbitrary $0 < c < 1$. Then the maximum likelihood estimator is

$$\begin{aligned} \hat{\alpha} &= c, \\ \hat{\lambda} &= \left(\frac{n}{\sum_{i=1}^{n}t_i}\right)^{\frac{1}{c}}. \end{aligned}$$

For further interesting discussion remains the optimal choice of c, but since the remark above, it should not influenced the estimator as far as $0 < c < 1$.

Time–changed Markov Chain Maximum Likelihood

Recall that a homogeneous continuous–time Markov chain can be defined through an increasing sequence of stopping times J_n with exponentially distributed inter–arrival times, and an embedded discrete–time Markov chain with transition probabilities from state i to j equal to $\frac{\lambda_{ij}}{\lambda_i}$. If we change the time evolution, the embedded discrete–time Markov chain remains untouched, but the exponential distribution will change in manner as described above. The log–likelihood of the time–unchanged chain with observations up to horizon T^* (assuming we are given by initial state i_0)

$$\begin{aligned} L(\mathbf{Q}) = q_{i_0} \exp(-q_{i_0} J_1) \\ \times \prod_{k=1}^{m-1} \frac{q_{i_{k-1} i_k}}{q_{i_{k-1}}} q_{i_k} \exp\big(- q_{i_k}(J_k - J_{k-1})\big) \\ \times \frac{q_{i_{m-1} i_m}}{q_{i_{m-1}}} \exp\big(- q_{i_m}(T^* - J_m)\big), \end{aligned}$$

will change to

$$\begin{aligned} L(\mathbf{Q}, \boldsymbol{\alpha}) = \eta'_{J_1}(q_{i_0}; \boldsymbol{\alpha}) \exp\big(- \eta_{J_1}(q_{i_0}; \boldsymbol{\alpha})\big) \\ \times \prod_{k=1}^{m-1} \frac{q_{i_{k-1} i_k}}{q_{i_{k-1}}} \eta'_{J_k - J_{k-1}}(q_{i_k}; \boldsymbol{\alpha}) \exp\big(- \eta_{J_k - J_{k-1}}(q_{i_k}; \boldsymbol{\alpha})\big) \\ \times \frac{q_{i_{m-1} i_m}}{q_{i_{m-1}}} \exp\big(- \eta'_{T^* - J_m}(q_{i_m}; \boldsymbol{\alpha})\big) \end{aligned} \tag{2.19}$$

For further computation we need particular choice of the subordinator distribution with some additional condition, for example on its expected value or some fixed value of the parameter to get a unique maximum likelihood estimation.

Again, let us compute the maximum likelihood estimator explicitly for an α-stable subordinator. We have to estimate q_{ij} and α. Same as in the case of an exponentially distributed variable the solution will not be unique, unless an additional condition is added. So let $\alpha = c$, where $0 < c < 1$. Recall that

$$\begin{aligned} \eta_t(u; \alpha) &= t u^\alpha, \\ \eta'_t(u; \alpha) &= u^\alpha. \end{aligned}$$

By substitution in (2.19) we get

$$\begin{aligned}
L(\mathbf{Q},\alpha) &= q_{i_0}^{\alpha}\exp(-J_1 q_{i_0}^{\alpha}) \\
&\quad\times \prod_{k=1}^{m-1}\frac{q_{i_{k-1}i_k}}{q_{i_{k-1}}} q_{i_k}^{\alpha}\exp\big(-(J_k-J_{k-1})q_{i_k}^{\alpha}\big) \\
&\quad\times \frac{q_{i_{m-1}i_m}}{q_{i_{m-1}}}\exp\big(-(T^*-J_m)q_{i_m}^{\alpha}\big) \\
&= q_{i_0}^{\alpha-1}\exp(-J_1 q_{i_0}^{\alpha}) \\
&\quad\times \prod_{k=1}^{m-1} q_{i_{k-1}i_k} q_{i_k}^{\alpha-1}\exp\big(-(J_k-J_{k-1})q_{i_k}^{\alpha}\big) \\
&\quad\times q_{i_{m-1}i_m}\exp\big(-(T^*-J_m)q_{i_m}^{\alpha}\big) \\
&= \prod_{i=1}^{K}\exp\big(-q_i^{\alpha}R_i(T^*)\big)\prod_{\substack{j=1\\ j\neq i}}^{K}\big(q_{ij}q_j^{\alpha-1}\big)^{n_{ij}(T^*)}.
\end{aligned}$$

The log–likelihood is then

$$\begin{aligned}
l(\mathbf{Q},\alpha) &= \sum_{i=1}^{K}\Big(-q_i^{\alpha}R_i(T^*) + \sum_{\substack{j=1\\ j\neq i}}^{K} n_{ij}(T^*)\big(\log(q_{ij}) + (\alpha-1)\log(q_j)\big)\Big) \\
&= -\sum_{i=1}^{K}\Big(\sum_{\substack{j=1\\ j\neq i}}^{K} q_{ij}\Big)^{\alpha} R_i(T^*) + \sum_{i=1}^{K}\sum_{\substack{j=1\\ j\neq i}}^{K} n_{ij}(T^*)\log(q_{ij}) \\
&\quad + (\alpha-1)\sum_{i=1}^{K}\sum_{\substack{j=1\\ j\neq i}}^{K} n_{ij}(T^*)\log\Big(\sum_{\substack{k=1\\ k\neq j}}^{K} q_{jk}\Big),
\end{aligned}$$

where we used that $q_i = \sum_{\substack{j=1\\ j\neq i}}^{K} q_{ij}$. This relation we will use also the other way around.

If we differentiate the log–likelihood with respect to q_{ij} and set equal to 0 we get equations

$$\frac{\partial l(\mathbf{Q},\alpha)}{\partial q_{ij}} = -\alpha q_i^{\alpha-1}R_i(T^*) + \frac{n_{ij}(T^*)}{q_{ij}} + \frac{(\alpha-1)}{q_i}\sum_{\substack{k=1\\ k\neq i}}^{K} n_{ki}(T^*) = 0, \qquad i\neq j,$$

which we can adjust into form

$$\alpha q_i^{\alpha-1}R_i(T^*) - \frac{(\alpha-1)}{q_i}\sum_{\substack{k=1\\ k\neq i}}^{K} n_{ki}(T^*) = \frac{n_{ij}(T^*)}{q_{ij}}. \tag{2.20}$$

From (2.20) we see that for fixed i the ratio $\frac{n_{ij}(T^*)}{q_{ij}}$ is constant for every j. Let $m \in \{1,2,\ldots,i-1,,i+1,\ldots,K\}$ be an arbitrary index different than i. Then it holds

$$q_{ij} = \frac{n_{ij}(T^*)}{n_{im}(T^*)}q_{im}, \qquad j\neq i. \tag{2.21}$$

If we substitute (2.21) into the equation (2.20) associated with indices i and m and adjust it a bit we will get

$$\alpha q_i^{\alpha-1} R_i(T^*) - \frac{(\alpha-1)}{q_i} \sum_{\substack{k=1\\k\neq i}}^{K} n_{ki}(T^*) = \frac{n_{im}(T^*)}{q_{im}}$$

$$\alpha \Big(\sum_{\substack{j=1\\j\neq i}}^{K} q_{ij} \Big)^{\alpha-1} R_i(T^*) - \frac{(\alpha-1)}{\Big(\sum_{\substack{j=1\\j\neq i}}^{K} q_{ij} \Big)} \sum_{\substack{k=1\\k\neq i}}^{K} n_{ki}(T^*) = \frac{n_{im}(T^*)}{q_{im}}$$

$$\alpha \Big(\sum_{\substack{j=1\\j\neq i}}^{K} \frac{n_{ij}(T^*)}{n_{im}(T^*)} q_{im} \Big)^{\alpha-1} R_i(T^*) - \frac{(\alpha-1)}{\sum_{\substack{j=1\\j\neq i}}^{K} \frac{n_{ij}(T^*)}{n_{im}(T^*)} q_{im}} \sum_{\substack{k=1\\k\neq i}}^{K} n_{ki}(T^*) = \frac{n_{im}(T^*)}{q_{im}}$$

$$\alpha \Big(\frac{q_{im}}{n_{im}(T^*)} \Big)^{\alpha} \Big(\sum_{\substack{j=1\\j\neq i}}^{K} n_{ij}(T^*) \Big)^{\alpha-1} R_i(T^*) - (\alpha-1) \frac{\sum_{\substack{k=1\\k\neq i}}^{K} n_{ki}(T^*)}{\sum_{\substack{j=1\\j\neq i}}^{K} n_{ij}(T^*)} = 1.$$

After few more adjustments we get

$$q_{im}^{\alpha} = \frac{\big(n_{im}(T^*)\big)^{\alpha} \Big(\sum_{\substack{j=1\\j\neq i}}^{K} n_{ij}(T^*) - (1-\alpha) \sum_{\substack{k=1\\k\neq i}}^{K} n_{ki}(T^*) \Big)}{\alpha R_i(T^*) \Big(\sum_{\substack{j=1\\j\neq i}}^{K} n_{ij}(T^*) \Big)^{\alpha}}$$

$$q_{im} = \frac{n_{im}(T^*)}{\sum_{\substack{j=1\\j\neq i}}^{K} n_{ij}(T^*)} \sqrt[\alpha]{\frac{\sum_{\substack{j=1\\j\neq i}}^{K} n_{ij}(T^*) - (1-\alpha) \sum_{\substack{k=1\\k\neq i}}^{K} n_{ki}(T^*)}{\alpha R_i(T^*)}}. \tag{2.22}$$

There is one thing which we need to take care of. Under the root there might be negative number, which depend on our choice of $\alpha = c$. Hence we need to choose c sufficiently small to q_{im} be well defined for all $i \neq m$. Then we have maximum likelihood estimator

$$\hat{\alpha} = c,$$

$$\hat{q}_{im} = \frac{n_{im}(T^*)}{\sum_{\substack{j=1\\j\neq i}}^{K} n_{ij}(T^*)} \sqrt[c]{\frac{\sum_{\substack{j=1\\j\neq i}}^{K} n_{ij}(T^*) - (1-c) \sum_{\substack{k=1\\k\neq i}}^{K} n_{ki}(T^*)}{c R_i(T^*)}}, \qquad i \neq m,$$

$$\hat{q}_{ii} = -\sum_{\substack{m=1\\m\neq i}}^{K} \hat{q}_{im}, \qquad i = 1, \ldots, K.$$

Note that for $c = 1$ we get the classical maximum likelihood estimator for continuous–time Markov chain

$$\hat{q}_{im} = \frac{n_{im}(T^*)}{R_i(T^*)}.$$

2.5 Generator Estimator from Partially Observed Data

Let us imagine the situation when we want to model some system using a continuous–time Markov chain. We have some past observations of the system from which we need to estimate the generator matrix $\mathbf{Q}$. If we have continuous–time observations of the system, it is easy to compute the maximum likelihood estimator. Problem raise up if we do not have continuous observations and we have just discretely (partially) observed data. Without loss of generality let us assume that we have yearly data observations.

Using the cohort method (2.9) we can estimate the yearly transition probability matrix and denote it by $\hat{\mathbf{P}}$. Then we would like to derive the estimator of the generator $\mathbf{Q}$ which generate the transition matrix $\hat{\mathbf{P}}$. This problem is called the *embedability problem*. Not every probability matrix is embeddable and if there exists the generator $\mathbf{Q}$ it has not to be unique.

Under the assumption that the process can change the state just once a year Jarrow et al. (1997) shows how to derive $\hat{\mathbf{Q}}$ from $\hat{\mathbf{P}}$. In Israel et al. (2001), authors work in a more general setting without any restrictive assumptions and they provide sufficient conditions on a transition matrix $\hat{\mathbf{P}}$ for the existence or for the failure of the existence of the generator $\hat{\mathbf{Q}}$. Their idea is to compute the logarithm of matrix $\hat{\mathbf{P}}$. The resulting matrix does not need to be a valid generator.[13] Israel et al. (2001) come up with two ad–hoc methods how to get the nearest possible valid generator. Both method set negative off–diagonal elements equal to zero and then adjust other elements to get zero row–sum. We describe them in Section 2.5.1

Other method how to get a valid generator matrix, which generates the transition probability matrix as near as possible to $\hat{\mathbf{P}}$, is from Kreinin and Sidelnikova (2001) (QOG method) and described in Section 2.5.2.

In Bladt and Sørensen (2005) they choose a bit different approach. They did not try to find the generator of the matrix $\hat{\mathbf{P}}$, but rather estimate the generator matrix $\mathbf{Q}$ using all available information, i.e., partial observed data. As a tool how to deal with the incomplete information for the maximum likelihood estimator, they used the EM algorithm (Section 2.5.3) and MCMC (Section 2.5.4).

A comprehensive comparison of these five methods can be found in Inamura (2006) who shows that the best performance gives the MCMC method.

2.5.1 DA and WA Method

Israel et al. (2001) suggest to use the matrix logarithm through the Taylor series for computing the estimator

$$\hat{\mathbf{Q}} = \log \hat{\mathbf{P}} = (\hat{\mathbf{P}} - \mathbf{I}) - \frac{(\hat{\mathbf{P}} - \mathbf{I})^2}{2} + \frac{(\hat{\mathbf{P}} - \mathbf{I})^3}{3} - \frac{(\hat{\mathbf{P}} - \mathbf{I})^4}{4} + \dots,$$

where $\mathbf{I}$ is the identity matrix. Israel et al. (2001) also provide some necessary conditions for the existence of the real matrix logarithm of the matrix $\hat{\mathbf{P}}$. Nevertheless, even if the real logarithm exists, the resulting $\hat{\mathbf{Q}}$ does not need to be a valid generator. Israel et al. (2001) propose two ad–hoc methods how to fix it — the *diagonal adjustment method* (DA) and the *weighted adjustment method* (WA). Methods proceed in two steps. In the first step they fix all off–diagonal elements and in the second step they adjust other elements of $\hat{\mathbf{Q}}$ to be a valid generator.

[13] Nonpositive elements on the diagonal, nonnegative elements otherwise and zero row–sum.

1. Set all off–diagonal negative elements equal to 0,

$$\hat{\lambda}_{ij} = \max\left\{\hat{\lambda}_{ij}, 0\right\}, \qquad i \neq j.$$

2. For DA method set the diagonal element equal to minus sum of off–diagonal elements

$$\hat{\lambda}_{ii} = -\sum_{j \neq i} \hat{\lambda}_{ij}, \qquad i = 1, \ldots, K.$$

For WA method adjust all nonzero elements relatively to their magnitudes

$$\hat{\lambda}_{ij} = \hat{\lambda}_{ij} - |\hat{\lambda}_{ij}| \frac{\sum_{j=1}^{K} \hat{\lambda}_{ij}}{\sum_{j=1}^{K} |\hat{\lambda}_{ij}|}, \qquad i, j = 1, \ldots, K.$$

2.5.2 Quasi–optimization Method

We are given by the matrix $\hat{\mathbf{P}}$ and we would like to find a generator which generates the matrix $\hat{\mathbf{P}}$ or generate a matrix which is as close as possible to $\hat{\mathbf{P}}$. Let us denote a set of all possible $K \times K$-dimensional generators by $\mathcal{Q}$, that is

$$\mathcal{Q} := \left\{ \mathbf{Q} \in \mathbb{R}^{K \times K} \middle| \sum_{j=1}^{K} q_{ij} = 0, q_{ij} \geq 0, i \neq j \right\}.$$

We are trying to find the element $\mathbf{Q}^* \in \mathcal{Q}$, which minimize

$$\left\| \exp(\mathbf{Q}) - \hat{\mathbf{P}} \right\|.$$

That is not an easily tractable problem. Kreinin and Sidelnikova (2001) suggest that if $\exp(\mathbf{Q})$ is near to $\hat{\mathbf{P}}$, then also their logarithms should be near to each other. Therefore they formulate a similar problem, where they are looking for $\mathbf{Q}^* \in \mathcal{Q}$, which minimize

$$\left\| \mathbf{Q} - \log \hat{\mathbf{P}} \right\|.$$

Since conditions on a valid generator are conditions on every row we can split the optimization problem into K separate problems and perform for each row following optimization algorithm

Let us denote the particular row of $\hat{\mathbf{P}}$, for which we are performing the algorithm, by $\boldsymbol{a} = (a_1, \ldots, a_K)$ and assume the cone $\mathcal{C}$ given by

$$\mathcal{C} = \left\{ z \in \mathbb{R}^K \middle| \sum_{i=1}^{K} z_i = 0, \sum_{i=1}^{K} \mathbf{1}_{\{z_i < 0\}} \in \{0, 1\} \right\}.$$

Note that $\sum_{i=1}^{K} \mathbf{1}_{\{z_i<0\}} = 0$ implies $\boldsymbol{z} = \boldsymbol{0}$.

Now we are looking for $\boldsymbol{z}^* \in \mathcal{C}$ which is the argument of

$$\min_{z \in \mathcal{C}} \sum_{i=1}^{K} (a_i - z_i)^2.$$

Kreinin and Sidelnikova (2001) provide fast algorithm for finding $\boldsymbol{z}^*$ based on the algorithm of Tuenter (2001). The algorithm proceeds as following

1. If $\boldsymbol{a} \in \mathcal{C}$ then $\boldsymbol{z}^* = \boldsymbol{a}$ and stop.
2. Construct $\boldsymbol{b} = (b_1, \ldots, b_K)$ as
$$b_i = a_i - \frac{1}{K}\sum_{i=1}^{K} a_i.$$
3. Let $\pi(\cdot)$ be a permutation which orders $\boldsymbol{b}$ in the increasing order, that is $b_i \leq b_{i+1}$. Then compute $\tilde{\boldsymbol{a}} = \pi(\boldsymbol{b})$.
4. Find the smallest m^*, for $1 \leq m \leq (K-1)$ which satisfies[14]
$$(K-m+1)\tilde{a}_{m+1} - \left(\tilde{a}_1 + \sum_{i=m+1}^{K} \tilde{a}_i \geq 0\right).$$
5. Let $\mathcal{J}$ be given by
$$\mathcal{J} := \{i \in \mathbb{N} | 2 \leq i \leq m^*\}.$$
Note that $\mathcal{J}$ can be an empty set if $m^* = 1$.
6. Construct $\boldsymbol{z}^*$ in as
$$z_i = \begin{cases} 0, & i \in \mathcal{J}, \\ \tilde{a}_i - \frac{1}{K-m^*+1}\left(\sum_{j \notin \mathcal{J}} \tilde{a}_j\right), & \text{otherwise.} \end{cases}$$
Note that in the case of $m^* = 1$, we get simply $\boldsymbol{z}^* = \tilde{\boldsymbol{a}}$.
7. As the optimal solution for this row return $\pi^{-1}(\boldsymbol{z}^*)$, where π^{-1} is the inverse permutation to π.

If we perform the algorithm for every row, we get the solution to QOG problem.

2.5.3 Expectation Maximization Method

The idea of the expectation maximization method (EM) is quite simple and appears to be useful in the situation of incomplete data, where the maximum likelihood could not be applied directly. For more general information about EM algorithm see McLachlan and Krishnan (1997). The application for the presented problem was proposed by Bladt and Sørensen (2005).

We assume that we have discrete (one year) observation of n realizations of the continuous–time Markov chain. We can not apply maximum likelihood since we do not observe these realizations continuously, but we know in which state the chain started and where it ended. For computing the maximum likelihood we would need to know the number of transitions from i to j through the year denoted by n_{ij} and time spend in state i by all realizations denoted by R_i.

The EM algorithm consists of two steps. In the first step n_{ij} and R_i is replaced by their expected values, given the partial observation x^{obs} (starting and ending point). In the second step we compute the maximum likelihood estimator of the generator $\mathbf{Q}$ using n_{ij} and R_i from the first step. Formally the algorithm proceeds as follows

[14] In the original paper Kreinin and Sidelnikova (2001) or in Inamura (2006) they request $m \geq 2$, which we think is wrong and do not work in some special cases (matrices in the credit risk are usually not the case). We do not have the theoretical proof of this adjustment, just numerical. The theoretical proof remain for further research.

1. Let $\mathbf{Q}_0$ be some starting point of the algorithm. The possible choice is estimator $\mathbf{Q}_{DA}$ or $\mathbf{Q}_{WA}$.
2. Calculate $\mathsf{E}\left[n_{ij}|x^{obs}, \mathbf{Q}_k\right]$ and $\mathsf{E}\left[R_i|x^{obs}, \mathbf{Q}_k\right]$ using $\mathbf{Q}_k$.
3. Calculate $\mathbf{Q}_{k+1}$, where ij-th element is given by

$$q_{ij} = \frac{\mathsf{E}\left[n_{ij}|x^{obs}, \mathbf{Q}_k\right]}{\mathsf{E}\left[R_i|x^{obs}, \mathbf{Q}_k\right]}.$$

4. If $||\mathbf{Q}_k - \mathbf{Q}_{k+1}|| < \varepsilon$ then return $\mathbf{Q}_k$ as the optimal solution, else go to step 2.

The only thing we need to clarify is how to compute expectations $\mathsf{E}\left[n_{ij}|x^{obs}, \mathbf{Q}\right]$ and $\mathsf{E}\left[R_i|x^{obs}, \mathbf{Q}\right]$. Note that

$$\mathsf{E}\left[n_{ij}|x^{obs}, \mathbf{Q}\right] = \sum_{k=1}^{n} \mathsf{E}\left[n_{ij}^k|x^k, \mathbf{Q}\right],$$
$$\mathsf{E}\left[R_i|x^{obs}, \mathbf{Q}\right] = \sum_{k=1}^{n} \mathsf{E}\left[R_i^k|x^k, \mathbf{Q}\right],$$

where x^k is the partial observation of the k-th realization of the chain, n_{ij}^k is the number of transitions from state i to state j in the k-th realization and similar to R_i^k. Then it holds

$$\mathsf{E}\left[n_{ij}^k|x^k, \mathbf{Q}\right] = \frac{1}{D}\boldsymbol{e}_{x^k(0)}^\top \left(q_{ij}\int_0^1 \exp(s\mathbf{Q})\boldsymbol{e}_i\boldsymbol{e}_j^\top \exp\left((1-s)\mathbf{Q}\right)\mathrm{d}s\right)\boldsymbol{e}_{x^k(1)}, \tag{2.23}$$

$$\mathsf{E}\left[R_i^k|x^k, \mathbf{Q}\right] = \frac{1}{D}\boldsymbol{e}_{x^k(0)}^\top \left(\int_0^1 \exp(s\mathbf{Q})\boldsymbol{e}_i\boldsymbol{e}_i^\top \exp\left((1-s)\mathbf{Q}\right)\mathrm{d}s\right)\boldsymbol{e}_{x^k(1)}, \tag{2.24}$$

$$D = \boldsymbol{e}_{x^k(0)}^\top \exp(\mathbf{Q})\boldsymbol{e}_{x^k(1)}, \tag{2.25}$$

where $x^k(0)$ is the starting and $x^k(1)$ the ending state of the k-th realization, $\boldsymbol{e}_i$ is a vector of zeros with one on i-th place. For the detail derivation of formulas (2.23) – (2.25) see Bladt and Sørensen (2005) and/or Inamura (2006, Appendix B). To be able to compute any of formulas (2.23) – (2.25) we need to know ho to compute integrals involving matrix exponentials. An easy formula suggested by Van Loan (1978) use two square matrices $\mathbf{F}$ and $\mathbf{A}$ such that

$$\begin{pmatrix} \mathbf{F}_{11} & \mathbf{F}_{12} \\ \mathbf{0} & \mathbf{F}_{22} \end{pmatrix} = \exp\left[t\begin{pmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} \\ \mathbf{0} & \mathbf{A}_{22} \end{pmatrix}\right],$$

Then it holds that

$$\mathbf{F}_{11} = \exp(t\mathbf{A}_{11}),$$
$$\mathbf{F}_{22} = \exp(t\mathbf{A}_{22}),$$
$$\mathbf{F}_{12} = \int_0^t \exp\left((t-s)\mathbf{A}_{11}\right)\mathbf{A}_{12}\exp(s\mathbf{A}_{22})\,\mathrm{d}s.$$

If we substitute $\mathbf{A}_{11}$ and $\mathbf{A}_{22}$ by $\mathbf{Q}$, $\mathbf{A}_{12}$ by $\boldsymbol{e}_i\boldsymbol{e}_i^\top$ or $\boldsymbol{e}_i\boldsymbol{e}_j^\top$ and $t = 1$, we can easily compute $\exp(\mathbf{A})$ and get $\mathbf{F}_{12}$, which is what we are looking for.

2.5.4 Markov Chain Monte Carlo Method

The last method from the comparison of Inamura (2006) is the MCMC method proposed same as EM method by Bladt and Sørensen (2005). The idea is also very simple. Let $\mathbf{Q}$ be a random matrix satisfying conditions on the generator matrix with the distribution $p(\mathbf{Q})$. Let us draw some particular $\mathbf{Q}$ from $p(\mathbf{Q})$ and simulate the run X of the chain given the generator matrix $\mathbf{Q}$ in such way, that the partial observation (starting and ending state) agrees with data which we have observed x^{obs}. Then we compute the maximum likelihood from the run X and adjust the distribution of $\mathbf{Q}$ according to that. The setting is Bayesian and from the Bayes theorem we have

$$\begin{aligned} p(\mathbf{Q}|X, x^{obs}) &\propto p(Q)p(X|\mathbf{Q}) \\ &\propto p(\mathbf{Q})L(\mathbf{Q}) \\ &= p(\mathbf{Q}) \prod_{i=1}^{K} \prod_{\substack{j=1 \\ j\neq i}}^{K} q_{ij}^{n_{ij}} \exp(-q_{ij}R_i). \end{aligned} \tag{2.26}$$

Thus, $p(\mathbf{Q})$ is the prior distribution and $p(\mathbf{Q}|X, x^{obs})$ is the posterior distribution. This distribution adjustment of the matrix $\mathbf{Q}$ we iterate n-times and after that we forget first l iterations, which are called the *burn–in period*. Then we get the estimator

$$\hat{\mathbf{Q}} = \frac{1}{n-l} \sum_{i=l+1}^{n} \mathbf{Q}_i,$$

where $\mathbf{Q}_i$ is the matrix drawn in the i-th iteration. First l iterations are forgotten to enable the process to reach a stationarity. Bladt and Sørensen (2005) proposed to use a gamma distribution for off–diagonal elements of $\mathbf{Q}$ (diagonal elements are given by $q_{ii} = -\sum_{j\neq i} q_{ij}$). Then the prior distribution is

$$p(\mathbf{Q}) \propto \prod_{i=1}^{K} \prod_{\substack{j=1 \\ j\neq i}}^{K} q_{ij}^{\alpha_{ij}-1} \exp(-q_{ij}\beta_i),$$

where α_{ij} and β_i are parameters of the gamma distribution. The posterior distribution is then from (2.26) again the gamma distribution, but with different parameters

$$p(\mathbf{Q}|X, x^{obs}) \propto \prod_{i=1}^{K} \prod_{\substack{j=1 \\ j\neq i}}^{K} q_{ij}^{n_{ij}+\alpha_{ij}-1} \exp\big(-q_{ij}(R_i + \beta_i)\big).$$

Let α_{ij} and β_i are given parameters of the gamma distribution and x^{obs} is the partial observation. The algorithm proceeds as follows

1. Let $k = 1$
2. Draw $\mathbf{Q}_k$ from distribution $\Gamma(\boldsymbol{\alpha}, \boldsymbol{\beta})$.
3. Simulate the run X of the chain in such way that X agrees with the partial observation x^{obs}.

4. Compute n_{ij} and R_i from the run X and update α_{ij} and β_i,

$$\begin{aligned} \alpha_{ij} &= \alpha_{ij} + n_{ij}, \\ \beta_i &= \beta_i + R_i, \\ k &= k + 1. \end{aligned}$$

5. If $k = (n+1)$ return $\frac{1}{n-l}\sum_{i=l+1}^{n} \mathbf{Q}_i$, else go to step 2.

The simulation of the run X can be done easily by simulating a Markov chain with the initial state according to x^{obs}. Then the run is accepted or rejected if it will end in the desired ending point (according to x^{obs}).

Note that the generator estimators from EM and MCMC method can generate significantly different one–year transition matrix then $\hat{\mathbf{P}}$. It is caused from the similar reason as in the case of the ML estimator compared with the cohort estimator. EM and MCMC method incorporate more information then methods DA, WA and QOG. Therefore it is not easy to compare this two kinds of methods. Sometimes the first one might be more useful then the other one dependent on the observed information.

Inamura (2006) performed a simulation when he has some generator $\mathbf{Q}$ and he simulate the partial observation and compared estimators from these five methods with the original one. The best estimator was by MCMC method. Though *QOG* gives better fit if one is looking for the generator of $\hat{\mathbf{P}}$.

2.5.5 Componentwise Optimization

As an additional method to five previous we add the componentwise optimization (CO) which has similar goal as DA, WA or QOG method — find $\mathbf{Q}$ which generate $\hat{\mathbf{P}}$. The idea is very simple — divide the problem of finding $\mathbf{Q}$ into $(K-1)\times(K-1)$ steps. In each step we are dealing with a one–dimensional problem, where we find

$$\hat{q}_{ij} = \arg\min_{q_{ij}\in[0,c]} \left\| \exp\left(\mathbf{Q}(q_{ij})\right) - \hat{\mathbf{P}} \right\|, \qquad i \neq j, \tag{2.27}$$

where $\mathbf{Q}(q_{ij})$ expresses the dependency of the generator on q_{ij}. Constant c is chosen in conservative way from empirical expectations.[15] The smaller the constant c is, the faster the algorithm proceeds. Constant c can be also different for different choices of indices i and j.

We see that in each step we have fixed all elements of $\mathbf{Q}$ except particular $q_{ij}, i \neq j$ and we perform a one–dimensional optimization. The important thing is, that simultaneously as we are moving q_{ij} we need to adjust other elements of i-th row of $\mathbf{Q}$ to remain valid generator. This one dimensional problem is solvable numerically using the wide variety of methods.[16]

Formally the algorithm is following

1. Find some initial $\mathbf{Q}_0$ using DA or WA method.
2. While convergence is not reached do
   - For $i = 1, \ldots, K$ do
     - For $j = 1, \ldots, K$ do

[15] In the credit risk, the highest intensities are around 0.1, therefore possible conservative choice would be $c = 1$.

[16] We have used the function `optimize` from the statistical software `R`, where is implemented the well–known Fortran subroutine `fmin` based on ALGOL 60 procedure `localmin` given in Brent (1973).

- If $i \neq j$, find $\hat{q}_{ij}$ as

$$\hat{q}_{ij} = \arg \min_{q_{ij} \in [0,c]} \left\| \exp\left(\mathbf{Q}(q_{ij})\right) - \hat{\mathbf{P}} \right\|,$$

  Simultaneously with changing q_{ij} adjust other row elements to $\mathbf{Q}$ remain the valid generator.

3. Return $\mathbf{Q}$ as the optimal solution.

The CO algorithm does not need to lead to the optimal solution, since it converge to some local minimum of one dimension and since we do not know anything about convexity, we can not ensure that the found minimum is a global minimum. Hence we know, that we will get "some" result, but the result can be far away from the optimal solution. What we can do, is to compare it to solutions from previous five methods and then say if it is better solution or it is not. Important thing is also the choice of the initial point since we never converge to worse solution then the initial. Hence if we choose as a starting point the best solution which we get from previous five methods we can just improve using CO. The standalone usage without any reference results can not be done without further discussion about the convexity of the algorithm.

2.6 From m Transition Probability Matrices to One Generator under Stochastic Time

In this section we will try to extend CO method from Section 2.5 in a way of Section 2.4.2 and give the motivation for this extension.

2.6.1 Idea and Motivation

All six previous methods assume that we are given by the one–year partial observation of the system (using cohort method we can estimate matrix $\hat{\mathbf{P}}$) and these methods find the generator estimator $\hat{\mathbf{Q}}$. Let us extend this problem.

We assume that we are given by discrete yearly observations (similar as in the previous section) from sequence of m years. If we assumed that the system does not change its dynamics in any way we can easily use previous methods with an advantage of more observations. We are going to suggest a bit different approach. Let us assume that the system runs under some stochastic transformation of the time which is independent of the Markov chain. Let us explain it in a more detail way.

Let $\mathbf{Q}$ be the valid generator matrix and $T_1, \ldots, T_m$ be some nonnegative random variables. We do not require to be independent or same distributed. Let $t_1, \ldots, t_m$ be their realizations. Then we assume, that in the i-th year, the system follows the continuous–time Markov chain with the generator $t_i\mathbf{Q}$. Hence we are dealing with a problem of estimation of $\hat{t}_1, \ldots, \hat{t}_m$ and matrix $\hat{\mathbf{Q}}$ conditioned on partially observed data. Note that $\hat{t}_1, \ldots, \hat{t}_m$ and $\hat{\mathbf{Q}}$ are not unique since if we multiply $\hat{\mathbf{Q}}$ by any positive constant c and $\hat{t}_1, \ldots, \hat{t}_m$ by $1/c$ we get same Markov chain with generator t_iQ. Therefore we need to add an additional condition on $\hat{t}_1, \ldots, \hat{t}_m$, for example $\sum_{i=1}^{m} \hat{t}_i = m$.

Methods DA and WA do not give us any simple idea how to extend them to the stochastic time setting. QOG also can not be used since the problem can not be split into problems for each row. On the contrary CO method can be extended very easily.

In stochastic time CO method (STCO) we want to minimize

$$\min\left(\sum_{i=1}^{m}\left\|\exp(t_i\mathbf{Q})-\hat{\mathbf{P}}_i\right\|\right),$$

subject to

$$\begin{aligned} q_{ii} &\leq 0, \qquad i=1,\ldots,n\\ q_{ij} &\geq 0, \qquad i\neq j\\ \sum_{j=1}^{K} q_{ij} &= 0,\\ t_i &\geq 0,\\ \sum_{i=1}^{m} t_i &= m. \end{aligned}$$

where $||\cdot||$ is the Euclidean norm. Full description of the algorithm is in Section 2.6.3.

Before we will describe STCO method we will derive the maximum likelihood estimator if we assume that t_i are parameters of the model, not random variables like in Section 2.4.3. The maximum likelihood estimator can be used if we have fully–observed data. For partially observed data we think that EM and MCMC method can be extended to stochastic time setting, but it remains for further research. Nevertheless, for both stochastic time EM or MCMC method, the knowledge of the maximum likelihood is the first step, hence result from next subsection will be useful.

2.6.2 Maximum Likelihood Estimation

Recall that the maximum likelihood of the continuous–time Markov chain is

$$L(\mathbf{Q})=\prod_{i=1}^{K}\prod_{\substack{j=1\\ j\neq i}}^{K} q_{ij}^{n_{ij}(T)}\exp\big(-q_{ij}R_i(T)\big).$$

Let us have observations from m years and let times $t_1,\ldots,t_m$ be not random, but rather some nonnegative parameters. The maximum likelihood will take the form

$$L(\mathbf{Q},\boldsymbol{t})=\prod_{k=1}^{m}\prod_{i=1}^{K}\prod_{\substack{j=1\\ j\neq i}}^{K}(t_k q_{ij})^{n_{ij}(k)}\exp\big(-t_k q_{ij}R_i(k)\big),$$

where $\boldsymbol{t}=(t_1,\ldots,t_m)$, $n_{ij}(k)$ is the number of transitions from i to j in the k-th year and $R_i(k)$ is the total time spend by the process in state i in the k-th year. The log–likelihood is then

$$l(\mathbf{Q},\boldsymbol{t})=\sum_{k=1}^{m}\sum_{i=1}^{K}\sum_{\substack{j=1\\ j\neq i}}^{K}\Big(n_{ij}(k)\log(t_k q_{ij})-t_k q_{ij}R_i(k)\Big).$$

After differentiating we get

$$0 = \frac{\partial l(\mathbf{Q}, \boldsymbol{t})}{\partial q_{ij}} = \sum_{k=1}^{m} \left(\frac{n_{ij}(k)}{q_{ij}} - t_k R_i(k) \right),$$
$$0 = \frac{\partial l(\mathbf{Q}, \boldsymbol{t})}{\partial t_k} = \sum_{i=1}^{K} \sum_{\substack{j=1 \\ j \neq i}}^{K} \left(\frac{n_{ij}(k)}{t_k} - q_{ij} R_i(k) \right),$$

and after few adjustments

$$q_{ij} = \frac{\sum_{k=1}^{m} n_{ij}(k)}{\sum_{k=1}^{m} t_k R_i(k)}, \tag{2.28}$$
$$t_k = \frac{\sum_{i=1}^{K} \sum_{\substack{j=1 \\ j \neq i}}^{K} n_{ij}(k)}{\sum_{i=1}^{K} \sum_{\substack{j=1 \\ j \neq i}}^{K} q_{ij} R_i(k)}. \tag{2.29}$$

From (2.28) and (2.29) we can derive

$$t_l = \frac{\sum_{i=1}^{K} \sum_{\substack{j=1 \\ j \neq i}}^{K} \sum_{k=1}^{m} t_k n_{ij}(l) R_i(k)}{\sum_{i=1}^{K} \sum_{\substack{j=1 \\ j \neq i}}^{K} \sum_{k=1}^{m} n_{ij}(k) R_i(l)}, \qquad l = 1, \ldots, m$$

which is a system of linear equations which can be easily solved by any algebraic method together with the condition

$$\sum_{i=1}^{m} t_i = m.$$

Once we have t_l for $l = 1, \ldots, K$ we can compute q_{ij} for $i \neq j$ through (2.28) and after that

$$q_{ii} = -\sum_{\substack{j=1 \\ j \neq i}}^{K} q_{ij}, \qquad i = 1, \ldots, K.$$

However, if we do not have the continuous–time observation we can not use maximum likelihood estimation. Hence let us try to extend CO method from Section 2.5 to the stochastic time setting.

2.6.3 Stochastic Time Componentwise Optimization

Let us recall that we have m empirical transition matrices $\hat{\mathbf{P}}_1, \ldots, \hat{\mathbf{P}}_m$ and we want to find the generator matrix $\mathbf{Q}$ and vector $\boldsymbol{t}$, such that the transition probability matrix $\exp(t_i \mathbf{Q})$ is as near as possible to $\hat{\mathbf{P}}_i$ for $i = 1, \ldots, m$.

Probably the first thing what one can do, is to compute the average generator

$$\hat{\mathbf{Q}} = \frac{1}{m} \sum_{i=1}^{m} \hat{\mathbf{Q}}_i,$$

where $\hat{\mathbf{Q}}_i$ is derived by any of previous six methods. Then we find the estimator of constant $t_i > 0$ as

$$\hat{t}_i = \arg\min_{t \geq 0} \left\| \exp(t\hat{\mathbf{Q}}) - \hat{\mathbf{P}}_i \right\|.$$

Since $|| \exp(t\hat{\mathbf{Q}}) - \hat{\mathbf{P}}_i ||$ is convex in t and for $t \to \infty$ norm goes to infinity, there exists the unique minimum which can be easily found. Then we need to rescale vector $\boldsymbol{t}$ to hold $\sum_{i=1}^{m} t_i = m$

$$D = \sum_{i=1}^{m} \hat{t}_i,$$

and then set

$$\boldsymbol{t} = \hat{\boldsymbol{t}} \frac{m}{D},$$
$$\mathbf{Q} = \hat{\mathbf{Q}} \frac{D}{m}.$$

This simple approach splits the optimization problem into two steps. In the first step we find the estimator of the generator matrix and in the second step we estimate t_i. Clearly it is not the optimal approach and one can get better result if he estimates the matrix $\hat{\mathbf{Q}}$ and constants $\hat{t}_i$ in one step. However, the estimator $\hat{\mathbf{Q}}$ can be used as a starting point for the stochastic time componentwise optimization (STCO) method.

STCO proceeds in the following way

1. Compute $\mathbf{Q}$ as an average generator as described above.
2. Until the convergence is reached repeat
   - For $i = 1, \ldots, m$ find t_i as
   $$t_i = \arg\min_{t \in [0,c]} || \exp\left(t\mathbf{Q} - \hat{\mathbf{P}}\right) ||$$
   - Compute
   $$D = \sum_{i=1}^{m} t_i.$$
   - Set
   $$\boldsymbol{t} = \boldsymbol{t} \frac{m}{D},$$
   $$\mathbf{Q} = \mathbf{Q} \frac{D}{m}.$$
   - For $i = 1, \ldots, K$ do
     - For $j = 1, \ldots, K$ do
       - If $i \neq j$, find q_{ij} as
       $$q_{ij} = \arg\min_{q_{ij} \in [0,c]} \sum_{k=1}^{m} \left\| \exp\left(t_k \mathbf{Q}(q_{ij})\right) - \hat{\mathbf{P}}_k \right\|,$$
       Simultaneously with changing q_{ij} adjust other row elements of $\mathbf{Q}$ to remain a valid generator.
3. Return $\mathbf{Q}$ as the optimal solution.

Here c is chosen in such way that it is safely higher than any intensity and also higher then possible values of t_i (depend on how volatile behavior of time evolution we expect). If we are not sure we can rather set c really high, for example $c = 100$.

2.7 Affine Processes

The Markov process X_t with some state space $\mathbb{R}_+^n \times \mathbb{R}^{d-n}$, $0 \leq n \leq d$, is called the *regular affine process* if and only if its conditional characteristic function is "exponential–affine", i.e.,

$$\mathsf{E}\left[\mathrm{e}^{iu\cdot X_t}|X_s\right] = \mathrm{e}^{\alpha(t-s,iu)+\beta(t-s,iu)\cdot X_s}, \qquad u \in \mathbb{R}^d,$$

for some functions $\alpha(t, iu)$ and $\beta(t, iu)$ which are differentiable in t and their derivatives are continuous at 0.

Let us assume $d = 1$ and the Markov process X_t follows dynamics

$$\mathrm{d}X_t = \mu(X_t)\,\mathrm{d}t + \sigma(X_t)\,\mathrm{d}W_t,$$

where W_t is a standard Brownian motion and functions $\mu(x)$ and $\sigma^2(x)$[17] are both affine in x.[18] Furthermore let $\lambda(x)$ be also affine in x, then there exist functions $A(t)$ and $B(t)$ such that

$$\mathsf{E}\left[\exp\left(-\int_0^t \lambda(X_s)\,\mathrm{d}s\right)\right] = \exp(A(t) - B(t)X_0).$$

If we assume $\lambda(x) = x$, $\mu(x) = \kappa(\theta - x)$ and $\sigma(x) = \sigma$ then we get the model of Vasicek (1977)

$$\mathrm{d}X_t = \kappa(\theta - X_t)\,\mathrm{d}t + \sigma\,\mathrm{d}W_t,$$

and

$$\begin{aligned} B(t) &= \frac{1}{\kappa}\left(1 - \mathrm{e}^{-\kappa t}\right), \\ A(t) &= \frac{\left(B(t) - t\right)\left(\kappa^2\theta - \frac{\sigma^2}{2}\right)}{\kappa^2} - \frac{\sigma^2 B(t)^2}{4\kappa}. \end{aligned}$$

If we replace in the Vasicek's model function $\sigma(x) = \sigma$ by $\sigma(x) = \sigma\sqrt{x}$ we get the CIR model of Cox et al. (1985)

$$\mathrm{d}X_t = \kappa(\theta - X_t)\,\mathrm{d}t + \sigma\sqrt{X_t}\,\mathrm{d}W_t,$$

and

$$\begin{aligned} B(t) &= \frac{2(\mathrm{e}^{\gamma t} - 1)}{(\gamma - \kappa)(\mathrm{e}^{\gamma t} - 1) + 2\gamma}, \\ A(t) &= \frac{2\kappa\theta}{\sigma^2}\log\left(\frac{2\gamma(\mathrm{e}^{(\gamma-\kappa)t} - 1)}{(\gamma - \kappa)(\mathrm{e}^{\gamma t} - 1) + 2\gamma}\right), \end{aligned}$$

where $\gamma = \sqrt{\kappa^2 + 2\sigma^2}$.

More mathematical theory about affine processes can be found in Duffie et al. (2003) for time homogeneous Markov processes and in Filipovič (2005) extended to time inhomogeneous Markov processes.

[17] Notice that we want $\sigma^2(x)$ to be affine in x not just $\sigma(x)$.

[18] We say that function $f : \mathbb{R}^d \to \mathbb{R}$ is affine if there exist $a \in \mathbb{R}^d$ and $b \in \mathbb{R}$, such that $f(x) = a \cdot x + b$.

2.8 Copulas

Let random variables X_i, $i = 1, \ldots, n$ from Ω to $(0, \infty)$ be some survival times and we are interested in way how X_i depends on each other. More precisely we know $\mathsf{P}[X_1 \leq x_1], \ldots, \mathsf{P}[X_n \leq x_n]$ and we would like to model the joint probability $\mathsf{P}[X_1 \leq x_1, \ldots, X_n \leq x_n]$. The answer for that is to introduce *copulas*.

Let $\boldsymbol{X}$ be a random vector from an arbitrary n–dimensional distribution $\mathbf{F}$ and let F_i be the marginal distribution of X_i. Then by Sklar (1959) there exists a function

$$C : [0,1]^n \to [0,1],$$

called *copula* such that

$$F(x_1, \ldots, x_n) = C\big(F_1(x_1), \ldots, F_n(x_n)\big).$$

If F_i are continuous for every i, then the copula is unique and

$$C(u_1, \ldots, u_n) = F\big(F_1^{-1}(u_1), \ldots, F_n^{-1}(u_n)\big),$$

where

$$F_i^{-1}(u) = \inf\left\{x \geq 0 | F_i(x) \geq 0\right\}.$$

For copula hold:

- $C(u_1, \ldots, u_n) = 0$ whenever $u_i = 0$ for some i.
- $C(u_1, \ldots, u_n) = u_i$ whenever $u_j = 1$ for all $j \neq i$.
- $\max\left(\sum_{i=1}^n u_i + 1 - n, 0\right) \leq C(u_1, \ldots, u_n) \leq \min_{i=1,\ldots,n} u_i$, where boundaries are called the *minimal copula* and the *maximal copula*.

Probably the most common used copula is a Gaussian copula given by

$$C(u_1, \ldots, u_n) = \Phi_{\mathbf{R}}\big(\Phi_1^{-1}(u_1), \ldots, \Phi_n^{-1}(u_n)\big),$$

where Φ is the standard normal distribution function and $\Phi_{\mathbf{R}}$ is the cumulative distribution function of n-dimensional normal distribution with zero mean and covariance matrix $\mathbf{R}$.

For more examples and theory about copulas see Nelsen (2006).

3
Credit Risk Models

First, we are going to introduce some basic concepts and ideas of credit risk modeling. Then we will show what are common used practices in modeling the loss distribution.

3.1 Idea of Credit Risk

One of the main businesses of banks is lending money. They lend money to people or firms that need money which they do not have. Of course banks lend money just to debtors which have a high probability to repay the money. Ability of repaying money is called the *credit worthiness*. For lending money, the bank expects some rewards in the form of interest. The interest is composed of reward for lending and some risk premium for the bearing risk that the debtor will fail to satisfy his obligation and a loss for the bank will occur.

Usually a contract develops in the following way. A future debtor applies for a loan. The bank checks his credit worthiness. It consists of checking the present financial situation, evaluation of present and estimation of future assets and liabilities etc.[19] If the debtor fulfills all internal requirements of bank loan, there are couple things that need to be established — the height of loan, the height of interest rate, the calendar of installment, the conditions of drawing money, etc. After signing the contract the debtor withdraw the money and starts paying regular instalments according to the calendar of installments until whole debt is repayed including the interest.

That is the optimal development of the contract, but sometimes deviations from the optimal development might occur. The debtor can get into the financial distress and fail to fulfill his obligation. Usually at the time when debt is granted the debtor has a high credit worthiness (is in a good financial condition), but during the duration of the contract the credit worthiness can change. The change in credit worthiness is often called a *credit event*. A situation when the debtor fails to fulfill his obligation is called a *default* and the random time at which it happens is called the *default time*.

A default of the debtor means a financial loss for the bank. This loss is called the *credit loss* and we will denote it by L. When a default occurs it does not mean that all outstanding money is lost, some fraction can be recovered. The fraction that is recovered is called the *recovery rate* (RR) and it is from the interval $[0, 1]$. The height of the credit loss, if a default occurs, is a random variable which depends on many factors, mainly on the *exposure at default* (EAD) and the *loss given default* (LGD). The EAD is the outstanding amount of drawn money at the time of default. The LGD is simply $1 - RR$, i.e., the fraction of the EAD that will be lost at default. The LGD depends on many circumstances like the presence of a collateral, the business sector of the debtor, the balance sheet of the debtor, etc.

Some debts has a valuable collateral which cover the big portion of the debt. Then it appears to be useful to estimate the total amount C, which we can cover from the collateral

[19] The most common used tool for that are so called scoring cards, which are models based on logistic regression or similar mathematic theory. Once all necessary details are given to model, the model gives a score for client. If the score is sufficiently high, the loan is granted. If it is to low, the loan is rejected, and if it is in the middle further personal examination is performed.

for sure and then define new $\widetilde{EAD}$ as

$$\widetilde{EAD} = (EAD - C)^+.$$

Then we assume just the new $\widetilde{EAD}$ in the model. Sometimes a collateral value is very volatile. In that case the estimation of the amount C should be very conservative even if we know that with the very high probability we will be able to cover the bigger amount then C at default.

The bank has a whole portfolio of loans denoted by $PF = \{1, \ldots, N\}$. One debtor can have more loans but we can look on it as on one loan with installments that cames up from more loans. The credit loss L^i on loan $i \in PF$ can be written in following way

$$L^i = EAD_i \times LGD_i \times \mathbf{1}_{\{\tau_i \leq T_i\}},$$

where τ_i is the default time of the debtor (possibly infinite if the debtor will not default) and T_i is the maturity of the loan i. The credit loss of the whole portfolio is simply a sum of losses over all loans. Thus,

$$L = \sum_{i \in PF} EAD_i \times LGD_i \times \mathbf{1}_{\{\tau_i \leq T_i\}}.$$

The credit portfolio loss L is also called the *aggregate loss*. A portfolio credit risk model is interested in the estimation of the probability distribution of L and the time evolution of the loss. Let us introduce the loss process of loan i

$$L_t^i = EAD_i \times LGD_i \times \mathbf{1}_{\{\tau_i \leq \min\{t, T_i\}\}}. \tag{3.1}$$

Then the loss process of the whole portfolio is

$$L_t = \sum_{i \in PF} EAD_i \times LGD_i \times \mathbf{1}_{\{\tau_i \leq \min\{t, T_i\}\}}.$$

Note that the loss process L_t is a nondecreasing stochastic process. Since L_t is a stochastic process we can try to compute its expected value $\mathsf{E}\, L_t$ which we call the *expected loss*. The expected loss is something which a bank should count with in its balance sheet and future cash flow planning. We can also define the *unexpected loss* as a standard deviation of L_t. The unexpected loss shows how volatile the credit loss is and how much it probably will differ from its expected value $\mathsf{E}\, L_t$. The bank should, and in a matter of fact it is obliged, to have reserves for covering the unexpected loss.

It is useful to know the whole distribution of the process L_t since the bank managers are often interested in other statistics then the expected and unexpected loss. The main example is the credit VaR — *Value at Risk* which is a loss boundary which will not be crossed with a very high probability, typically 99%. More rigorously the credit VaR_t^α is defined as

$$VaR_t^\alpha = \inf \left\{ l \in \mathbb{R} | \mathsf{P}[L_t > l] < 1 - \alpha \right\}.$$

The VaR is a very common used risk measure in practice, but it has a big disadvantage since it does not tell anything about the height of the loss if the loss is higher then VaR. It can be fixed by introducing the risk measure $CVaR$ — *Conditional Value at Risk* that is also often called the *Expected Shortfall* and is given by

$$CVaR_t^\alpha = \mathsf{E}\left[L_t | L_t > VaR_t^\alpha \right].$$

There are many ways to model the process L_t. One can try to model it as an aggregate process which we will refer to as *aggregate models*. The idea is similar to one where we would like to model the movement of a stock index that consists of many stocks. One way is to model the movement of every stock and the correlations between them. Another way is to model the index as some stochastic process. In that case some information is lost, but the modeling is easier. The aggregate loss process can be modeled using many approaches — time series, nondecreasing Markov chain, (Lévy) subordinator. We are not going to describe these models in this thesis and we refer for an overview to Giesecke (2008), where other references can be found.

Other models describe the evolution of every particular loan and the dependency between them. There are three main types of these models — *structural models* (Section 3.3), *reduced–form models* (Section 3.4), and incomplete information models (Section 3.5).

3.2 From Loan to Defaultable Zero–coupon Bond

Before we start describing portfolio credit risk models we should say few words about a pricing of credit derivatives. We expect that the reader is familiar with basic ideas and techniques of risk neutral pricing. For an introduction to the risk neutral pricing we refer to Baxter and Rennie (1996), Shreve (2004), Musiela and Rutkowski (2005), or any other financial stochastic introductionary text. Since we are going to mention pricing aspects of the credit risk only marginally, reader can skip this section.

3.2.1 Zero–coupon Bond

We assume the arbitrage free economy with the risk free interest rate r_t which can follow any short rate model as well as more complex HJM interest rate model. Then we can define a *saving account* B_t as

$$B_t = \exp\left(\int_0^t r_s \,\mathrm{d}s\right),$$

that expresses the time value of the money. If we have 1 unit of money at time 0, we will have without any risk B_t unit of money at time t. Conversely 1 unit of money at time t is worthed B_t^{-1} at time 0.

We know that an absence of the arbitrage opportunity is equivalent to the existence of the risk neutral probability measure P^* and also implies the uniqueness of the risk free interest rate r_t. For an overview of interest rate models we refer to Musiela and Rutkowski (2005) or Brigo and Mercurio (2006) where all models are with respect to Brownian motions. If we want to move to more general Lévy processes interest rate framework we should see for example Kluge (2005).

From the theory of the risk neutral pricing we know that an arbitrage free price $\pi_t(X)$ of the European contingent claim X settled at time T is given by pricing formula

$$\pi_t(X) = B_t \,\mathsf{E}_{\mathsf{P}^*}\left[B_T^{-1} X | \mathcal{F}_t\right]. \tag{3.2}$$

The zero–coupon bond is a security which has pay–off 1 at a maturity T. Using the formula (3.2) we can deduce that the arbitrage free price of the zero–coupon bond at time t is

$$B(t,T) = B_t \,\mathsf{E}_{\mathsf{P}^*}\left[B_T^{-1} | \mathcal{F}_t\right]. \tag{3.3}$$

Zero–coupon bonds are in fact very rarely traded in markets but they are very important modeling tools derived from coupon bonds or other interest rate securities. Bonds can be issued by governments, banks or companies. One of the reasons why bonds are so important is because any fixed–income security can be written as some portfolio of zero–coupon bonds with different maturities $T_1, \ldots, T_n$. A payoff of the security is then given by

$$S_t = \sum_{i=1}^{n} c_i \mathbf{1}_{\{t=T_i\}},$$

where c_i is the height of the payment at time T_i. We can think about every payment as about c_i zero–coupon bonds with maturity T_i. Hence the arbitrage free price of the security S at time t is

$$\pi_t(S) = \sum_{i=1}^{n} c_i B(t, t_i).$$

Defaultable Zero–coupon Bond

As far as bonds are issued by some issuer there is always a risk that the issuer will goes to default and will fail to pay the bond. For some issuers, such as the US government or big international banks, is the default (or credit) risk negligible and we call their bonds risk free bonds and formula (3.3) is valid for them. For the rest we need to incorporate the default risk into the valuation formula. Since the discussion above, it is enough, if we investigate just the pricing formula of the *defaultable zero–coupon bond*. Coupon bonds can be build from zero–coupon bonds. The defaultable zero–coupon bond has pay–off 1 if the default time τ of the issuer is higher then the maturity T and the recovery RR otherwise, i.e.,

$$\mathbf{1}_{\{\tau>T\}} + RR\,\mathbf{1}_{\{\tau\leq T\}}.$$

Hence using (3.2) the arbitrage free price of the defaultable zero–coupon bond is

$$\begin{aligned} D(t,T) &= B_t\,\mathsf{E}_{\mathsf{P}^*}\left[B_T^{-1}(\mathbf{1}_{\{\tau>T\}} + RR\,\mathbf{1}_{\{\tau\leq T\}})|\mathcal{F}_t\right] \\ &= B_t\,\mathsf{E}_{\mathsf{P}^*}\left[B_T^{-1}(1-(1-RR)\mathbf{1}_{\{\tau\leq T\}})|\mathcal{F}_t\right] \\ &= B(t,T) - B_t\,\mathsf{E}_{\mathsf{P}^*}\left[B_T^{-1}(1-RR)\mathbf{1}_{\{\tau\leq T\}}|\mathcal{F}_t\right] \end{aligned}$$

By comparing with (3.1) we see that the price of the defaultable zero–coupon bond is the price of the risk free zero–coupon bond minus a discounted expected loss computed with respect to risk neutral measure P^*. The valuation for any fixed–income security can be established in same manner as for risk free securities above.

Sometimes, bonds with the face value different that 1 will be mentioned. Bond with face value K is exactly same as K standard bonds with face value 1.

From the no arbitrage condition we know that there exists a risk neutral probability measure P^*, but it is not unique in general. If we model an uncertainty in model with the Brownian motion then market is complete, every contingent claim is replicable and the risk neutral measure is unique. But in a more general setting of Lévy processes there might exist infinitely many possible risk neutral measures (dependent if market is or is not complete). Then we need to determine which measure we should use. Very widely used approach to choose a measure P^* is the Esscher transform.

If we fix t and T and compare the yield to maturity on the risk free zero–coupon bond and the defaultable zero–coupon bond we can see that the defaultable zero–coupon bond is cheaper, hence if default did not occur, the defaultable zero–coupon bond has higher yield to maturity. The difference between yields of the defaultable zero–coupon bond and the risk free zero–coupon bond is called a *credit spread* $S(t,T)$ and is equal to

$$S(t,T) = -\frac{\log D(t,T) - \log B(t,T)}{T-t}.$$

3.2.2 Loan as Defaultable Zero–coupon Bond

Some loans has a fixed interest rate and hence we know the whole instalment schedule. We can see these loans as fixed–income securities and model it as a portfolio of defaultable zero–coupon bonds, where every installment is a zero–coupon bond.

Other loans have a floating interest rate. The floating interest rate is in the most cases interest rate from the wholesale market (where the bank takes the money) plus margin of the bank (incorporate the risk premium and margin for lending money). Future interest rate on the wholesale market can be estimated from forward rates which are observable on the market. The margin of the bank is constant in time in the most cases. Hence we can estimate the interest rate on loans with floating interest rate, from forward rates and then model it as a fixed–income security.

3.3 Structural Models

Structural models have roots in the early papers of Black and Scholes (1973) and Merton (1974). Structural models are very popular for their economic interpretation which is clearer than in reduced–form modeling. The debt can be seen as a contingent claim on a firm's asset. The market value of a firm is the main source of uncertainty and a default occurs if the market value process V_t falls below some threshold. The biggest assumption and the main disadvantage of the model is that the firm's value process V_t has to be observable. An advantage of structural models over reduced–form models is that we do not need to specify recovery rates since they follow implicitly from the model as the residual value of the firm's asset at maturity.

A firm value process V_t is often modeled via a Brownian motion. The assumption of normality is very convenient to work with, but not very realistic and empirical studies shows a significant deviation from normality. Continuous paths of a Brownian motion also imply a predictability of a default and vanishing credit spreads on bonds with near maturities. Empirical studies shows that even on short term bonds there is a significantly high credit spread. A way to fix it, is introduce jumps into a firm value process. For example Zhou (1997) assumed the Brownian part with an additional jump term. We will describe structural models in a more general framework of Lévy processes which involves a Brownian motion as a special case.

We assume that a firm value process follows a geometric Lévy process with respect to some real world measure P, i.e.,

$$V_t = V_0 \exp(X_t).$$

In all following models we are interested in expected values of future random variables. We could compute these expectations conditioned on knowledge of informations up to time $t \leq T$ (σ-algebra $\mathcal{F}_t$), but since in our models risk driver processes are Lévy processes with independent identically distributed increments it follows that for any integrable function g and $t \leq T$

$$\mathsf{E}\left[g(X_T - X_t)|\mathcal{F}_t\right] = \mathsf{E}\left[g(X_{T-t})\right].$$

Therefore we will assume that we are doing all computations for time 0, which allows a simpler notation.

3.3.1 Merton's Model

The model introduced by Merton (1974) is interested in a firm value process only at the time of maturity T. Let us assume that a firm is financed by an equity and a debt with profile of a zero coupon bond with face value K. A default occurs if the firm value process at the time of maturity is below the face value of the bond, hence the default time τ is defined as

$$\tau = \begin{cases} T, & V_T < K, \\ \infty, & V_T \geq K. \end{cases}$$

In the case of a default, the equity is useless and the remaining value of assets goes to the creditor, otherwise the debt is repaid in full amount K and amount $V_T - K$ belongs to stockholders. Therefore the pay-off of the defaultable zero–coupon bond at maturity is

$$\min(V_T, K) = K - (K - V_T, 0)^+,$$

and the pay–off of the equity is

$$(V_T - K)^+.$$

One can see that the bond's pay–off at maturity is the face value of the bond lowered by the pay–off of put option on the firm's values with strike K and the pay–off of equity is the pay–off of the call option on the firm's value. This approach is often called option theoretic approach or the firm value approach.

The probability of default is the probability that V_T will be below K,

$$\begin{aligned} DP &= \mathsf{P}[V_T < K] \\ &= \mathsf{P}[V_0 \exp(X_T) < K] \\ &= \mathsf{P}\left[X_T < \log \frac{K}{V_0}\right], \end{aligned}$$

which is equal to the cumulative distribution function F_{X_T} of X_T if F_{X_T} is continuous in the point $\log(K/V_0)$. Otherwise the left limit is chosen as its value.

The expected loss on the loan computed at time 0 is equal to the expected pay–off of the put option on the firm's value with respect to a real world probability measure P

$$\mathsf{E}\, L = \mathsf{E}\left[(K - V_T)^+\right]$$

Recall that we assume dynamics $V_T = V_0 \exp(X_T)$. If the Lebesgue density f of X_T exists, the expected loss is

$$\mathsf{E}\, L = \int_{-\infty}^{\infty} (K - V_0 \mathrm{e}^x)^+ f(x)\, \mathrm{d}x,$$

and the expected return of the bond is $K - \mathsf{E}\, L$.

From a pricing point of view we are interested in a fair present value of the defaultable zero–coupon bond using risk–neutral pricing techniques. In a similar manner as Section 3.2.1 we conclude that the value of the defaultable bond with face value K at time 0 is

$$D(0,T) = KB(0,T) - \mathsf{E}_{\mathsf{P}^*} \left[\frac{1}{B_T} (K - V_T)^+ \right]. \tag{3.4}$$

There remains a couple difficulties to solve. First of all we need to find a risk neutral measure P^*. Under the no arbitrage condition there exists exactly one risk neutral measure in complete market. Unfortunately a complete market is rather an exception when we use a general Lévy process X_t. In an incomplete market there exists infinitely many risk neutral measures. A widely–used approach to choose the risk neutral measure is the Esscher transform. Then we have a problem of the option pricing using the Esscher transform, see Gerber and Shiu (1994), Buhlmann et al. (1996), Miyahara (2001) and Elliott et al. (2005). Another problem comes up if we do not assume independence of B_T and V_T in (3.4). Then we need to compute their joint distribution, which is also not easy. In the case when the interest rate is a constant r and the risk neutral measure $\mathsf{P}^* = \mathsf{P}_\theta$ is chosen by Esscher transform we have

$$\mathsf{E}_{\mathsf{P}^*} \left[\frac{1}{B_T} (K - V_T)^+ \right] = \mathsf{E} \left[\frac{\exp(\theta X_T)}{\mathsf{E} \exp(\theta X_T)} \mathrm{e}^{-rT} \big(K - V_0 \exp(X_T)\big)^+ \right],$$

where θ is chosen such that the discounted firm value process V_t is a martingale with respect to the measure P_θ.

In the original paper of Merton (1974) the Lévy process X_t is assumed to be $X_t = (\mu - \sigma^2/2)t + \sigma W_t$, where $\mu \in \mathbb{R}$, $\sigma > 0$, and W_t is a standard Brownian motion. Since W_t is normally distributed with expected value 0 and variance t we have an equality in distribution

$$W_t \stackrel{d}{=} \sqrt{t} Y,$$

where Y is a standard normally distributed variable. In that setting the default probability DP is

$$\begin{aligned} DP &= \mathsf{P} \left[X_T < \log \frac{K}{V_0} \right] \\ &= \mathsf{P} \left[\left(\mu - \frac{\sigma^2}{2} \right) T + \sigma \sqrt{T} Y < \log \frac{K}{V_0} \right] \\ &= \mathsf{P} \left[Y < \frac{\log \frac{K}{V_0} - \left(\mu - \frac{\sigma^2}{2} \right) T}{\sigma \sqrt{T}} \right] \\ &= \Phi(d), \end{aligned}$$

where Φ is the cumulative distribution function of a standard normal distribution and

$$d = \frac{\log \frac{K}{V_0} - \left(\mu - \frac{\sigma^2}{2}\right) T}{\sigma\sqrt{T}}.$$

The value d is often called the *distance to the default*. The expected loss is then

$$\begin{aligned}
\mathsf{E}\, L &= \int_{-\infty}^{\infty} \left(K - V_0 \mathrm{e}^{\left(\mu - \frac{\sigma^2}{2}\right)T + \sigma\sqrt{T}x} \right)^+ \frac{1}{\sqrt{2\pi}} \mathrm{e}^{-\frac{x^2}{2}} \,\mathrm{d}x \\
&= K \int_{-\infty}^{d} \frac{1}{\sqrt{2\pi}} \mathrm{e}^{-\frac{x^2}{2}} \,\mathrm{d}x \\
&\quad - V_0 \int_{-\infty}^{d} \mathrm{e}^{\left(\mu - \frac{\sigma^2}{2}\right)T + \sigma\sqrt{T}x} \frac{1}{\sqrt{2\pi}} \mathrm{e}^{-\frac{x^2}{2}} \,\mathrm{d}x \\
&= K\Phi(d) - V_0 \mathrm{e}^{\mu T} \Phi\big(d - \sigma\sqrt{T}\big). \qquad (3.5)
\end{aligned}$$

Under a risk neutral measure P^* the firm value process evolves also as a geometric Brownian motion but with a different drift equal to the risk neutral interest rate, which is the result of Black and Scholes (1973). If we assume a constant interest rate r, the process X_t is

$$X_t = \left(r - \frac{\sigma^2}{2}\right) t + \sigma W_t,$$

with respect to the measure P^*. Hence using (3.5), the expected loss of the bond with respect to measure P^* is

$$\mathsf{E}_{\mathsf{P}^*}\, L = K\Phi(d) - V_0 \mathrm{e}^{rT} \Phi\big(d - \sigma\sqrt{T}\big),$$

where

$$d = \frac{\log \frac{K}{V_0} - \left(r - \frac{\sigma^2}{2}\right) T}{\sigma\sqrt{T}}.$$

The price of the defaultable zero–coupon bond is the price of the risk free zero–coupon bond minus the discounted expected loss with respect to risk neutral measure P^*. Therefore

$$\begin{aligned}
D(0,T) &= \mathrm{e}^{-rT} K - \mathrm{e}^{-rT} \left(K\Phi(d) - V_0 \mathrm{e}^{rT} \Phi\big(d - \sigma\sqrt{T}\big) \right) \\
&= \mathrm{e}^{-rT} K\Phi(-d) - V_0 \Phi\big(d - \sigma\sqrt{T}\big).
\end{aligned}$$

The presented case where we have one firm that is financed by an equity and a bond is very simple. If we would assume more firms we can easily compute the expected loss of the portfolio since the expected loss of portfolio is simply the sum of the expected losses of the particular loans. However, if we are interested in computing the unexpected loss, VAR or CVAR, of the portfolio we need to consider dependencies between the firm value processes. Also in practice, the financial structure of firm is much more complicated then just an equity and bonds. In that case we have to take into account which liabilities of the firm have higher priority and include it in the model. For a discussion of a firm's liabilities see Vasicek (1984). Determining of the loss distribution is often done by simulation.

Extensions to Merton's Model

Many extensions to Merton's model have been done. For an overview of these extensions see Bielecki and Rutkowski (2002, Section 2.4) and references cited in there. One of the main extensions is presented in next section.

3.3.2 First–Passage Model

In Merton's model a default can be observed only at maturity. Even if a firm's value almost vanishes a default is not triggered before maturity. To fix this Black and Cox (1976) added a barrier D_t such that if value process V_t falls below it, a default is triggered and the creditor overtakes the control of the firm. The barrier D_t can be exogenously or endogenously given and can be constant, deterministic, or even random, but tractability rapidly decreases. The time when a firm value process V_t hits the barrier is called the *hitting time*. The hitting time probability distribution of a Lévy process is not known explicitly apart in a few exceptions.

In a first–passage model the default can be triggered in two ways. The first possibility is the same as in Merton's model at maturity T:

$$\tau_1 = \begin{cases} T, & V_T < K, \\ \infty, & V_T \geq K. \end{cases}$$

The second possible default is by hitting the barrier D_t

$$\tau_2 = \inf\left\{t \middle| \, 0 < t < T, V_t < D_t\right\},$$

where we assume that the infimum of an empty set is equal to $+\infty$. Let us define a default for the first–passage model as

$$\tau = \min\left\{\tau_1, \tau_2\right\}.$$

We can also choose $D_T = K$ to default τ_1 and τ_2 agree at maturity T.

Let us assume a barrier as the face value of the bond discounted using some constant discount factor $\gamma > 0$, i.e.,

$$D_t = \mathrm{e}^{-\gamma(T-t)}K.$$

Then we have

$$\begin{aligned} \mathsf{P}[V_t \leq D_t] &= \mathsf{P}[V_0 \mathrm{e}^{X_t} \leq \mathrm{e}^{-\gamma(T-t)}K] \\ &= \mathsf{P}[V_0 \mathrm{e}^{X_t - \gamma t} \leq \mathrm{e}^{-\gamma T}K] \\ &= \mathsf{P}\left[X_t - \gamma t \leq \log\left(\mathrm{e}^{-\gamma T}\frac{K}{V_0}\right)\right] \\ &= \mathsf{P}\left[\tilde{X}_t \leq \log \tilde{K}\right], \end{aligned}$$

where

$$\begin{aligned} \tilde{X}_t &= X_t - \gamma t, \\ \tilde{K} &= \mathrm{e}^{-\gamma T}\frac{K}{V_0}. \end{aligned}$$

We see that the probability distribution of hitting the barrier D_t of the original process V_t is same as the probability distribution of hitting the constant barrier $\log \tilde{K}$ with Lévy process $\tilde{X}_t$. Let us define the process $\tilde{M}_t$ as the running minimum of the process $\tilde{X}_t$

$$\tilde{M}_t = \min_{0 \le s \le t} \tilde{X}_s.$$

A default occurs if and only if there exists $t \le T$ such that $V_t < D_t$ and it is equivalent with $\tilde{M}_t < \log \tilde{K}$. Hence the default probability is

$$DP = \mathsf{P}[\tilde{M}_T < \log \tilde{K}]. \tag{3.6}$$

If a default occurs the creditor takes control over the firm and his pay–off at maturity is V_T, otherwise the face value K is payed. Therefore the pay–off of the bond at maturity is

$$\begin{aligned} K\mathbf{1}_{\{\tilde{M}_T \ge \log \tilde{K}\}} &+ V_T \mathbf{1}_{\{\tilde{M}_T < \log \tilde{K}\}} \\ &= K - (K - V_T)^+ + (V_T - K)^+ \mathbf{1}_{\{\tilde{M}_T < \log \tilde{K}\}} \\ &= K - (K - V_T)^+ + \mathrm{e}^{\gamma T} V_0 \left(\mathrm{e}^{\tilde{X}_T} - \tilde{K}\right)^+ \mathbf{1}_{\{\tilde{M}_T < \log \tilde{K}\}}. \end{aligned} \tag{3.7}$$

One can see from (3.7) that the pay–off is at least as high as in Merton's model, furthermore, the pay–off can be even higher then the face value of the bond and an additional revenue $(V_T - K)^+$ can occur. From (3.7) it follows that the expected loss in the model is

$$\mathsf{E}\, L = \mathsf{E}\,(K - V_T)^+ - \mathrm{e}^{\gamma T} V_0 \mathsf{E}\, \left(\mathrm{e}^{\tilde{X}_T} - \tilde{K}\right)^+ \mathbf{1}_{\{\tilde{M}_T < \log \tilde{K}\}}. \tag{3.8}$$

In some cases the expected loss can be negative, i.e., the expected return on the bond is higher than its face value K. Whether it is the case depends on the parameters of the process X_t and the constant γ.

From (3.7) we see that the pay–off of the bond in the model is equivalent to the pay-off of the portfolio consisting of risk–free K zero–coupon bonds, a short European put option with strike K, and a $\mathrm{e}^{\gamma T} V_0$ long European down-and-in call option on the exponential process X_t with strike $\tilde{K}$. Therefore the price of the defaultable bond at time 0 is

$$D(0,T) = KB(0,T) - PUT(0,T,K) + \mathrm{e}^{\gamma T} V_0 DIC(0,T,\tilde{K}), \tag{3.9}$$

where PUT is the price of the European put option and DIC is the price of the European down-and-in call option with time–varying barrier. Barrier option pricing in a Lévy processes framework is very hard and only partial results are known. List of references can be found in Schoutens (2006).

In the original paper of Black and Cox (1976), the firm value process is assumed to follow a geometric Brownian motion, i.e.,

$$X_t = \left(\mu - \frac{\sigma^2}{2}\right) t + \sigma W_t.$$

Then

$$\begin{aligned} \tilde{X}_t &= \left(\mu - \gamma - \frac{\sigma^2}{2}\right) t + \sigma W_t \\ &= \nu t + \sigma W_t, \end{aligned}$$

where

$$\nu = \mu - \gamma - \frac{\sigma^2}{2}.$$

Using the reflection principle and the Girsanov's theorem is not hard to prove that the joint probability distribution of the Brownian motion $\tilde{X}_t$ and its running minimum $\tilde{M}_t$ is

$$\mathsf{P}[\tilde{X}_t \geq x, \tilde{M}_t \geq y] = \Phi\left(\frac{-x+\nu t}{\sigma\sqrt{t}}\right) - \exp\left(\frac{2\nu y}{\sigma^2}\right)\Phi\left(\frac{2y-x+\nu t}{\sigma\sqrt{t}}\right). \tag{3.10}$$

The proof can be found in Musiela and Rutkowski (2005, Appendix B.4). Since

$$\mathsf{P}[\tilde{X}_t \geq x, \tilde{M}_t \geq y] = 1 - P[\tilde{X}_t < x] - P[\tilde{M}_t < y] + \mathsf{P}[\tilde{X}_t < x, \tilde{M}_t < y],$$

we can obtain the joint density function of $\tilde{X}_T$ and $\tilde{M}_T$ from (3.10) by differentiation. If we differentiate with respect to x we get

$$-\frac{1}{\sigma\sqrt{T}}\varphi\left(\frac{-x+\nu T}{\sigma\sqrt{T}}\right) + \frac{1}{\sigma\sqrt{T}}\exp\left(\frac{2\nu y}{\sigma^2}\right)\varphi\left(\frac{2y-x+\nu T}{\sigma\sqrt{T}}\right),$$

where φ is the density function of the standard normal distribution. Then by differentiating with respect to y we get the joint density function

$$f_{\tilde{X}_T,\tilde{M}_T}(x,y) = \frac{-2(2y-x)}{\sigma^3\sqrt{T^3}}\exp\left(\frac{2\nu y}{\sigma^2}\right)\varphi\left(\frac{2y-x+\nu T}{\sigma\sqrt{T}}\right).$$

The default probability (3.6) takes the form

$$\begin{aligned}
DP &= \mathsf{P}[\tilde{M}_T < \log\tilde{K}] \\
&= 1 - \mathsf{P}[\tilde{M}_T \geq \log\tilde{K}] \\
&= 1 - \mathsf{P}[\tilde{M}_T \geq \log\tilde{K}, \tilde{X}_T \geq \log\tilde{K}] \\
&= 1 - \Phi\left(\frac{-\log\tilde{K}+\nu T}{\sigma\sqrt{T}}\right) + \tilde{K}^{\frac{2\nu}{\sigma^2}}\Phi\left(\frac{\log\tilde{K}+\nu T}{\sigma\sqrt{T}}\right) \\
&= \Phi\left(\frac{\log\tilde{K}-\nu T}{\sigma\sqrt{T}}\right) + \tilde{K}^{\frac{2\nu}{\sigma^2}}\Phi\left(\frac{\log\tilde{K}+\nu T}{\sigma\sqrt{T}}\right).
\end{aligned}$$

Now let us compute the expected loss in the first–passage model. The first term in (3.8) is same as the expected loss in Merton's model. For the valuation of the second term in (3.8) note that expression

$$\left(\mathrm{e}^{\tilde{X}_T} - \tilde{K}\right)^+ \mathbf{1}_{\{\tilde{M}_T < \log\tilde{K}\}}$$

is nonzero on the set

$$A = \left\{\mathrm{e}^{\tilde{X}_T} > \tilde{K}, \tilde{M}_T < \log\tilde{K}\right\} = \left\{\tilde{X}_T > \log\tilde{K}, \tilde{M}_T < \log\tilde{K}\right\}.$$

Therefore we can compute

$$\begin{aligned}\mathsf{E}\left(\mathrm{e}^{\tilde{X}_T}-\tilde{K}\right)^+\mathbf{1}_{\left\{\tilde{M}_T<\log\tilde{K}\right\}}&\\ &=\int_A\left(\mathrm{e}^x-\tilde{K}\right)f_{\tilde{X}_T,\tilde{M}_T}(x,y)\,\mathrm{d}x\,\mathrm{d}y\\ &=\int_A \mathrm{e}^x f_{\tilde{X}_T,\tilde{M}_T}(x,y)\,\mathrm{d}x\,\mathrm{d}y-\tilde{K}\int_A f_{\tilde{X}_T,\tilde{M}_T}(x,y)\,\mathrm{d}x\,\mathrm{d}y. \qquad (3.11)\end{aligned}$$

The first term in (3.11) is

$$\begin{aligned}&\int_A \mathrm{e}^x f_{\tilde{X}_T,\tilde{M}_T}(x,y)\,\mathrm{d}x\,\mathrm{d}y\\ &\quad=\exp\left(\frac{2\nu\log\tilde{K}}{\sigma^2}\right)\int_{\log\tilde{K}}^\infty \mathrm{e}^x\frac{1}{\sqrt{2\pi\sigma^2T}}\exp\left(-\frac{(-x+2\log\tilde{K}+\nu T)^2}{2\sigma^2T}\right)\mathrm{d}x\\ &\quad=\exp\left(\frac{4\log\tilde{K}(\nu+\sigma^2)+\sigma^2T(\sigma^2+2\nu)}{2\sigma^2}\right)\\ &\qquad\times\int_{\log\tilde{K}}^\infty\frac{1}{\sqrt{2\pi\sigma^2T}}\exp\left(-\frac{(-x+2\log\tilde{K}+\nu T+\sigma^2T)^2}{2\sigma^2T}\right)\mathrm{d}x\\ &\quad=\tilde{K}^{\frac{2\nu+\sigma^2}{\sigma^2}}\mathrm{e}^{\frac{T}{2}(2\nu+\sigma^2)}\Phi\left(\frac{\log\tilde{K}+\nu T}{\sigma\sqrt{T}}+\sigma\sqrt{T}\right).\end{aligned}$$

The second term in (3.11) is equal to

$$\begin{aligned}&\tilde{K}\mathsf{P}[\tilde{X}_T>\log\tilde{K},\tilde{M}_T<\log\tilde{K}]\\ &\quad=\tilde{K}\left(\mathsf{P}[\tilde{X}_T>\log\tilde{K}]-\mathsf{P}[\tilde{X}_T>\log\tilde{K},\tilde{M}_T\geq\log\tilde{K}]\right)\\ &\quad=\tilde{K}\left(1-\Phi\left(\frac{\log\tilde{K}-\nu T}{\sigma\sqrt{T}}\right)-\Phi\left(\frac{-\log\tilde{K}+\nu T}{\sigma\sqrt{T}}\right)+\tilde{K}^{\frac{2\nu}{\sigma^2}}\Phi\left(\frac{\log\tilde{K}+\nu T}{\sigma\sqrt{T}}\right)\right)\\ &\quad=\tilde{K}^{\frac{2\nu+\sigma^2}{\sigma^2}}\Phi\left(\frac{\log\tilde{K}+\nu T}{\sigma\sqrt{T}}\right).\end{aligned}$$

Putting it all together we get the expected loss

$$\begin{aligned}\mathsf{E}\,L&=K\Phi(d_1)-V_0\mathrm{e}^{\mu T}\Phi(d_1-\sigma\sqrt{T})\\ &\quad-K\exp\left(\frac{2\nu}{\sigma^2}\left(\log\frac{K}{V_0}-\gamma T\right)\right)\left(\mathrm{e}^{\frac{T}{2}(2\nu+\sigma^2)}\Phi(d_2+\sigma\sqrt{T})-\Phi(d_2)\right),\end{aligned}$$

where

$$\begin{aligned}d_1&=\frac{\log\frac{K}{V_0}-(\gamma+\nu)T}{\sigma\sqrt{T}},\\ d_2&=\frac{\log\frac{K}{V_0}-(\gamma-\nu)T}{\sigma\sqrt{T}},\\ \nu&=\mu-\gamma-\frac{\sigma^2}{2}.\end{aligned}$$

If we want to price a defaultable zero–coupon bond in the case of a geometric Brownian motion we can use (3.9), where for pricing formulas of the European put option and especially the down-and-in call option we refer to Hull (2005) or Musiela and Rutkowski (2005).

A Step Further

The first–passage model is an extension of the Merton's model and can be further extended by an excursion approach. Some authors argue that if the firm's value falls below the barrier D_t the creditor does not take the control over the firm's assets immediately but he let the firm reorganize and operate for a while. If the firm value does not rise the creditor still takes control over the firm's assets. Therefore a default in this approach occurs after the firm value process spends some given time below the barrier. For the excursion approach overview see Giesecke (2004, Section 2.3). For more structural models and pricing of derivatives we refer to Bielecki and Rutkowski (2002).

3.4 Reduced–form Models

In reduced–form modeling the idea of default is different than in structural models where the default depends on the firm's value. The default time is modeled here as some random variable τ using the theory explained in Chapter 2. The most common approaches of modeling the default time τ are through a Poisson process, an inhomogeneous Poisson process or a Cox process.

Let λ_t is the hazard rate of the default time τ and $N_t = \mathbf{1}_{\{\tau \leq t\}}$ is the associated counting process. The hazard rate of the default time τ is in the credit risk often called a *default intensity*.

Poisson process In this model a hazard rate λ is constant and the default time τ has an exponential distribution with parameter λ. The exponential distribution is memory–less, which is not a very realistic assumption, but constant intensities are easy to estimate and compute with. The default probability in the model is

$$DP_t = 1 - \mathsf{P}[N_t = 0] = 1 - \mathrm{e}^{-\lambda t}.$$

Time inhomogeneous Poisson process If one wants to fix the unrealistic assumption of constant default intensities he can assume a default intensity λ_t as a function of time. The time dependency can be estimated from historical data using some econometrics model (regression analysis) or given exogenously. The default probability is then

$$DP_t = 1 - \mathsf{P}[N_t = 0] = 1 - \exp\left(-\int_0^t \lambda_s \,\mathrm{d}s\right).$$

Cox process Finally, we incorporate the future uncertainty and assume stochastic default intensities. In the Cox process setting it is assumed that there exists some risk driver factor X_t which is often modeled via *affine models* (Section 2.7). Default intensity λ_t is then deterministic function of X_t, i.e., $\lambda_t = \lambda(X_t)$. Thus, conditioned on realization of X_t we get an inhomogeneous Poisson process. The default probability in the Cox model is then

$$DP_t = 1 - \mathsf{P}[N_t = 0] = 1 - \mathsf{E}\left[\exp\left(-\int_0^t \lambda(X_s) \,\mathrm{d}s\right)\right].$$

Often choice for function $\lambda(x)$ is $\lambda(x) = x$. Furthermore, if we assume that risk factor X_t follows some affine process, we can get closed–form expression in form

$$1 - \exp\big(A(t) - B(t)X_0\big),$$

for some $A(t)$ and $B(t)$. For example if we assume model of Vasicek (1977) as in Section 2.7 we have

$$DP_t = 1 - \exp\big(A(t) - B(t)X_0\big),$$

where

$$\begin{aligned} B(t) &= \frac{1}{\kappa}\left(1 - \mathrm{e}^{-\kappa t}\right), \\ A(t) &= \frac{(B(t) - t)(\kappa^2\theta - \frac{\sigma^2}{2})}{\kappa^2} - \frac{\sigma^2 B(t)^2}{4\kappa}. \end{aligned}$$

Another possibility is to use time subordination which we discussed in Section 2.4.2.

3.4.1 Recovery Rates

The essential part of the risk management modeling are recovery rates RR. Unlike in structural models recovery rates do not follow implicitly from reduced–form models. The big problem of the recovery modeling is a lag of data for estimation. Only some banks have sufficient historical data about recoveries. Therefore some simplifying assumptions are necessary to enable any modeling.

Recall that the loss process of the defaultable zero–coupon bond with face value 1 has dynamics

$$L_t = (1 - RR_\tau)\mathbf{1}_{\{\tau \le t\}}.$$

Then we can compute the expected loss in general as

$$\begin{aligned} \mathsf{E}\, L_t &= \mathsf{E}\,\left[(1 - RR_\tau)\mathbf{1}_{\{\tau \le t\}}\right] \\ &= \mathsf{E}\,\left[\mathsf{E}\,[(1 - RR_\tau)\mathbf{1}_{\{\tau \le t\}}|\tau]\right] \\ &= \mathsf{E}\,\left[\mathbf{1}_{\{\tau \le t\}}\,\mathsf{E}\,[(1 - RR_\tau)|\tau]\right]. \end{aligned}$$

The recovery rate RR_t can be a deterministic function of time and then RR_τ is a random variable because it is a value of the function in a random time. In general, recovery RR_t can be a stochastic process with values in $[0, 1]$.[20]

If recovery rates are independent of the random default time τ, i.e., the random variable $RR_t = RR$ has same distribution for every t, then

$$\mathsf{E}\, L_t = DP_t\,\mathsf{E}\,[1 - RR].$$

In literature for pricing credit derivatives there are three main approaches to recoveries. We will mention each approach and compute the expected loss of the bond in this model when we assume a Cox process as a driver of the default time τ.

[20] Rarely we can allow recovery rates higher than 1.

Recovery of Face Value (RFV) — The recovery is assumed to be the exogenously given fraction c of the face value of the bond. Then recovery rates are constant and independent of τ

$$RR_\tau = c,$$

and the expected loss is then

$$\begin{aligned}\mathsf{E}\, L_t &= \mathsf{E}\left[\mathbf{1}_{\{\tau\le t\}}\,\mathsf{E}\left[(1-c)|\tau\right]\right]\\ &= (1-c)\mathsf{E}\,\mathbf{1}_{\{\tau\le t\}}\\ &= (1-c)DP_t.\end{aligned}$$

Recovery of Treasury (RT) — The recovery is assumed to be the exogenously given fraction c of an equivalent risk free zero–coupon bond. Then we need to specify dynamics of the risk free zero–coupon bonds. The recovery rate is a random variable

$$RR_\tau = cB(\tau,T),$$

and the expected loss is then

$$\begin{aligned}\mathsf{E}\, L_t &= \mathsf{E}\left[\mathbf{1}_{\{\tau\le t\}}\,\mathsf{E}\left[\big(1-cB(\tau,T)\big)|\tau\right]\right]\\ &= DP_t - c\,\mathsf{E}\left[\mathbf{1}_{\{\tau\le t\}}B(\tau,T)\right].\end{aligned}$$

In the case when τ is exponentially distributed with parameter λ and the risk free interest rate r is constant, we have

$$\begin{aligned}RR_\tau &= c\mathrm{e}^{-r(T-\tau)}\\ \mathsf{E}\, L_t &= DP_t - c\mathrm{e}^{-rT}\mathsf{E}\left[\mathbf{1}_{\{\tau\le t\}}\mathrm{e}^{r\tau}\right]\\ &= 1-\mathrm{e}^{-\lambda t} - c\mathrm{e}^{-rT}\int_0^\infty \mathbf{1}_{\{x\le t\}}\,\mathrm{e}^{rx}\,\lambda\mathrm{e}^{-\lambda x}\,\mathrm{d}x\\ &= 1-\mathrm{e}^{-\lambda t} - \frac{\lambda c}{\lambda-r}\mathrm{e}^{-rT}\int_0^t(\lambda-r)\mathrm{e}^{-(\lambda-r)x}\,\mathrm{d}x\\ &= 1-\mathrm{e}^{-\lambda t} - \frac{\lambda c}{\lambda-r}\mathrm{e}^{-rT}\left(1-\mathrm{e}^{-(\lambda-r)t}\right).\end{aligned}$$

Recovery of Market Value (RMV) — The recovery is assumed to be an exogenously given fraction of the predefault market value of the bond, i.e.

$$RR_\tau = cD(\tau_-,T).$$

The expected loss is then

$$\mathsf{E}\, L_t = DP_t - c\,\mathsf{E}\left[\mathbf{1}_{\{\tau\le t\}}D(\tau_-,T)\right].$$

For further computation we need to know the dynamics of the defaultable zero–coupon bonds and how to price them. Pricing of defaultable coupon bonds is very wide topic and beyond this scope of this work. We refer to Duffie and Singleton (1999) and Collin-Dufresne et al. (2004) for that.

In all three cases we mentioned the constant c. This constant is estimated from historical data. We can go further and assume the fraction c is not constant anymore but is a random variable C with values on $[0, 1]$.

For purpose of modeling random fraction C we could use the uniform distribution on $[0, 1]$, but empirical studies shows the assumption of the uniform distribution very unrealistic. In Schuermann (2004) there are some empirical distributions of recoveries of American bonds and loans. These empirical distributions are highly right–skewed.

Much better distribution than the uniform distribution is very flexible Beta distribution with the density function

$$f(x) = \frac{1}{\mathbf{B}(a,b)} x^{a-1}(1-x)^{b-1},$$

where $a, b > 0$ and $\mathbf{B}$ is the Beta function defined as

$$\mathbf{B}(a,b) = \int_0^1 x^{a-1}(1-x)^{b-1}\,\mathrm{d}x.$$

Parameters a and b can be estimated from data using the moment method or the maximum likelihood estimation. Note that the uniform distribution is special kind of the Beta distribution with parameters $a = b = 1$.

The Beta distribution has mean value

$$\frac{a}{a+b}.$$

Therefore if we assume RFV approach we get the expected loss

$$\mathsf{E}\, L_t = \left(1 - \frac{a}{a+b}\right) DP_t.$$

Dependent Recoveries

Some empirical studies indicates that recovery rates and default intensities are not independent, but they tend to be negatively correlated. They shows that in the recession of the economy default intensities tends to be higher and recoveries lower.

One of the usual way of modeling dependent recovery and default intensity λ_t is to introduce an additional possibly multidimensional process X_t that can be interpreted as a state of the economy. Random fraction C and default intensity λ_t both depends on X_t, but conditioned on realization X_t they are independent on each other.

In our Beta distribution setting it can be for example such that parameters a and b are deterministic functions of X_t. If we assume RFV approach we have the expected loss

$$\begin{aligned}
\mathsf{E}\, L_t &= \mathsf{E}\,\left[\big(1 - C(X_t)\big)\mathbf{1}_{\{\tau \le t\}}\right] \\
&= \mathsf{E}\,\left[\mathsf{E}\,\left[\big(1 - C(X_t)\big)\mathbf{1}_{\{\tau \le t\}} | X_t\right]\right] \\
&= \mathsf{E}\,\left[\big(1 - \mathsf{E}\,[C(X_t)|X_t]\big)\,\mathsf{E}\,[\mathbf{1}_{\{\tau \le t\}}|X_t]\right]
\end{aligned}$$

In RT and RMV models things become more complicated since for these approaches we need a dynamics of interest rates for a zero coupon bond modeling, and interest rates can be correlated with default intensities and recoveries. The interest rate is also often taken as a dimension of the process X_t. Sometimes even the process X_t is chosen as $X_t = r_t$.

We should also mention the dependency of recovery rates on the character of the debtors business. Imagine that we have two debtors. The first one is a real estate agent with only one debt and a big equity and the second one is a financial derivative trader with a minimal equity. One could hardly expect that recovery rates of both of them have same probability distribution. Therefore we can divide debtors to some groups and estimate the distribution of recovery rates inside these groups. In dividing into groups we are limited by accessible historical data.

3.4.2 Credit Rating Migration

Sometimes it is useful to describe the credit worthiness of the obliger in more details than just a default or a non–default. Way to do this is to introduce credit ratings. A credit rating is a grade on a finite scale $\{1, \ldots, K\}$ expressing the credit worthiness of the obliger. The rating 1 is the best and the rating K is a default. The default rating K is absorbing, i.e., once the default state is reached it lasts until the end of assumed horizon. As a special case when $K = 2$ we get a model where just a default and a non–default are assumed (as in the previous scope). In some credit rating systems there might be more credit ratings for a default.[21] We will speak about default only in case of rating K if not stated otherwise. The *rating agency* is a company which measures the credit worthiness of the obliger and then assigns a credit rating from its rating systems. Probably the most well–known agencies are Standard&Poors (S&P) and Moody's. These agencies assign credit rating only to big international companies or to states. Hence many banks developed their own rating systems which they use for internal purposes.

If the bank uses its own internal rating system it knows the present credit worthiness of the obliger, but it also need to model the future development of obliger's credit worthiness, i.e., how the obliger will migrate between different ratings. The credit rating migration is often modeled using Markov chains as was introduced in Jarrow et al. (1997). The credit rating process is assumed to be a discrete–time or a continuous–time Markov chain. It is quite expensive for bank to update the credit rating for all their obliger's continuously and hence they update it semi–annually in the most cases. Therefore it make sense to model the credit rating process as a discrete Markov chain. From the other point of view, the credit worthiness of the obliger changes continuously and assigning ratings on a semi–annual base is just a discrete observation of a continuous process and hence one should use a continuous–time Markov chain. The continuous approach is also more tractable and use more information about transitions then the discrete–time Markov chain.

The continuous–time homogeneous Markov chain model implies few not very realistic features that real data do not approve.

1. Constant rating intensities — Real data shows that intensities change through the time.
2. The exponential distribution of sojourn times — The exponential distribution is memoryless, but data shows some rating "momentum", i.e., the time to the next rating change depends on the time already spend in a current rating.
3. The Markov property — Transitional probabilities should depend only on a current rating, but empirically there are evidence that after a rating downgrade, there is a

[21] For example 3 rating grades are default and they differ from each other in recovery rates. The first one assume constant recovery 66 %, the second one 33 % and the last one 0 recovery.

higher probability of another downgrade than if a current rating was reached by an upgrade.

For more see studies Carty and Fons (1993), Kavvathas (2001) and Lando and Skødeberg (2002).

The last mentioned problem can be solved by an extending the state space from K possible ratings $\{1,\dots,K\}$ to $2K-2$ in the following way

$$1,2,2-,3,3-,\dots,(K-1)-,K,$$

where rating k (resp. $k-$) means that rating k was reached by an upgrade (resp. downgrade). Note that state 1 can not be reached by a downgrade and state K can be reached only by a downgrade, hence $2K-2$ states. After this extension we can renumber ratings and work with the state space $\{1,\dots,2K-2\}$. A disadvantage of the method is that we need more data for estimation of transition probabilities. Note that the number of transition probabilities, which we need to estimate, rises with the second power of the number of states. In practice there is often not enough data and some ratings are joining together for lower the data demand for estimation.

The other two problems of a time–homogeneous Markov chains can be solved introducing a time–inhomogeneous continuous Markov chain, time homogeneous semi–Markov chain, a time–inhomogeneous semi–Markov chain or even transition intensities are model as a stochastic processes.

In general, we are interested in modeling transition probabilities

$$\mathbf{P}(s,t)=\Big(p_{ij}(s,t)\Big)_{i,j=1}^{K},$$

where $p_{ij}(s,t)$ is the probability that the debtor will be in rating j at time t conditioned that he is in rating i at time s. We assume that there exists a $K\times K$-dimensional process $\mathbf{Q}(t)$ such that

$$\mathbf{P}(s,t)=\exp\left(\int_s^t \mathbf{Q}(u)\,\mathrm{d}u\right),$$

if $\mathbf{Q}(t)$ is deterministic matrix function of time and

$$\mathbf{P}(s,t)=\mathsf{E}\left[\exp\left(\int_s^t \mathbf{Q}(u)\,\mathrm{d}u\right)\right],$$

if $\mathbf{Q}(t)$ is a stochastic process. In both cases the integral is assumed componentwise and we assume that these integrals exist and for every t, $\mathbf{Q}(t)$ is a valid generator.

The last column of the matrix $\mathbf{P}(s,t)$ is a vector of default probabilities for different ratings. Let us denote it

$$\boldsymbol{DP}(s,t)=\big(DP_i(s,t)\big)_{i=1}^{K-1},$$

where

$$DP_i(s,t)=p_{iK}(s,t),$$

is the probability that the debtor will default up to time t conditioned he is in rating i at time s. The probability matrix $\mathbf{P}(t)$ can be written in the form

$$\mathbf{P}(s,t) = \begin{pmatrix} \tilde{\mathbf{P}}(s,t) & \boldsymbol{DP}(s,t) \\ \boldsymbol{0} & 1 \end{pmatrix}, \tag{3.12}$$

where the matrix $\tilde{\mathbf{P}}(s,t)$ is a $(K-1)\times(K-1)$ matrix and $\boldsymbol{0}$ and $\boldsymbol{DP}(s,t)$ are $(K-1)$ dimensional vectors. Since $\mathbf{P}(s,t)$ is a stochastic matrix it holds for every $0 \le t_1 \le t_2 \le \ldots \le t_n$

$$\mathbf{P}(t_1,t_n) = \mathbf{P}(t_1,t_2)\cdot\mathbf{P}(t_2,t_3)\cdot\ldots\cdot\mathbf{P}(t_{n-1},t_n). \tag{3.13}$$

Using (3.12) and (3.13) it follows an intuitive relation

$$\boldsymbol{DP}(t_1,t_n) = \boldsymbol{DP}(t_1,t_2) + \tilde{\mathbf{P}}(t_1,t_2)\boldsymbol{DP}(t_2,t_3) + \ldots + \tilde{\mathbf{P}}(t_1,t_{n-1})\boldsymbol{DP}(t_{n-1},t_n),$$

which is useful if we are interested in some particular discrete time evolution of default probabilities.

If we use a time–homogeneous continuous–time Markov chain we can compute transition probabilities and especially default probabilities by very straitforward application of the theory from Section 2.2.2. These kinds of models are the most common used in practice.

If one wants to use more complicated structure of the inhomogeneous continuous–time Markov chain he needs to specify the structure of inhomogeneity and justify it by some arguments. That is not easy at all. Bluhm and Overbeck (2007) suggest to use an inhomogeneous continuous–time Markov chain with the generator

$$\mathbf{Q}_t = \mathbf{D}(t)\mathbf{Q}, \qquad t \ge 0,$$

where $\mathbf{Q}$ is a constant valid generator and $\mathbf{D}(t)$ is a diagonal matrix with diagonal elements

$$d_{ii} = \frac{\left(1-\mathrm{e}^{-\alpha_i t}\right)t^{\beta_i-1}}{1-\mathrm{e}^{-\alpha_i}}, \qquad i = 1,\ldots,K,$$

and $\alpha_i, \beta_i > 0$ are nonnegative parameters of the model. Note that the generator $\mathbf{Q_t}$ came up from constant $\mathbf{Q}$ by multiplying the i-th row by d_{ii}. By this inhomogeneity structure they reached significantly better fit to default probabilities on data from S&P. Bluhm and Overbeck (2007) also provide some discussion, where they pointed out that the choice of the structure is not completely random.

The main result of this paper is usage of time changed continuous–time Markov chains described in Section 2.4 and 2.6, which is simpler than the structure of Bluhm and Overbeck (2007) and in some cases leads to the homogeneous continuous–time Markov chain. It also gives us more possibilities in predicting the future development. We will show the real usage and performance of these methods in Chapter 4.

3.5 Incomplete Information Models

In the Section 3.3 we have described structural models in a general Lévy processes setting, but in the most of cases just a Brownian motion as the risk driver is used. As a corollary of the continuity of a Brownian motion paths there is a vanishing credit spread for short term

debts, because it is very unlikely for a continuous process to steeply decrease in a short period. Furthermore, since the continuity, the default time is a predictable stopping time.

A pioneer work in incomplete information models is Duffie and Lando (2001), where they argue that the firm's value process is not observable directly and continuously to public, but is observable to the management of the firm. The information available to company managers forms the filtration $\mathcal{F}_t$ which is different from the filtration $\mathcal{G}_t$ which is formed by the information available to market.

Duffie and Lando (2001) assumed that the information from filtration $\mathcal{F}_t$ is revealed to public ("transfered" to filtration $\mathcal{G}_t$) in some discrete times (for example earnings announcement) and even then it is revealed with some additional noise. A default time τ is then a predictable stopping time with respect to the filtration $\mathcal{F}_t$, but is a totally inaccessible stopping time[22] with respect to the filtration $\mathcal{G}_t$. This idea was motivated by accounting scandals in American companies Enron and WorldCom. Both of them had mistakes in their accounting which is by Duffie and Lando (2001) viewed as an additional noise.

Kusuoka (1999) assumes similar as Duffie and Lando (2001) that information is revealed to the market with an additional noise, but in a continuous way.

In Collin-Dufresne et al. (2003) they choose different approach. They assume that the information about company is revealed to public with some delay, but continuously and completely.

Çetin et al. (2004) suggest an approach where the information available to the management is not completely revealed to public, but just some fraction of it.

Giesecke (2006) described a first–passage model where the default barrier is random. That is also the incomplete information approach.

In Guo et al. (2008) they try to present some unified framework for incomplete information models using the delayed filtration. Especially they reconcile the approach by Duffie and Lando (2001) and Collin-Dufresne et al. (2003).

Behind every credit risk model (does not matter if structural, reduced–form or incomplete information model), there is a counting process $N_t = \mathbf{1}_{\{\tau \leq t\}}$ for which according to Doob–Mayer decomposition there exists a compensator A_t such that $N_t - A_t$ is a martingale. The compensator A_t is often called a *default drift* in this context. A reduced–form model assumes an existence of the default intensity and default intensity is major tool in reduced–form models. An incomplete information model uses rather directly the compensator process A_t then intensity itself. If the intensity exists the model is unified with a reduced–form model. So incomplete information models can be seen as a generalization of a reduced–form model based on the structural model ideas. The default probability is then under some technical conditions equal to

$$DP_t = 1 - \mathsf{E}\left[\mathrm{e}^{-A_t}\right].$$

For further details about particular incomplete information model see article referenced above.

3.6 Dependent Default Modeling

There is no need to investigate the dependency between particular debtors in the expected loss modeling, since the expected loss is just a sum of expected losses on particular loans. But

[22] A stopping time τ is totally inaccessible if and only if the probability that τ is a limit of any sequence of predictable stopping times τ_n is zero.

if we want to compute the variance of the credit loss of the portfolio, dependencies between debtors comes into play as a major factor.

Let us assume two big companies from the same industry and location, and one of them has default. Then one could expect that the default probability of the second company will raise, since both companies are influenced by very similar risk factors on the market.

Recall that the one–year credit loss of the portfolio is given by

$$L = \sum_{i=1}^{n} EAD \times LGD \times \mathbf{1}_{\{\tau_i \le 1\}}.$$

For demonstrating the idea let us assume that $EAD = LGD = 1$, then

$$L = \sum_{i=1}^{n} \mathbf{1}_{\{\tau_i \le 1\}}.$$

The expected one–year loss is then simply

$$\mathsf{E}\, L = \sum_{i=1}^{n} \mathsf{E}\, \mathbf{1}_{\{\tau_i \le 1\}} = \sum_{i=1}^{n} DP_1^i,$$

where DP_1^i is the one–year default probability on the i-th loan. The different situation is if we want to compute the variance of the credit loss

$$\begin{aligned} \mathsf{Var}\,(L) &= \mathsf{Var}\,\left(\sum_{i=1}^{n} \mathbf{1}_{\{\tau_i \le 1\}}\right) \\ &= \sum_{i=1}^{n} \mathsf{Var}\,\left(\mathbf{1}_{\{\tau_i \le 1\}}\right) + 2\sum_{i=1}^{n-1}\sum_{j=i+1}^{n} \mathsf{Cov}\,\left(\mathbf{1}_{\{\tau_i \le 1\}}, \mathbf{1}_{\{\tau_j \le 1\}}\right), \end{aligned} \tag{3.14}$$

where

$$\begin{aligned} \mathsf{Cov}\,\left(\mathbf{1}_{\{\tau_i \le 1\}}, \mathbf{1}_{\{\tau_j \le 1\}}\right) &= \mathsf{E}\,\left[\mathbf{1}_{\{\tau_i \le 1\}}, \mathbf{1}_{\{\tau_j \le 1\}}\right] - \mathsf{E}\,\mathbf{1}_{\{\tau_i \le 1\}} \mathsf{E}\,\mathbf{1}_{\{\tau_j \le 1\}} \\ &= \mathsf{P}[\tau_i \le 1, \tau_j \le 1] - DP_i DP_j. \end{aligned}$$

Also for other statistics like VaR or CVaR we need to know how default events of particular debtors depend on each other. In a matter of fact we should go even further and model the joint distribution of defaults and recoveries, but it is beyond the scope of the presented work. In the following sequel we assume independent recovery rates and default events unless stated otherwise.

Le us imagine the situation when the bank loan portfolio consists of n loans and a new $(n+1)$-st loan is considered. What will be a contribution of a new loan to the current portfolio loss variance? From (3.14) we see that the contribution will be

$$\mathsf{Var}\,\left(\mathbf{1}_{\{\tau_{n+1} \le 1\}}\right) + 2\sum_{i=1}^{n} \mathsf{Cov}\,\left(\mathbf{1}_{\{\tau_i \le 1\}}, \mathbf{1}_{\{\tau_{n+1} \le 1\}}\right),$$

which can be even negative if the default event of the new debtor is negatively correlated with default events of current debtors. In that case it is very desirable to accept this loan, since it diversifies the risk.

3.6.1 Unit Loss and Homogeneous Portfolio

The larger the loan portfolio is the harder and more time consuming the calculations are. It is also more complicated to estimate all necessary parameters (default probabilities, covariances, recovery rates,...) to every obliger. Hence a concept of the *homogeneous portfolio* is often assumed. It is an assumption that exposure, default probability, recovery rates and also dependencies are same all over the portfolio.[23] Without loss of generality we can take exposure equal to 1. Then the number of defaults divided by the number of loans in the portfolio is the fraction of the whole portfolio exposure, which will be lost.

3.6.2 Factor Models

As we mentioned above, two companies doing similar business are sensitive to similar risk drivers. Also one can imagine that a location of theses two companies play significant role (risk factor) for the default event. Finally, let there be an economy recession and people are not willing to spend much money. It will influence many different businesses in many differently ways and we would like to know how.

The *factor model* is a dependency structure of a particular firm risk on some common risk factors such as business factor, country factor etc. Part of the firm's riskiness that can not be explained by common risk factors is called an *idiosyncratic risk* or *residual risk*.

Let $\boldsymbol{Z}_t$ be a d-dimensional risk factor common to all firms in the portfolio. Let us take an arbitrary firm i. Now we are interested in a way and how much riskiness of the firm i depends on the risk factor $\boldsymbol{Z}_t$.

Let us assume that we use a structural model. Recall that we assume that the firm value process of the firm i has dynamics

$$V_t^{(i)} = V_0^{(i)} \exp\left(X_t^{(i)}\right),$$

where $X_t^{(i)}$ is some Lévy process. In factor models we assume that the risk driver $X_t^{(i)}$ is a linear function of the common risk factor $\boldsymbol{Z}_t$ and some idiosyncratic process $\varepsilon_t^{(i)}$ which is specific to firm i, i.e.,

$$X_t^{(i)} = \boldsymbol{\beta}^{(i)} \cdot \boldsymbol{Z}_t + \varepsilon_t^{(i)},$$

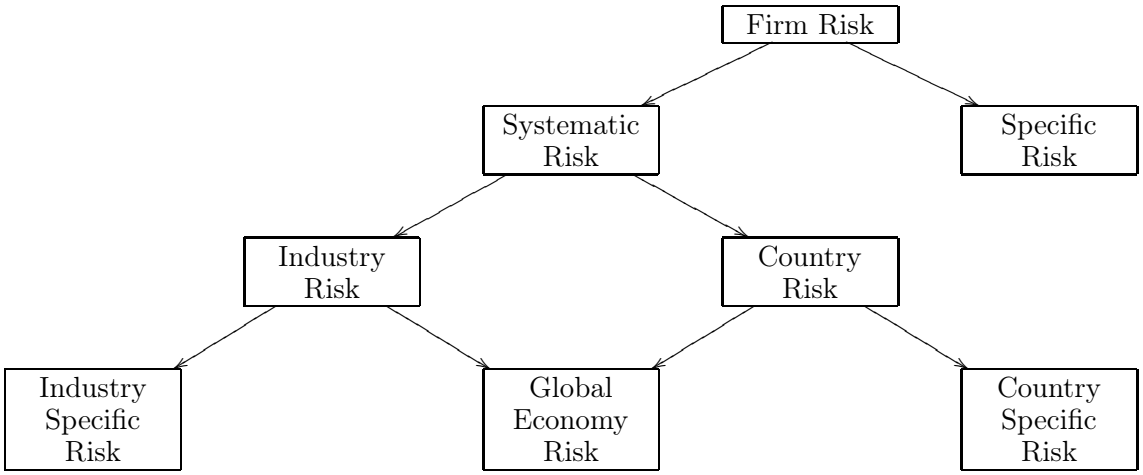
where processes $\boldsymbol{Z}_t, \varepsilon_t^{(1)}, \ldots, \varepsilon_t^{(m)}$ are independent of each other. Since the process $\boldsymbol{Z}_t$ is common to all firms in the economy (or portfolio), default events are dependent if $\boldsymbol{\beta}^{(i)} \neq \boldsymbol{0}$. Conditionally on the realization of the common risk factor $\boldsymbol{Z}_t$ processes $X_t^{(1)}, \ldots, X_t^{(n)}$ are independent. This fact is often used in computations or simulations. The d-dimensional real parameter $\boldsymbol{\beta}^{(i)}$ is estimated from historical data using linear regression methods and is often called *loads*.

The common risk factor can have for example structure as in diagram (3.15) where the three–level factor model of KMV[24] is described as presented in Bluhm et al. (2002, Figure

[23] The homogeneity assumption is useful especially in retail segment, where a lot of very similar loans exist and they behave in very same pattern. The bigger loans and smaller number of loans (big corporate loans), the less accurate results models based on the homogeneous assumption will give.

[24] KMV is a part of Moody's and is one of the biggest provider of credit risk solutions in world.

1.7).



(3.15)

We see that the riskiness of the firm was decomposed into three common factors (industry specific risk, global economy risk and country specific risk) and a firm specific risk (idiosyncratic risk).

The special case of factor models is *one–factor model* where $\boldsymbol{Z}_t$ is one–dimensional. This model is very often used in practice since it is very simple.

In reduced–form modeling a factor model can be used to decompose intensities $\lambda_t^{(i)}$. Then we have

$$\lambda_t^{(i)} = \boldsymbol{\beta}^{(i)} \cdot \boldsymbol{Z}_t + \varepsilon_t^{(i)}, \tag{3.16}$$

where $\varepsilon_t^{(i)}$ are independent of $\boldsymbol{Z}_t$ and each other. In a reduced form model there is just need to take care of λ_t to be nonnegative. It can be done by truncating $\lambda_t^{(i)}$, which will come from model (3.16).

For further discussion about factor models see Bluhm et al. (2002, Section 1.2.3).

3.6.3 Bernoulli Models

Let us fix the time horizon on one year and let us assume a unit exposure and zero recovery on every loan in the portfolio. Then the aggregate credit loss is

$$L = \sum_{i=1}^{n} \mathbf{1}_{\{\tau_i \leq 1\}}.$$

We will describe deterministic Bernouli model and its extension to the general Bernouli mixture model.

"Deterministic" Bernoulli Model

In the Bernoulli model we think about a default event $Y_i = \mathbf{1}_{\{\tau_i \leq 1\}}$ as about a Bernoulli random variable which equals 1 with probability $p_i = DP_i$ and 0 with probability $1 - p_i$,

where DP_i is the one–year default probability of loan i. Furthermore, default events are assumed to be independent of each other. Then we get

$$\begin{aligned}\mathsf{E}\, L &= \sum_{i=1}^{n} \mathsf{E}\, Y_i = \sum_{i=1}^{n} p_i, \\ \mathsf{Var}\,(L) &= \sum_{i=1}^{n} \mathsf{Var}\,(Y_i) = \sum_{i=1}^{n} p_i(1-p_i).\end{aligned}$$

In the case of the homogeneous portfolio, where we assume same default probability DP on every loan, the portfolio loss L is the binomial random variable with parameters n and $p = DP$. Thus,

$$\begin{aligned}\mathsf{E}\, L &= n\, p, \\ \mathsf{Var}\,(L) &= n\, p(1-p).\end{aligned}$$

General Bernoulli Mixture Model

Independent default events is not very realistic assumption, hence we extend the model to the general Bernoulli mixture model. Default events are still the Bernoulli random variables with parameter $P_i = DP_i$, but the parameter P_i is not deterministic anymore. Parameters $(P_1, \ldots, P_n)$ are random variables with the joint n-dimensional distribution $\boldsymbol{F}$ on $[0,1]^n$. Conditioned on the realization $(p_1, \ldots, p_n)$ of $(P_1, \ldots, P_n)$, the default event Y_i is a Bernoulli random variable with parameter p_i independent of $Y_1, \ldots, Y_{i-1}, Y_{i+1}, \ldots, Y_n$. Then it follows

$$\mathsf{P}[Y_1 = y_1, \ldots, Y_n = y_n] = \int_{[0,1]^n} \prod_{i=1}^{n} p_i^{y_i}(1-p_i)^{1-y_i}\, \mathrm{d}\boldsymbol{F}(p_1, \ldots, p_n), \tag{3.17}$$

where $y_i \in \{0, 1\}$. The expected loss is then

$$\begin{aligned}\mathsf{E}\, L &= \sum_{i=1}^{n} \mathsf{E}\,[Y_i] \\ &= \sum_{i=1}^{n} \mathsf{E}\,\left[\mathsf{E}\,[Y_i|P_i]\right] \\ &= \sum_{i=1}^{n} \mathsf{E}\,[P_i].\end{aligned}$$

Before we will compute the variance of the credit loss note that

$$\begin{aligned}\mathsf{Var}\,(Y_i) &= \mathsf{Var}\,\left(\mathsf{E}\,[Y_i|P_i]\right) + \mathsf{E}\,\left[\mathsf{Var}\,(Y_i|P_i)\right] \\ &= \mathsf{Var}\,(P_i) + \mathsf{E}\,\left[P_i(1-P_i)\right] \\ &= \mathsf{E}\,[P_i^2] - (\mathsf{E}\,[P_i])^2 + \mathsf{E}\,[P_i] - \mathsf{E}\,[P_i]^2. \\ &= \mathsf{E}\,[P_i]\,(1 - \mathsf{E}\,[P_i])\end{aligned} \tag{3.18}$$

and

$$\begin{aligned}\mathsf{Cov}\,(Y_i, Y_j) &= \mathsf{E}\,[Y_i Y_j] - \mathsf{E}\,[Y_i]\mathsf{E}\,[Y_j] \\ &= \mathsf{E}\left[\mathsf{E}\,[Y_i Y_j|P_i, P_j]\right] - \mathsf{E}\,[P_i]\mathsf{E}\,[P_j] \\ &= \mathsf{E}\,[P_i P_j] - \mathsf{E}\,[P_i]\mathsf{E}\,[P_j] \\ &= \mathsf{Cov}\,(P_i, P_j).\end{aligned}$$

We see that the whole correlation structure of default events is equipped by the correlation structure of random parameters $P_i, i = 1, \ldots, n$. The variance of the aggregate credit loss then follows from (3.14)

$$\mathsf{Var}\,(L) = \sum_{i=1}^{n} \mathsf{E}\,[P_i]\,(1 - \mathsf{E}\,[P_i]) + 2\sum_{i=1}^{n-1}\sum_{j=i+1}^{n} \mathsf{Cov}\,(P_i, P_j).$$

Homogeneous Portfolios In the case of the homogeneous credit portfolio, the default probability on every loan is same. Therefore, we have just one random parameter P with the probability distribution F. We can rewrite (3.17) into the form

$$\mathsf{P}[Y_1 = y_1, \ldots, Y_n = y_n] = \int_0^1 p^k (1-p)^{n-k}\,\mathrm{d}F(p),$$

where $k = \sum_{i=1}^{n} y_i$, $y_i \in \{0, 1\}$, is the number of defaults in the portfolio. The probability that exactly k defaults occur is then

$$\mathsf{P}[L = k] = \binom{n}{k} \int_0^1 p^k (1-p)^{n-k}\,\mathrm{d}F(p).$$

Let us denote the uniform default probability by $\bar{p}$. Then

$$\bar{p} = \mathsf{P}[Y_i = 1] = \mathsf{E}\,[Y_i] = \mathsf{E}\,[P] = \int_0^1 p\,\mathrm{d}F(p),$$

further using (3.18) we get

$$\begin{aligned}
\mathsf{Corr}\,(Y_i, Y_j) &= \frac{\mathsf{Cov}\,(Y_i, Y_j)}{\bar{p}(1-\bar{p})} \\
&= \frac{\mathsf{E}\,[Y_i Y_j] - \mathsf{E}\,Y_i \mathsf{E}\,Y_j}{\bar{p}(1-\bar{p})} \\
&= \frac{\mathsf{E}\left[\mathsf{P}[Y_i = 1, Y_j = 1|P]\right] - \bar{p}^2}{\bar{p}(1-\bar{p})} \\
&= \frac{\int_0^1 p^2\,\mathrm{d}F(p) - \bar{p}^2}{\bar{p}(1-\bar{p})} \\
&= \frac{\mathsf{Var}\,(P)}{\bar{p}(1-\bar{p})}. \qquad (3.19)
\end{aligned}$$

From (3.19) follows few interesting consequences. First of all, it implies that the correlation between default events vanishes only when the variance of random parameter P goes to 0, that is F is the Dirac measure with mass in $\bar{p}$. The second immediate consequence of (3.19) is, that since the variance is always a positive number, it is impossible to capture the negative correlation structure by this model.

Now let us assume that we use the Merton's model in a Brownian motion setting, where we are interested if firm's asset value is at maturity below the face value of the debt or it is not. The default probability is

$$\bar{p} = \mathsf{P}[V_T < K] = \mathsf{P}\left[X_T < \log\frac{K}{V_0}\right].$$

We us assume that X_T is normally distributed with mean value μT and variance $\sigma^2 T$. Then the default probability can be rewritten as

$$\bar{p} = \mathsf{P}\left[\frac{X_T - \mu T}{\sigma\sqrt{T}} < \frac{\log\frac{K}{V_0} - \mu T}{\sigma\sqrt{T}}\right] = \mathsf{P}\left[\tilde{X} < \tilde{K}\right],$$

where $\tilde{X}$ is a standard normal random variable with the cumulative distribution function Φ and $\tilde{K}$ is the normalized face value of the debt. Then it follows

$$\tilde{K} = \Phi^{-1}(\bar{p}), \tag{3.20}$$

because in the homogeneous portfolio the default probability of every debtor is same.

Let us assume that the risk driver $\tilde{X}$ follows a one–factor model, i.e.,

$$\tilde{X} = \sqrt{\varrho}Z + \sqrt{1-\varrho}\,\varepsilon, \tag{3.21}$$

where $Z \sim N(0,1)$ is a systematic risk factor and $\varepsilon \sim N(0,1)$ an idiosyncratic risk independent of Z, and $\sqrt{\varrho}$ is load describing how much is the debtor sensitive to the systematic risk factor. Let

$$p(Z) = \mathsf{P}[Y = 1|Z] = \mathsf{E}\,[Y|Z],$$

be the default probability of debtor conditioned by the realization of the systematic risk factor Z. Then using (3.20) and (3.21) follows

$$\begin{aligned} p(Z) &= \mathsf{P}\left[\sqrt{\varrho}Z + \sqrt{1-\varrho}\varepsilon < \tilde{K}\right] \\ &= \mathsf{P}\left[\varepsilon < \frac{\Phi^{-1}(\bar{p}) - \sqrt{\varrho}Z}{\sqrt{1-\varrho}}\right] \\ &= \Phi\left(\frac{\Phi^{-1}(\bar{p}) - \sqrt{\varrho}Z}{\sqrt{1-\varrho}}\right). \end{aligned} \tag{3.22}$$

The conditioned default probability $p(Z)$ is then a random variable with values in [0,1] and expected value $\mathsf{E}\,p(Z) = \bar{p}$. Now we can calculate the joint probability distribution of default events Y_i as

$$\mathsf{P}[Y_1 = y_1, \ldots, Y_n = y_n] = \int_{-\infty}^{\infty} p(z)^k \big(1 - p(z)\big)^{n-k} \varphi(z)\,\mathrm{d}z,$$

where $k = \sum_{i=1}^{n} y_i$, $p(z)$ is given by (3.22) and φ is the standard normal density function.

Heterogeneous Portfolios Let us try to extend the result of the homogeneous portfolio to the heterogeneous one. Let the default probability of the debtor i be p_i and let the risk driver $X_T^{(i)}$ be given by the multi–factor model

$$X_T^{(i)} = \alpha^{(i)} \cdot \boldsymbol{Z} + \beta^{(i)}\varepsilon^{(i)},$$

where $\boldsymbol{Z}$ is a random vector normally distributed with a zero mean value and a covariance matrix Σ, and $\varepsilon^{(i)}$ are independent standard random variable, $\boldsymbol{\alpha^{(i)}}$ is a vector of constant factor loads and β_i is also constant. Then the conditioned default probability is

$$p_i(\boldsymbol{Z}) = \Phi\left(\frac{\Phi^{-1}(p_i) - \alpha^{(i)} \cdot \boldsymbol{Z}}{\beta^{(i)}}\right),$$

and the joint default probability is

$$\mathsf{P}[Y_1 = y_1, \dots, Y_n = y_n] = \int_{\mathbb{R}^n} \prod_{i=1}^{n} p_i^{y_i}(\boldsymbol{z})\big(1 - p_i(\boldsymbol{z})\big)^{1-y_i} \varphi(\boldsymbol{z}; \Sigma)\, \mathrm{d}\boldsymbol{z},$$

where $\varphi(\boldsymbol{z}; \Sigma)$ is the n-dimensional normal density function with zero mean and covariance matrix Σ. We see that the joint probability distribution is modeled same as in the case of the homogeneous portfolio by the Gaussian copula (see Section 2.8).

In the case of reduced–form models we will end up also with some copula (sometimes called *survival copulas*). For more see Giesecke (2004, Section 3.6).

3.6.4 Monte Carlo Simulation

Let us assume that we have chosen some model dynamics, calibrate its parameters as well as the correlation structure and recoveries. Now we are interested in the probability distribution of the aggregate loss of the whole portfolio in some given horizon (for example one year). In some simple models we have seen that we can get a closed formula, but by adding more realistic features to the model and allowing more randomness and a dependency between different random drivers (like dependent recoveries and default events) we get more and more complex problem, where it is very hard to get a closed formula. In that case the simulation seems to be the only possibility.

Every run of the simulation gives us a realization of the random credit loss. We should perform sufficient many runs of the simulation to get image of the aggregate loss distribution. The complex simulation can take many days of the computation. Therefore, the number of runs of the simulation must be the compromise between the time consumption of the simulation and the precision of the resulting probability distribution.

To lower time requirements of the simulation some approximation methods can be used.

Approximations of Aggregate Loss Distribution

After n runs of the simulation we get n realizations of the random credit loss variable. We can assume that the loss distribution is very close or even belong to some well–known parametric distribution family. Then we think about n realizations of the simulation as about a random sample from the distribution F. Parameters of the distribution F can be estimate using the moment method or the maximum likelihood method.

Often used distribution is a gamma distribution with the density function

$$f(x) = \frac{a^b}{\Gamma(b)} e^{-ax} x^{b-1}, \qquad x > 0,$$

which is right–skewed and easily parametrized. If we are interested in modeling the number of defaults, i.e., modeling of random variable

$$L = \sum_{i=1}^{m} \mathbf{1}_{\{\tau_i \le 1\}},$$

often used distribution is the negative Binomial distribution. The integer valued negative Binomial distribution can be interpreted as the gamma mixture of Poisson distributions which

is consistent with the idea of reduced–form models where a default is first jump of a Poisson process. If every defaulter has a Poisson process as its own default process and parameter λ is random variable from the gamma distribution, then the random variable L has a negative Binomial distribution. Confidence intervals about parameters of the distribution can be estimated through the *bootstrap*. Other possibility for the aggregate loss distribution estimation is to use nonparametric kernel estimators.

4
Real Data Study

In this chapter we will show methods from Section 2.6 used on real data.

4.1 Data Analysis and Estimations

We use data from Standard&Poor's (2006), where yearly transition matrices for period 1981–2005 are given same as the number of firms starting within a particular rating. Let us denote the transition matrices by $\mathbf{P}_1, \dots, \mathbf{P}_{25}$. We assume that transition matrices $\mathbf{P}_1, \dots, \mathbf{P}_{25}$ were computed using the cohort method, i.e., for fixed k the ij-th element p_{ij}^k of matrix $\mathbf{P}_k$ was computed as

$$p_{ij}^k = \frac{n_{ij}^k}{n_{i.}^k},$$

where n_{ij}^k is the number of companies which started the k-th year in rating i and ended in rating j, and $n_{i.}^k$ is the number of companies started in rating i. Since we are given by $\mathbf{P}_k$ and $n_{i.}^k$ we can compute elements of frequency matrix $\mathbf{N}_k$ as

$$n_{ij}^k = p_{ij}^k n_{i.}^k. \tag{4.1}$$

This is of course valid only under the assumption that transition matrices were really computed by the cohort method.[25] The rating scale of S&P consists of 8 elements

$$\{AAA, AA, A, BBB, BB, B, CCC, D\},$$

and one special rating NR (not rated). The best rating is AAA and D is a default. S&P use for their purposes more finer rating scale with much more ratings, but in publicly available data they are grouped into 8 ratings above. Since we do not know what happened with companies which migrate to NR we simply ignored these transitions, that is, we took just first 8 columns of frequency matrices $\mathbf{N}_1, \dots, \mathbf{N}_{25}$. Data after this cleaning can be found in Appendix A.

Now let us recall what we want to show on real data. We assume that system (hundreds of companies around the world monitored by S&P) follows a continuous–time Markov chain under some stochastic transformation, which enters into the system as parameters $\boldsymbol{t} = (t_1, \dots, t_{25})$. In other words we assume that the generator matrix in k-th year is $t_k\mathbf{Q}$, where $\mathbf{Q}$ is same for every $k = 1, \dots, 25$. For uniqueness of the solution we need to add some additional condition. We add

$$\sum_{k=1}^{25} t_k = 25.$$

[25] Frequency data which we get by (4.1) was always very near to integer value (6.99, 81.01,...), from which we can speculate that really the cohort method was used.

This condition make sense to us, since we would like to have t_k near to natural value 1. We are looking for matrix $\mathbf{Q}$ and time parameters $\boldsymbol{t}$ such that matrices $\exp(t_k\mathbf{Q})$ will be as near as possible to matrices $\mathbf{P}_k$ for every $k = 1, \ldots, 25$. We use stochastic time componentwise optimization method (STCO) from Section 2.6.3 and compare the results with situation when we assume that the system follows a regular continuous–time Markov chain (without time change). This method without time change will be referred as finding a constant generator. Further, if we will speak about distance of two matrix we mean by that Euclidean distance.

In an accompanied file `fce.r` can be found the implementation of all six "one–year" methods from Section 2.6 and also the implementation of stochastic time componentwise optimization (STCO) procedure. All methods were implemented in statistical software `R`.

For the estimator of the constant generator matrix we used the componentwise optimization when as a starting point was chosen the best solution from first five methods (DA, WA, QOG, EM, MCMC).[26] A constant solution from this CO method is denoted by $\mathbf{Q}^{CO}$ and can be found in table 4.1.

	AAA	AA	A	BBB	BB	B	CCC	D
AAA	−0.09034	0.08730	0.00177	0.00064	0.00062	0.00000	0.00000	0.00000
AA	0.00666	−0.09938	0.08711	0.00390	0.00034	0.00114	0.00021	0.00001
A	0.00045	0.02174	−0.09197	0.06463	0.00322	0.00139	0.00029	0.00025
BBB	0.00018	0.00139	0.04526	−0.11012	0.05238	0.00668	0.00200	0.00224
BB	0.00043	0.00042	0.00149	0.06696	−0.18587	0.09711	0.01095	0.00851
B	0.00000	0.00062	0.00219	0.00133	0.07522	−0.20462	0.07073	0.05453
CCC	0.00000	0.00000	0.00407	0.00583	0.01348	0.18703	−0.62151	0.41110
D	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000

Table 4.1. Constant generator $\mathbf{Q}^{CO}$.

	AAA	AA	A	BBB	BB	B	CCC	D
AAA	−0.08299	0.07984	0.00196	0.00061	0.00057	0.00000	0.00000	0.00001
AA	0.00619	−0.09934	0.08852	0.00297	0.00059	0.00084	0.00022	0.00000
A	0.00019	0.01908	−0.08746	0.06318	0.00328	0.00126	0.00023	0.00026
BBB	0.00020	0.00157	0.04510	−0.11196	0.05428	0.00691	0.00163	0.00227
BB	0.00034	0.00056	0.00098	0.07153	−0.18255	0.08787	0.01210	0.00917
B	0.00000	0.00084	0.00249	0.00100	0.07326	−0.19530	0.06061	0.05708
CCC	0.00000	0.00000	0.00257	0.00808	0.01163	0.15970	−0.55413	0.37215
D	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000

Table 4.2. Multi–year generator $\mathbf{Q}^{STCO}$ with changed time.

As a second step we computed the estimation of the generator $\mathbf{Q}$ (see table 4.2) and the time evolution $\boldsymbol{t}$ (see table 4.3) using STCO method. Graph 4.1 shows us the Euclidean distance of matrices $\mathbf{P}_k$ and $\exp(t_k\mathbf{Q})$ which is the distance of theoretical transition matrix from the empirical one.

[26] The best performance if we compare distances from empirical transition matrices $\mathbf{P}_k$ was given by QOG method. It is not surprising since DA and WA methods are very simple and not a real optimization methods and aim of EM and MCMC method is not to produce generator which generate transition matrix as near as the empirical one, but rather estimate the real generator by incorporating more information (similar as in the comparison of the cohort method and the maximum likelihood method in the case of continuously observed data).

year	T_t	year	T_t	year	T_t	year	T_t	year	T_t
1981	0.501	1986	0.809	1991	1.290	1996	0.641	2001	1.471
1982	0.843	1987	0.675	1992	1.189	1997	0.742	2002	1.675
1983	0.694	1988	1.032	1993	1.219	1998	1.439	2003	1.177
1984	0.832	1989	1.198	1994	0.780	1999	0.998	2004	0.720
1985	1.042	1990	1.119	1995	0.984	2000	1.044	2005	0.885

Table 4.3. Time evolution estimation.

Let us denote the one–year transition matrix, which generate the generator $\mathbf{Q}^{CO}$ (resp. $\mathbf{Q}^{STCO}$), by $\mathbf{P}^{CO}$ (resp. $\mathbf{P}^{STCO}$). In the graph 4.1 we see the distances of matrices $\mathbf{P}_k^{CO}$ and $\mathbf{P}_k^{STCO}$ from the empirical $\mathbf{P}_k$. Better fit is almost in every year for $\mathbf{P}^{STCO}$ matrix. Detailed distances and percentage improvements are in table 4.4. The overall improvement is 35 %. Notice that we are comparing CO and STCO method. In practice there is often used maximum likelihood (ML) method though the partial observations. This wrong usage of ML method would give much worse fit then CO method.

The graph 4.1 also shows the estimated evolution of time. We can clearly see three periods where the time evolution was significantly higher (time ran faster) than in remain years. The first period are years 1990–1993 which can be connected to the U.S. recession in years 1990–

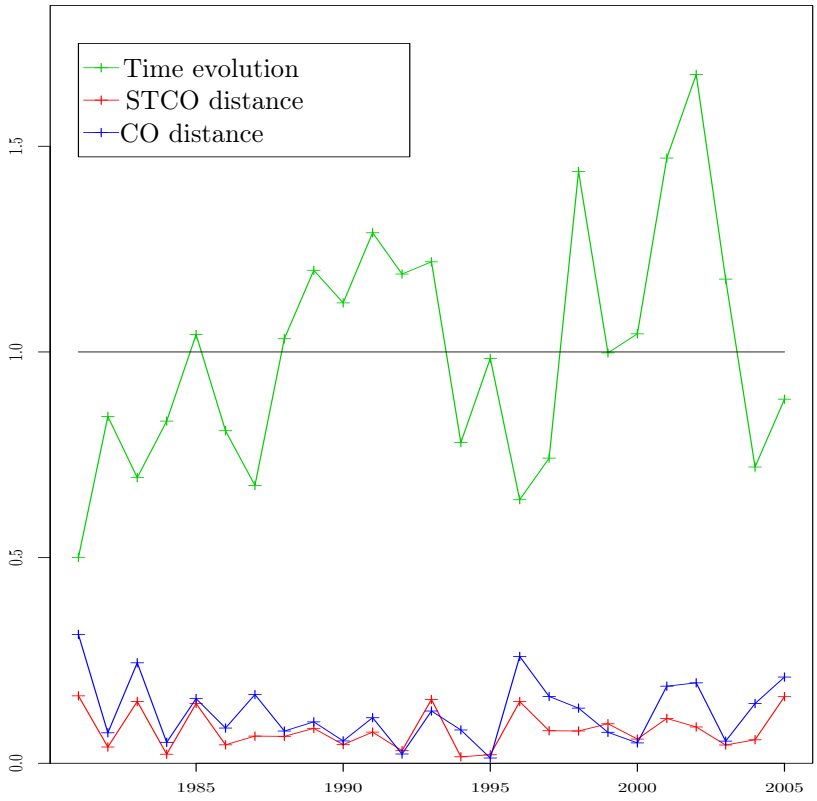


Figure 4.1. STCO method.

year	$\|\|\mathbf{P}-\mathbf{P}^{CO}\|\|$	$\|\|\mathbf{P}-\mathbf{P}^{STCO}\|\|$	Improvement
1981	0.313	0.164	−47.64 %
1982	0.074	0.040	−46.63 %
1983	0.244	0.150	−38.49 %
1984	0.051	0.022	−56.79 %
1985	0.157	0.145	−7.55 %
1986	0.086	0.045	−47.54 %
1987	0.167	0.066	−60.44 %
1988	0.078	0.065	−16.52 %
1989	0.101	0.085	−15.61 %
1990	0.054	0.045	−16.54 %
1991	0.111	0.076	−31.61 %
1992	0.023	0.031	36.08 %
1993	0.127	0.155	22.05 %
1994	0.081	0.016	−80.53 %
1995	0.013	0.021	62.43 %
1996	0.260	0.150	−42.25 %
1997	0.162	0.079	−51.30 %
1998	0.134	0.078	−41.49 %
1999	0.075	0.096	27.72 %
2000	0.050	0.059	18.83 %
2001	0.187	0.109	−41.85 %
2002	0.196	0.088	−54.85 %
2003	0.054	0.044	−17.50 %
2004	0.145	0.057	−60.65 %
2005	0.210	0.162	−22.72 %
1981–2005	3.153	2.049	−35.00 %

Table 4.4. Comparison of the fit by CO and STCO method.

1991. The second period is year 1998 in which the Russian crisis was and the last period is 2001–2002 when the Internet bubble burst.

These observations suggest an idea that if the economy is in recession the time evolution is quicker and companies defaults more often. On contrary when there is an economic expansion the time evolution slows down and companies default less often.

We would like to find some evidence of the connection between the time evolution and the state of economy. As indicators of the state of the economy we took the federal fund target rate (FFTR)[27] and the American GDP growth. Though the data are about companies all around the world we took both indicators of the American economy, because the American economy has great influence on the world economy. We get FFTR data from Federal Reserve Bank of New York (2008) and GDP data from Economic Research Division of Federal Reserve Bank of St. Louis (2008). Both are in table 4.5.

We have computed the correlation of t_k and FFRT, resp. GDP growth. In both cases the high negative correlation was the result. Precise values are

$$\mathsf{Corr}\,_{T_t,FFTR} = -0.574,$$
$$\mathsf{Corr}\,_{T_t,GDP} = -0.608.$$

[27] The interest rate announced by American FED. It is a major tool for regulation of the economy.

year	t_k	FFTR	GDP growth
1981	0.501	12.00	12.15
1982	0.843	8.50	4.05
1983	0.694	9.50	8.65
1984	0.832	8.25	11.21
1985	1.042	7.75	7.30
1986	0.809	6.00	5.75
1987	0.675	6.75	6.20
1988	1.032	9.00	7.69
1989	1.198	8.25	7.46
1990	1.119	7.00	5.81
1991	1.290	4.00	3.32
1992	1.189	3.00	5.70
1993	1.219	3.00	5.04
1994	0.780	5.50	6.23
1995	0.984	5.50	4.60
1996	0.641	5.25	5.67
1997	0.742	5.50	6.24
1998	1.439	4.75	5.33
1999	0.998	5.50	5.96
2000	1.044	6.50	5.92
2001	1.471	1.75	3.17
2002	1.675	1.25	3.37
2003	1.177	1.00	4.69
2004	0.720	2.25	6.86
2005	0.885	4.25	6.35

Table 4.5. Time evolution, FFTR and GDP growth data.

Further research can try to explain and predict the time evolution using some advanced econometric models. Of course for that we need better and more data.

Now let us try to model the time evolution $\boldsymbol{t}$ as a Lévy subordinator. We choose the Gamma process as our model. Using the maximum likelihood we estimate parameters of one–year gamma distributed increments with the density function

$$f(x) = \frac{a^{bt}}{\Gamma(bt)} e^{-ax} x^{bt-1}, \qquad x > 0,$$

Interesting is that the estimation of both parameters are same

$$a = b = 12.5095.$$

Then we would like to know how good is our hypothesis that time evolves according to the Gamma process. We tested using the goodness–of–fit test (see for example Lehmann and Romano (2005, Section 14.3)) whether we can reject the hypothesis of the Gamma distribution. With very high

$$p\text{-value} = 0.8187,$$

we did not reject the Gamma distribution. On picture 4.2 we can see the histogram of our observations and the theoretical density function.

4.2 Conclusion

Provided extension of a continuous–time Markov chain to the stochastic time is very easy doable and its performance is at least as good as the original model. In our data it showed 35 % better fit. We think it is not a neggligable improvement. Fur further research remain finding the dependence of the time evolution on some observable variables using some econometric models. With this done, one can predict more precisely future transition matrices.

Another very promising way is to extend EM and MCMC method to stochastic time and then perform complex simulation study and compare results.

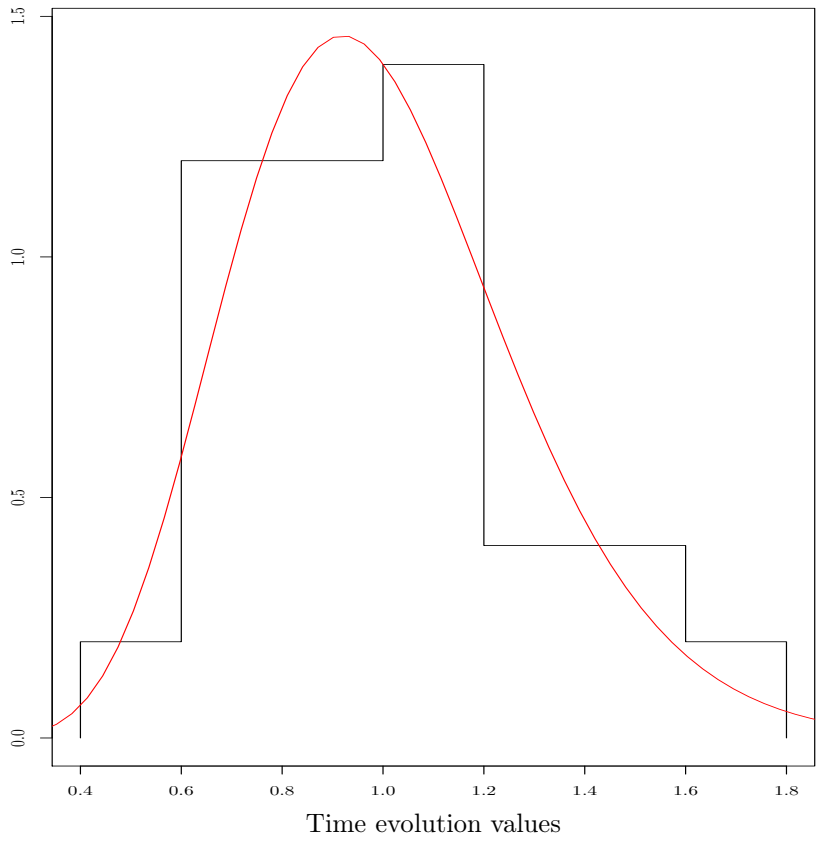


Figure 4.2. Time evolution histogram and theoretical density.

A
Frequency Data

Frequency data from Standard&Poor's (2006).

	AAA	AA	A	BBB	BB	B	CCC	D
AAA	77	10	0	0	0	0	0	0
AA	4	189	17	0	0	0	0	0
A	0	22	434	32	1	0	0	0
BBB	0	0	13	244	14	0	0	0
BB	0	0	2	12	134	66	1	0
B	0	0	1	0	4	72	2	2
CCC	0	0	0	0	0	1	10	0

Table A.1. S&P frequency matrix for year 1981.

	AAA	AA	A	BBB	BB	B	CCC	D
AAA	80	4	2	0	0	0	0	0
AA	1	204	15	1	3	0	0	0
A	0	18	409	45	3	0	0	1
BBB	1	0	6	243	26	1	0	1
BB	0	1	0	6	123	13	0	7
B	0	0	1	1	4	121	6	5
CCC	0	0	0	0	0	1	8	3

Table A.2. S&P frequency matrix for year 1982.

	AAA	AA	A	BBB	BB	B	CCC	D
AAA	78	17	1	0	0	0	0	0
AA	1	228	13	2	0	0	0	0
A	3	19	404	20	3	0	0	0
BBB	1	2	19	249	18	2	0	1
BB	0	1	2	6	125	20	0	2
B	0	0	1	1	5	121	1	7
CCC	0	0	0	0	0	3	11	1

Table A.3. S&P frequency matrix for year 1983.

	AAA	AA	A	BBB	BB	B	CCC	D
AAA	79	29	1	0	0	0	0	0
AA	5	271	13	3	0	0	0	0
A	0	13	426	16	3	0	0	0
BBB	0	1	35	232	18	6	0	2
BB	0	0	2	14	143	9	0	2
B	0	0	0	2	9	150	0	6
CCC	0	0	0	0	0	0	11	4

Table A.4. S&P frequency matrix for year 1984.

	AAA	AA	A	BBB	BB	B	CCC	D
AAA	85	6	0	0	1	0	0	0
AA	1	282	27	6	0	4	1	0
A	0	10	440	34	6	1	0	0
BBB	0	2	23	216	18	12	0	0
BB	0	0	2	11	153	21	3	3
B	0	0	3	0	5	167	1	13
CCC	0	0	0	0	0	4	7	2

Table A.5. S&P frequency matrix for year 1985.

	AAA	AA	A	BBB	BB	B	CCC	D
AAA	99	8	0	0	0	0	0	0
AA	4	311	16	6	0	2	0	0
A	1	26	436	50	8	9	0	1
BBB	0	0	21	224	25	8	1	1
BB	0	0	0	15	174	14	3	3
B	0	0	0	1	11	197	30	24
CCC	0	0	0	0	0	0	10	3

Table A.6. S&P frequency matrix for year 1986.

	AAA	AA	A	BBB	BB	B	CCC	D
AAA	122	4	0	1	0	0	0	0
AA	6	318	18	1	0	0	0	0
A	0	7	429	30	2	6	0	0
BBB	0	2	17	249	21	11	0	0
BB	0	0	1	19	192	21	0	1
B	0	0	3	0	17	267	9	11
CCC	0	0	0	1	1	4	35	7

Table A.7. S&P frequency matrix for year 1987.

	AAA	AA	A	BBB	BB	B	CCC	D
AAA	129	8	3	0	1	0	0	0
AA	5	297	38	10	2	1	0	0
A	0	8	444	23	4	3	0	0
BBB	0	1	29	244	17	7	2	0
BB	0	0	3	21	206	22	5	3
B	0	1	0	1	19	304	12	16
CCC	0	0	0	2	2	5	27	11

Table A.8. S&P frequency matrix for year 1988.

	AAA	AA	A	BBB	BB	B	CCC	D
AAA	144	10	0	0	0	0	0	0
AA	2	324	24	0	0	0	0	0
A	0	8	474	36	12	1	0	0
BBB	0	0	23	260	20	2	2	2
BB	0	0	2	35	187	15	2	2
B	0	1	0	0	30	289	16	14
CCC	0	0	1	0	1	0	24	18

Table A.9. S&P frequency matrix for year 1989.

	AAA	AA	A	BBB	BB	B	CCC	D
AAA	146	7	0	0	0	0	0	0
AA	2	336	39	0	0	0	0	0
A	0	12	478	42	7	1	0	0
BBB	0	0	17	292	18	3	0	2
BB	0	0	0	18	184	28	9	10
B	0	2	0	2	12	240	18	31
CCC	0	0	0	0	1	2	27	15

Table A.10. S&P frequency matrix for year 1990.

	AAA	AA	A	BBB	BB	B	CCC	D
AAA	146	20	1	0	0	0	0	0
AA	1	356	30	0	0	0	0	0
A	1	3	519	35	3	0	0	0
BBB	0	3	14	313	19	3	0	3
BB	0	0	0	14	182	18	4	4
B	0	1	0	1	16	197	10	40
CCC	0	0	0	1	2	4	28	21

Table A.11. S&P frequency matrix for year 1991.

	AAA	AA	A	BBB	BB	B	CCC	D
AAA	156	17	0	0	0	0	0	0
AA	5	408	29	5	0	0	0	0
A	0	7	585	26	1	1	0	0
BBB	0	1	18	341	15	3	0	0
BB	0	0	0	29	181	10	7	0
B	0	0	1	3	25	150	9	16
CCC	0	0	0	0	2	6	24	14

Table A.12. S&P frequency matrix for year 1992.

	AAA	AA	A	BBB	BB	B	CCC	D
AAA	160	5	3	0	0	0	0	0
AA	0	440	28	1	0	0	0	0
A	2	6	612	25	0	0	0	0
BBB	0	0	20	379	27	0	1	0
BB	0	1	1	23	200	22	1	1
B	0	0	1	2	30	147	4	6
CCC	0	0	0	0	1	12	13	6

Table A.13. S&P frequency matrix for year 1993.

	AAA	AA	A	BBB	BB	B	CCC	D
AAA	156	15	1	0	0	0	0	0
AA	2	434	43	0	0	0	1	0
A	0	9	681	30	1	1	1	1
BBB	0	1	14	446	10	2	0	0
BB	0	0	0	27	305	10	0	1
B	0	0	0	1	16	264	9	10
CCC	0	0	0	0	0	2	11	4

Table A.14. S&P frequency matrix for year 1994.

	AAA	AA	A	BBB	BB	B	CCC	D
AAA	163	14	0	0	0	0	0	0
AA	3	436	46	1	0	0	0	0
A	0	20	793	31	2	0	0	0
BBB	0	2	25	518	21	0	0	1
BB	0	0	2	25	333	20	0	4
B	0	0	1	2	29	301	8	18
CCC	0	0	0	1	0	2	13	7

Table A.15. S&P frequency matrix for year 1995.

	AAA	AA	A	BBB	BB	B	CCC	D
AAA	157	11	0	0	0	0	0	0
AA	3	445	26	0	0	0	0	0
A	0	28	859	19	1	0	0	0
BBB	1	0	40	613	12	1	0	0
BB	0	0	2	37	348	21	2	3
B	0	0	1	2	34	299	8	12
CCC	0	0	0	0	2	3	13	1

Table A.16. S&P frequency matrix for year 1996.

	AAA	AA	A	BBB	BB	B	CCC	D
AAA	156	8	0	0	0	0	0	0
AA	4	469	19	4	0	1	0	0
A	0	18	901	42	2	3	0	0
BBB	0	1	28	693	19	6	0	2
BB	0	0	1	40	400	24	0	1
B	0	0	3	0	31	340	12	16
CCC	0	0	0	0	0	4	13	3

Table A.17. S&P frequency matrix for year 1997.

	AAA	AA	A	BBB	BB	B	CCC	D
AAA	149	11	0	1	0	0	0	0
AA	2	465	27	0	0	0	0	0
A	1	15	897	59	3	0	0	0
BBB	0	0	24	798	50	7	2	4
BB	2	1	1	31	466	40	14	6
B	0	1	1	3	39	507	34	31
CCC	0	0	1	0	0	5	9	12

Table A.18. S&P frequency matrix for year 1998.

	AAA	AA	A	BBB	BB	B	CCC	D
AAA	134	9	0	0	0	0	0	0
AA	1	479	38	3	0	0	0	1
A	0	24	898	61	1	1	0	1
BBB	0	2	30	855	40	0	0	2
BB	0	1	0	21	575	51	5	7
B	0	0	1	2	22	644	35	62
CCC	0	0	0	0	0	2	35	23

Table A.19. S&P frequency matrix for year 1999.

	AAA	AA	A	BBB	BB	B	CCC	D
AAA	126	5	2	0	0	0	0	0
AA	4	465	60	1	0	0	0	0
A	0	25	890	79	4	1	0	1
BBB	0	3	27	934	38	5	3	4
BB	0	0	1	31	648	51	9	10
B	0	0	2	2	31	670	38	67
CCC	0	0	0	0	1	5	43	27

Table A.20. S&P frequency matrix for year 2000.

	AAA	AA	A	BBB	BB	B	CCC	D
AAA	120	7	0	0	0	0	0	0
AA	1	447	59	0	0	0	0	0
A	0	25	916	74	2	0	4	2
BBB	0	1	37	990	52	7	11	4
BB	0	0	3	23	608	81	18	26
B	0	0	0	0	26	581	72	98
CCC	0	0	0	0	0	8	44	49

Table A.21. S&P frequency matrix for year 2001.

	AAA	AA	A	BBB	BB	B	CCC	D
AAA	107	15	0	1	0	0	0	0
AA	1	394	89	12	1	3	0	0
A	0	6	931	118	9	2	1	1
BBB	0	1	27	1043	77	28	7	13
BB	1	0	3	26	623	61	11	22
B	0	0	0	2	35	515	69	61
CCC	0	0	1	0	2	13	58	75

Table A.22. S&P frequency matrix for year 2002.

	AAA	AA	A	BBB	BB	B	CCC	D
AAA	92	9	2	0	0	0	0	0
AA	2	362	47	2	0	0	0	0
A	0	7	998	74	2	0	0	0
BBB	0	0	19	1158	68	4	0	3
BB	0	0	0	27	694	90	6	5
B	0	0	0	1	50	548	29	31
CCC	0	0	0	0	1	19	75	54

Table A.23. S&P frequency matrix for year 2003.

	AAA	AA	A	BBB	BB	B	CCC	D
AAA	88	6	0	0	0	0	0	0
AA	1	368	15	1	0	0	0	0
A	0	13	1071	36	1	0	0	0
BBB	0	1	29	1311	27	2	0	0
BB	1	0	1	39	786	52	2	4
B	0	0	1	1	56	647	19	13
CCC	0	0	1	0	1	21	73	20

Table A.24. S&P frequency matrix for year 2004.

	AAA	AA	A	BBB	BB	B	CCC	D
AAA	87	9	1	0	0	0	0	0
AA	0	369	20	2	0	0	0	0
A	1	20	1088	54	0	0	0	0
BBB	0	3	91	1290	47	7	0	1
BB	0	0	0	58	779	70	2	2
B	0	0	1	6	86	713	38	16
CCC	0	0	0	1	1	32	59	11

Table A.25. S&P frequency matrix for year 2005.

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