Vrije Universiteit Amsterdam
Department of Mathematics

Master's Thesis

The Basel I and Basel II Accords. Comparison of the models and economical conclusions.

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Abstract

In this thesis we will compare the Basel I (1988) and the Basel II accord (2004), including the rules regulating operations of the present banks and financial companies. We will present these documents giving a general overview and mention some details. But our concentration will be mostly focused on measures, which give information about the contract, are exploited to compare contracts, and enable to make a decision whether the contract is attractive or not.

We will start from the *contract* analysis. We are going to bring closer the contract's structure by introducing the special notations, for instance, $\mathbb{P}$D, LGD, EAD, etc, and explain their meaning. We will present also the computational methods based on the *Merton model* and designed to determine *capital requirements*, called *Regulatory* or *Economic Capital*. Behind the mathematical approach we will outline economical background concerning the problems, which we are going to study. For illustrative purposes, we will apply the theory to real example.

At the end of thesis, we will draw conclusions and give an explanation why Basel II is better.

Chapter 1

Introduction

In the light of permanent development of financial markets and continuously changing market situation, banks and other financial companies need rules which will regulate financial agreements between them and their clients. Usually, conditions of such agreements are formalized in the form of *contract*. The bank and the customer, accepting the contract, commit themselves to obeying the contract's conditions. It means that bank lends the customer money and the customer is obliged to pay them back. However, sometimes the customer losses financial liquidity, and then the contract goes to *default*. Hence, for all financial institutions it is very difficult to quote whether the customer is 'good' or 'bad', whether he will be solvent at the contract duration. Seeing that, banks and financial companies would like to have a very good measure that will give them an unambiguous answer to their questions.

On the other hand, banks will only be able to generate new transactions if the customers do not doubt their ability to remain solvent. For the sake of complexity of the financial market and interactions between all components, it is difficult to consider all possible events which can occur in the future. The market volatility renders that each move or investment is related to risk.

Unfortunately, models which are usually used to measure the risk level do not give exact results. The reason is that, it is very difficult to calculate the risk, because of unforeseen events, like customer's default. Hence, the models only include tools to identify concentrations of risk and give opportunities for diversification within a disciplined and objective framework. Regardless of the shortcomings, these methods are commonly used because all companies are under an obligation to have some fixed frame of calculating risk and decision making whether to take this risk or not.

One of the tools used by banks is the determination of a *capital requirement*. The definition of the capital requirement says that this is an amount of money required to cover monetary losses due to the unexpected bad events, in other words, to hedge the contract. Each company should keep reserves of money to hedge contracts. Furthermore, it seems to be obvious that the amount of money is not the same for each company, but the question is: how should we calculate this amount? Are there some methods which let us determine the capital requirements? Unfortunately, the problem is not easy solvable.

During recent years, almost the whole banking and financial world has been concentrated on research concerning capital allocation. Everybody would like to find the method which gives the most precise results or the best estimation.

The most involved in this research is the Basel Committee on Banking Supervision. This committee, established by the Bank of International Settlements, has played a leading role in formalizing the relationship between credit risk, different forms of market risk, and the capital requirements. It undertook a detailed study of methods to set the regulatory capital. During a long time of considerations, the Basel Committee found some solutions which were published as the Basel I and Basel II accords. In general, the Basel documents are sets of rules for banking regulation and supervisory. In particular, they set the global capital adequacy standards. They are international agreements that describe the risk sensitive framework for the assessment of regulatory capital and oblige financial companies to take adequate hedging actions.

The Basel I accord was introduced in 1988. The main aims of this agreement were 'leveling the playing field' for the competition in terms of costs between internationally active banks and reduction of the probability that such competition would lead to bidding down of capital ratios to extremely low levels. It wanted to eliminate unfair advantages of banks in countries without a minimum of capital requirements. Hence, the Basel Committee set this minimum according to a given measure of the total credit risk outstanding amount.

At the beginning, it introduced calculation of the required capital using a mathematical model based on the *Merton model* and the properties of the *Beta distribution*. This model will be described in detail in Chapter 3. Unfortunately, methodology of this model was too advanced to apply in reality and caused a lot of problems in computations. Hence, the Basel Committee applied an approach that relied on historical data. It determined, required to hedge contract, amount of money as 8% of the capital contribution, regardless of the customer's credibility. In practice it looked as if the financial institutions had to keep 8% of outstanding amount of money as the capital requirement.

Application of this approach showed that it is not correct. The basic problem of the Basel I was that it focused on costs, overlooking the considerations of risk and financial stability. It took into consideration the risk, which is conected with given contract. Regardless of the customer's risk level, it requires 8% to secure the contract, whereas the capital requirements should depend on the customer's risk that one takes. Moreover, Basel I exhibited a fundamental weakness – it based on a model that was becoming obsolete quickly. During a very short time, it was recognized that the risk and unexpected events should have a significant impact on the investments. Namely, every company has different vulnerability to risk and every contract is related with a different risk level, so standardization of the requirements for all companies and contracts is a naive approach. On account of necessary corrections of the shortcomings, they prepared the *New Capital Accord*.

The Basel II accord was presented in 2004. In contrast to Basel I, the new agreement is mostly an instrument of prudential regulations. It puts a pressure on things, which are specific to each institution and defines instruments to deal with this diversity and idiosyncrasy.

The New Capital Accord extended the old method of calculating the capital requirements based on the Merton model and introduced a new method based on the *Asymptotic Risk Factor Model*. It takes into account the company's situation on the market, expressed by the *rating* and the *probability of default*, describing the risk level. Basel II makes financial institutions obligor to hedging all their contracts by different amounts of capital contributions depending on the customer's credibility. We will present the details in Chapter 4.

Talking about Basel I and Basel II we have to pay particular attention to the *Internal Return Capacity*, called the IRC and *Risk-adjusted Return on Capital*, called the RaRoC. These measures are strictly linked with the Basel environment.

Generally, both of them express a fraction of amount of money earned by the company to the amount of money, which company has to keep for hedging the contract. The IRC is provided by the Basel I accord and the RaRoC by the Basel II accord. Hence, the methods of computing the IRC are imposed by the Basel Committee. However, in the Basel II approach the choice of the method of the RaRoC computations is to the financial company to specify. Because the capital requirements can be computed in different way by each company, the same contract can be described by different values of the RaRoC, depending on the financial company.

The IRC and the RaRoC are used by banks to making the decision about a rejection or an acceptance of a given contract. Each financial institution sets its own IRC or RaRoC threshold, called in the RaRoC case *target RaRoC*, and according to that they make a decision about the contract. If the threshold is equal to for instance 20%, then it means that all contracts with the IRC and the RaRoC above this amount are accepted and below this amount are rejected. Hence the IRC and the RaRoC frameworks are very useful and commonly applicable by banks. The details and the application in practice will be seen in Chapter 5.

Chapter 2

Structure of the contract

Let us consider a hypothetical situation in which we have a contract between the bank and the customer. The contract is characterized by components. Some of them, such as the amount of money, which the bank is going to lend the customer or an *interest rate* are set by the bank or the customer in the beginning of the contract. But some of them, such as the *probability of default* or the *funding rate* are fixed and are usually dependent on the quality of the bank and the customer. Because we are interested in getting an economical overview, we will consider a very easy contract, without any additional components, such as *taxes*, and any other events, such as *automatical lease extension.* All components which we are going to consider are presented below, along with explanation of their meaning.

2.1 Periodical Installments

Let us assume that the financial company lends the customer some amount of money (a *principal amount*) for some time (a *contract's term*). In this situation, the customer becomes obliged to pay a fixed amount of money every month (year). This fixed payment is the sum of *principal payments* (the part of borrowed money which equals the principal amount divided by the number of periods) and *interest payment* (the amount of money paid to the bank for a service). The interest payment is determined by a monthly (yearly) percentage rate called *active rate*, which can be split into *funding rate* and *margin rate.*

The funding rate reflects the cost of funds and the margin rate determines the company's profit received as a result of giving a loan. In other words, the funding rate is the monthly (yearly) cost of getting money for the customer by the financial company. The basis rule is that the better company, the

cheaper they can get money, therefore the funding rate is lower.
The margin rate is a percentage rate reflecting the amount of the company's *income*. According to the definition, the income is the amount of money earned by the company as a result of normal business activities.

If we denote the contract's term as T, the principal amount as PA, the fixed monthly active rate as r_a^m, then the total monthly payment P is expressed by formula:

$$P := \frac{r_a^m}{1 - (1 + r_a^m)^{-12T}} \cdot PA$$

Because the monthly active rate is constant, hence the total monthly payment is constant in each month as well.

2.2 Default

When a financial company decides to accept a contract, it does not know what will happen in the future. There are a lot of internal and external factors which have a vast influence. We already know that all investments are related to risk. The company does not know if the customer will be solvent during the whole contract's duration or if he will default.

According to the Basel Committee on Bank Supervision:

"*A default is considered to have occurred with regard to a particular obligor when either or both of the two following events has taken place.*

The bank considers that the obligor is unlikely to pay its credit obligations to the banking group in full, without recourse by the bank to actions such as realizing security (if held).

The obligor is past due more than 90 days on any material credit obligation to the banking group. Overdrafts will be considered pas due once the customer has breached an advised limit or been advised of a limit smaller than current outstanding." (Basel Committee on Banking Supervision, 2003)

Hence we can conclude that a default does not mean that a bank will loose its money. It means that the customer has temporary problems with solvency, but after paying all arrears back, the contract finishes normally. By arrears we understand the amount of money, which the customer has to pay extra in case of not refunding at the fixed time.

The default, which occurs with a probability of default is not typical for a normal course of the contract. It entails the consequences for the client and its potential occurrence causes an uncertainty for the bank.

2.3 Probability of Default

Given a specific contract, we are not able to predict whether a lessee will default. However, experience teaches us how often a similar lessee has defaulted. Basing on historical data we can adjust the frequency of going default for a specific contract. This frequency is expressed by the *probability of default* and it depends on the customer and the vintage of the contract.
The "quality of the lessee" is usually asserted by *ratings* in the *Standard & Poor's* ranking. They change from R1 till R21, and each of them denotes a different financial situation and the ability to default for the company. The best is the R1 rating, and the R21 means a default.

2.4 Exposure

During the contract time the customer pays money back in periodical payments, causing reduction of the outstanding amount of money. According to Basel II, the outstanding amount of money is called an *Exposure* and is denoted by EXP.
Usually, the exposure is computed annually and at the start of the contract, is equal to the principal amount. As long as the customer pays the money back, the exposure equals to the principal amount at time t subtracted by the amount of money paid by the customer back and at the end of the contract it is equal to 0. For the sake of a time value of money, the principal amount at time t is equal the principal amount at time 0 multiplied by the annual interest factor.

We mentioned above that the exposure is computed yearly. Because an active rate is a monthly rate, hence we have to change it into a yearly rate. We can do this using follow expression:

$$(1 + r_a^m)^{12} = (1 + r_a^y)$$

Therefore, the formula expressing the exposure of the nth contract at time t, where t means the t'th year of the contract term, is as follows:

$$EXP_{n,t} := \begin{cases} PA, & t = 0 \\ PA \cdot (1 + r_a^m)^{12t} - P \cdot \frac{(1+r_a^m)^{12t}-1}{r_a^m} & \text{for } t = 1, \ldots, T-1 \\ 0 & t = T. \end{cases}$$

With respect to the contract, we use an *Exposure at Default*, denoted by EAD. By analogy, it means the amount of money, which the customer owes the bank at default time. In accordance with the definition of default introduced by the Basel Committee, this money includes the exposure at this time and arrears, accumulated during 90 days. Only the first year of the contract constitutes an exception.
At the start of the contract, arrears are unavailable, so the EAD is equal to the exposure. For future points, the assumption is made that in case of default three monthly payments of arrears have been added. This results in the following EAD equation:

$$EAD_{n,t} := \begin{cases} EXP_{n,t}, & \text{for } t = 0, \\ EXP_{n,t} + P \cdot \frac{(1+r_a^m)^3}{r_a^m}, & \text{for } t = 1, \ldots, T-1, \end{cases}$$

and finally:

$$EAD_{n,t} := \begin{cases} EXP_{n,t}, & \text{for } t = 0, \\ PA \cdot (1+r_a^m)^{12t} - P \cdot \frac{(1+r_a^m)^{12t} - (1+r_a^m)^3}{r_a^m}, & \text{for } t = 1, \ldots, T-1. \end{cases}$$

2.5 Costs

Each contract is strictly related to different kinds of *costs*, which the company has to incur. Based upon Activity Based Costing, cost allocations occur in the *Front Office Costs* and the *Back Office Costs*. A further distinction is made between fixed and variable costs.

The fixed costs are represented by a constant value and are incurred at the start of the contract. Similarly, the variable Front Office Costs occur at the start of the contract, but they are computed as a percentage of the principal amount. The variable Back Office Costs are a percentage of the average exposure over the periods. Hence:

$$C_{n,t} := \begin{cases} C_{n,t}^{FOC} + C_{n,t}^{BOC} + (c_{n,t}^{FOC} + c_{n,t}^{BOC}) \cdot \frac{1}{2}(EXP_{n,t} + EXP_{n,t+1}), & t = 0, \\ c_{n,t}^{BOC} \cdot \frac{1}{2} \cdot (EXP_{n,t} + EXP_{n,t+1}), & t = 1, \ldots, T-1. \end{cases}$$

where $C_{n,t}^{FOC}$, $C_{n,t}^{BOC}$ denote fixed Front and Back Office Costs, and $c_{n,t}^{FOC}$, $c_{n,t}^{BOC}$ denote variable Front and Back Office Costs for nth contract at time t, respectively.

2.6 Cash Flow

The term *Cash Flow* is used to describe all flows of money. It is defined as the difference between the income and the expenses of the company. If we assume that the contract does not go into default, then the expenses include only costs, but when the customer defaults then the expenses include the costs till default time and the loss caused by default.

The loss for nth contract is expressed as the fraction of the remaining exposure at default moment, and this fraction is determined by the percentage called *Loss Given Default*.

$$L_{n,t} = LGD_n \cdot EAD_{n,t}.$$

The loss given default (LGD) is given in advance and it is the contract's and customer's specification.

2.7 Regulatory and Economic Capital

In general, *Regulatory* and *Economic Capital* are the amount of capital allocated and held by the financial company in order to protect it from the unexpected losses with a reasonable degree of confidence. In other words, they are the sorts of capital requirements, which were provided in the introduction. Hence, they are determined by the confidence interval from the *Loss Distribution*. We will explain it in greater detail in the next chapter, but for now we would like to mention that the confidence level specifies how much of the unexpected losses should be covered by the economical or regulatory capital. Usually, this amount depends on the rating of the bank, for instance the banks with rating R1 should cover $99,99\%$ of the unexpected losses.

The difference between regulatory and economic capital concerns choices of the confidence level and the time horizon. For economic capital banks choose it, whereas for regulatory capital supervisors set it. Hence, usually the capital requirements provided by Basel I are called regulatory capital, whereas provided by Basel II are called economic capital. We will keep this terminology, and additionally we will denote regulatory capital by $RECAP$ and economic capital by $ECAP$.

The idea of using the confidence level in the computations of the capital requirements appears in the Basel I document, and it was extended in the Basel II document. The documents include some settlements concerning the confidence level and according to them the banks are asked to calculate their regulatory capital requirements to an αth confidence interval. This issue is elaborated in Chapter 4.

Chapter 3

The Basel I Capital Accord

3.1 Introduction

The Basel I accord was revolutionary in that it sought to develop the single risk-adjusted capital standard that would be applied by international banks. The heart of the Basel Accord was the establishment of similar capital requirements for the banks to eliminate unfair advantages of banks in the countries, where a minimum of capital was not required. Hence, Basel I defines a standard methodology for calculating the capital requirements.

3.2 The Basel I approach on the contract level

In the previous chapter, we gave a definition of the capital requirements and we affirmed also that they are determined by the confidence level of the loss distribution. Now we will explain it in greater detail.

The default and, linked with it, loss appear to be unforeseen. Because of that, we do not know whether it will occur and how much the loss will be. Hence, the idea is to consider the default and the loss, as a random variables. In aftermath of this, we can talk about the probability of an event and the distribution.
The definition of the probability of default was given in the previous chapter. Now, we will introduce the formula expressing this probability. Furthermore, we will use this formula to determine the loss distribution and, in the end, the αth percentile of this distribution.

Without loss of generality, we can consider discrete random variables $\mathrm{D}_{n,t}$, given as follows:

$$\mathrm{D}_{n,t} = \begin{cases} 1, & \text{when default occurs at time } t, \\ 0, & \text{otherwise,} \end{cases}$$

where n denotes the n'th contract and $t = 0, \ldots, T-1$ – t'th year of the contract term.

This variable has two-point distribution and values, which are taken by this variable, depend on the default occurrence. Because the default occurs at time t with the probability of default $\mathbb{P}\mathrm{D}_{n,t}$, we have

$$\mathrm{D}_{n,t} = \begin{cases} 1, & \text{with} \quad \mathbb{P}\mathrm{D}_{n,t} \\ 0, & \text{with} \quad 1 - \mathbb{P}\mathrm{D}_{n,t} \end{cases}$$

Furthermore, using properties of the two-point distribution, we can directly conclude that the expected value of $\mathrm{D}_{n,t}$ is expressed by a formula:

$$\mathbb{E}(\mathrm{D}_{n,t}) = \mathbb{P}\mathrm{D}_{n,t} \tag{3.1}$$

and the standard deviation is given as follows:

$$\sigma(D_{n,t}) = \sqrt{\mathbb{P}D_{n,t}(1 - \mathbb{P}D_{n,t})}. \tag{3.2}$$

In that case, it is also possible to change the loss formula, which is in accordance to the formula presented in the previous chapter, as follows:

$$L_{n,t} = LGD_n \cdot EAD_{n,t}. \tag{3.3}$$

Because the random variable $D_{n,t}$ takes only two values 0 or 1, and LGD_n and $EAD_{n,t}$ for all contracts and $t = 0, \ldots, T-1$ are the constants given in advance, we can write that:

$$L_{n,t} = D_{n,t} \cdot LGD_n \cdot EAD_{n,t}. \tag{3.4}$$

In the aftermath of this, we can consider the loss as a random variable and determine the distribution called the loss distribution.

With respect to the loss distribution we can talk about expected value called *Expected Loss* and standard deviation called *Unexpected Loss*.
The expected loss is a part of the loss, which is expected by banks to incur it in the future. For the sake of that, it is not related to the risk, so usually it is not covered by the economic capital. Because mathematically it is expressed by the expected value of the loss distribution, hence

$$\mathbb{E}(L_{n,t}) = \mathbb{E}(\mathrm{D}_{n,t}) \cdot LGD_n \cdot EAD_{n,t} = \mathbb{P}\mathrm{D}_{n,t} \cdot LGD_n \cdot EAD_{n,t}.$$

The unexpected loss, causing by unforseen events, is used to reflects uncertainty. Because it is the result of the risk taking, hence the Basel Committee requires keeping regulatory capital to cover it. Mathematically, the unexpected loss is understanding as the standard deviation of loss, so we can write directly that

$$\sigma(L_{n,t}) = \sqrt{\mathbb{P}\mathrm{D}_{n,t}(1 - \mathbb{P}\mathrm{D}_{n,t})} \cdot LGD_n \cdot EAD_{n,t}.$$

Usually, the expected loss at time t is denoted by $\mathbb{E}L_{n,t}$, the unexpected loss at time t by $\mathbb{U}L_{n,t}$ and we will keep this notation.

The construction of random variable for the loss guarantees existence of the loss distribution, which in this case is estimated by a Beta distribution with the mean that equals to the expected loss, and the standard deviation equals the unexpected loss.

Moreover, if we denote the confidence level by α, then in accordance with the model the capital requirements are represented by the αth percentile of the Beta distribution, which is approximated with eight multiplied by the unexpected loss.

$$Economic\ Capital_{n,t} \approx 8 \cdot \mathbb{U}L_{n,t}. \tag{3.5}$$

Both the above approximation and the estimation of the loss distribution are the results of plenty tests and numerical simulations that have been carried out by the Basel Committee. We refer interested readers to more advanced documents.

However, let us mention one issue, which is related to the computation of the capital requirements.

Namely, if we look at Formula (3.5), we see that only LGD, EAD, and the probability of default are necessary to compute the regulatory capital. As we stated above, LGD and EAD are given in advance, so only the probability of default is needed to get the value of capital requirements. Basel I admits several methods of computations, but the most widely used is based on the Merton model.

3.2.1 The Merton model

In 1974, Merton introduced Black and Scholes (1973) option pricing model to evaluate corporate liabilities, focusing mostly on the computations of the probability of default. Assuming that the firm's structure of capital can be expressed as the sum of equities and values of debt, he proposed to consider the firm's assets in the option framework. More precisely, he showed that the equity is equivalent to a *call option*. Using some examples, he explained legitimacy of his idea.

To present of Merton's methodology we will start from introducing the following notation: $A_{n,t}$ – the firm's asset value at time t, $E_{n,t}$ – the equity at time t, $B_{n,t}$ – the debt at time t and n – the number of the company.
In accordance with the Merton model assumptions, we have

$$A_{n,t} = E_{n,t} + B_{n,t} \text{ for } t = 0, \dots, T-1. \tag{3.6}$$

Further, let us assume that we have the contract between the bank and the client, which says that if the client does not repay his debt, then the bank holds the assets. In this situation, the assets exemplify a *guarantee*, that the client will give the money back. Therefore, we can consider two situations:

- $A_{n,t} > B_{n,t}$
  In this situation $E_{n,t} = A_{n,t} - B_{n,t}$ is positive when the client repaid his debt (because the asset's value, which he will get, is greater than the amount of repaid to bank money) or equal zero otherwise (because of insolvency of the client the bank kept the assets, so the customer do not earn anything).

- $A_{n,t} < B_{n,t}$
  In this situation $E_{n,t}$ is negative if the client repaid his debts (because the asset's value, which he will get, is smaller than the amount of repaid to bank money) or equal zero otherwise.

As a conclusion, we get that a repayment of the debt in the first case , however keeping the debt in the second situation is the most profitable action for the customer. In the first situation he will earn money. In the second he does not earn anything, but at least he does not loss anything. Let us place our results in a figure.

Now, if we look at the graph of payments, we will see that it is the same as the graph of call option payments. Hence, we state that the equity can be expressed as the call option on the asset with *strike price* equal to $B_{n,t}$ and *maturity* T.
It is also very easily noticeable that the threshold is determined by equality between the equity and the debts, and moreover it denotes the moment of default occurrence. Namely, it was showed above that the default occurs when the value of the firm's assets is less than the amount of debts. Seeing that, the frequency of the default's occurrence is expressed by the probability of default, we can express the probability of default as follows:

$$\mathbb{P}D_{n,t_D} = \mathbb{P}(A_{n,t_D} < B_{n,t_D}), \tag{3.7}$$

where t_D denotes the default time.

Above we also concluded that the equity is represented by the call option. We already know that the tool which lets us price the call option is the Black-Scholes model. The simple conclusion is that we can directly use the same model to the firm's asset pricing and indirectly to compute the probability of default.
Assume that we can express the dynamics of the firm's assets as

$$dA_{n,t} = \mu_n dt + \sigma_n dW_{n,t}, \tag{3.8}$$

where μ_n is the total expected return on the asset, σ_n is the asset volatility, and $W_{n,t}$ is a Brownian motion. Using Itô's formula, we can give a solution to the differential equation (3.8), which is as follows:

$$A_{n,t_D} = A_{n,t} \cdot \exp\left(\left(\mu_n - \frac{\sigma_n^2}{2}\right) \cdot (t_D - t) + \sigma_n \cdot \sqrt{t_D - t} \cdot X_{n,t,t_D}\right), \tag{3.9}$$

where

$$X_{n,t,t_D} = \frac{W_{n,t_D} - W_{n,t}}{\sqrt{t_D - t}},$$

and in accordance with the properties of a Brownian motion it is standard normally distributed. Let us make also a remark that $A_{n,t}$ denotes the current asset value of nth company.
In reference of Formula (3.7):

$$\mathbb{P}(A_{n,t_D} < B_{n,t_D}) \Leftrightarrow \mathbb{P}(A_{n,t} \cdot \exp\left(\left(\mu_n - \frac{\sigma_n^2}{2}\right) \cdot (t_D - t) + \sigma_n \cdot \sqrt{t_D - t} \cdot X_{n,t,t_D}\right)$$

$$< B_{n,t_D}) \Leftrightarrow \mathbb{P}\left(\left(\left(\mu_n - \frac{\sigma_n^2}{2}\right) \cdot (t_D - t) + \sigma_n \cdot \sqrt{t_D - t} X_{n,t,t_D}\right) < \ln \frac{B_{n,t_D}}{A_{n,t}}\right)$$

$$\Leftrightarrow \mathbb{P}\left(X_{n,t,t_D} < \frac{\ln \frac{B_{n,t_D}}{A_{n,t}} - (\mu_n - \frac{\sigma_n^2}{2}) \cdot (t_D - t)}{\sigma_n \cdot \sqrt{t_D - t}}\right).$$

Using assumption about the standard normal distribution of X_{n,t,t_D} we finally get:

$$\mathbb{P}D_{n,t_D} = \Phi\left(\frac{\ln B_{n,t_D} - \ln A_{n,t} - (\mu_n - \frac{\sigma_n^2}{2})(t_D - t)}{\sigma_n \sqrt{t_D - t}}\right). \tag{3.10}$$

Now, if we come back to Formula (3.5), and we insert all computed values then the regulatory capital on the contract level is expressed by

$$RECAP_{n,t} \approx 8 \cdot LGD_n \cdot EAD_{n,t} \cdot \sqrt{\Phi\left(\frac{\ln\frac{B_{n,t_D}}{A_{n,t}} - (\mu_n - \frac{\sigma_n^2}{2})(t_D - t)}{\sigma_n\sqrt{t_D - t}}\right)} \cdot$$

$$\cdot \sqrt{\left(1 - \Phi\left(\frac{\ln\frac{B_{n,t_D}}{A_{n,t}} - (\mu_n - \frac{\sigma_n^2}{2})(t_D - t)}{\sigma_n\sqrt{t_D - t}}\right)\right)}.$$

3.3 The Basel I approach on the portfolio level

In the previous section we regulatory capital on the contract level. Now using the same approach we will present an extension on the portfolio level, where the portfolio means the set of contracts.

According to the model, the formula describing the regulatory capital remains the same. Further, we accept that the expected loss at time t on portfolio level is equal to the sum of expected losses of single contracts at time t.

$$\mathbb{E}L_t^p = \sum_{n=1}^{N} \mathbb{E}L_{n,t}, \tag{3.11}$$

where N is the number of contracts in portfolio p.
Unfortunately, with the unexpected losses we can not do this and there are mathematical and economical reasons for this.
The first reason, results directly from non-linearity of the variance.

$$Var\left(\sum_{n=1}^{N} X_n\right) = \sum_{i=1}^{N}\sum_{j=1}^{N}\left((Var(X_i)) + Var(X_i)Var(X_j)\rho_{ij} + (Var(X_j))\right). \tag{3.12}$$

And the second is given by the economy.
Namely, in the real world we can notice a dependence between the contracts. The interactions between them are expressed by the so-called *correlation*. There exists a coefficient, denoted usually by ρ_{nm}, which expresses the correlation level between the nth and the mth contract.
The coefficient can take positive values as well as negative ones and the general rule is that the smaller coefficient the greater independence between

contracts. Particularly, it is very important with reference to default and unexpected loss. Because, when one of the contracts, positive correlated defaults, it is very probable that the second one defaults as well. However, in the case of negative correlated assets, default of one of them does not influence on the second asset. Therefore, using the definition of the unexpected losses and (3.12), we can evaluate the unexpected loss for N contracts as follows:

$$\mathbb{U}L_{p,t} = \sqrt{\sum_{n=1}^{N}\sum_{m=1}^{N}(\mathbb{U}L_{n,t} + \rho_{nm}\mathbb{U}L_{n,t}\mathbb{U}L_{m,t} + \mathbb{U}L_{m,t})}, \tag{3.13}$$

where ρ_{nm} is the correlation coefficient for contracts n and m and $t = 0, \ldots, T-1$. Let us make a remark, that in the case of default the correlation coefficient is called the default correlation coefficient.

The correlation coefficient is given by a formula:

$$\rho_{nm} = \frac{Cov(X_n, X_m)}{\sigma_n \sigma_m}, \tag{3.14}$$

where $Cov(X_n, X_m)$ expresses the covariance of random variables X_n and X_m and σ_n, σ_m are standard deviations of random variables X_n and X_m respectively. The same is in this case. The coefficient of the default correlation is given by

$$\rho_{nm} = \frac{Cov(L_{n,t}, L_{m,t})}{\mathbb{U}L_{n,t}\mathbb{U}L_{m,t}}. \tag{3.15}$$

Furthermore,

$$Cov(L_{n,t}, L_{m,t}) = (\mathbb{P}(\mathrm{D}_{n,t}\mathrm{D}_{m,t}) - \mathbb{P}\mathrm{D}_{n,t}\mathbb{P}\mathrm{D}_{m,t}) \cdot LGD_n \cdot EAD_{n,t} \cdot LGD_m \cdot EAD_{m,t}.$$

Hence final formula is as follows

$$\rho_{nm} = \frac{\mathbb{P}(\mathrm{D}_{n,t}\mathrm{D}_{m,t}) - \mathbb{P}\mathrm{D}_{n,t}\mathbb{P}\mathrm{D}_{m,t}}{\sqrt{\mathbb{P}\mathrm{D}_{n,t}(1-\mathbb{P}\mathrm{D}_{n,t})} \cdot \sqrt{\mathbb{P}\mathrm{D}_{m,t}(1-\mathbb{P}\mathrm{D}_{m,t})}}. \tag{3.16}$$

Now, we can notice that everything is given, and we can simply calculate it, except *joint probability of default.* The joint probability of default, denoted by $\mathbb{P}(D_{n,t}D_{m,t})$ in Formula (3.16), expresses the probability that company n and m default simultaneously at time t. In the next sections we will present the method of computing the joint probability, but before we will look at the single probability of default.

3.3.1 Probability of default

To determine the probability of default we will come back to the Merton model, which was presented in Section 3.2.1. According to this model, the default occurs at time t when the firm's asset value is smaller than the value of debts at this time. It means that the asset value of the company and its debt change in time. The changes caused mainly by various market factors, can be expected and unexpected. And accordingly we can consider the expected and the unexpected part of value of assets.

Usually, the expected changes are deterministic. They can occur only in cases when we have some new information about the market. Otherwise, the expected changes do not occur. This has a direct bearing on the expected market value of assets, because if we do not have the expected changes then the expected value of assets remains the same.
A different situation is with the unexpected changes in the risk factor. They occur very often and they cause an uncertainty in the asset value. Because of them, everything that can happen with assets is not predictable.

In addition to the market factors, we consider also the risk coefficients which are reserved separately for each asset. Those coefficients are in charge of level risk of investment in this asset and they enable distinction between more and less risky assets. In comparison to the market risk factors, which show us the general tendency of risk for all assets, they determine the risk level specify for each single asset.
Hence, the chances are that we can express the market asset value of firm n at time t as follows:

$$A_{n,t} = \delta_n + \sum_{k=1}^{K} (\phi_n^k \theta_{k,t}) + \psi_n \varepsilon_{n,t}, \tag{3.17}$$

for $n = 1, \ldots, N$, where n denotes the firm's number, $\delta_{n,t}$ expresses the expected part of market risk factor at time t, $\theta_{k,t}$ expresses the unexpected part of market risk factor at time t, $\varepsilon_{n,t}$ express the firm's risk factor at time t. Because the influence of the risk factors on each asset is different, we consider the coefficient of the firm's sensitivity to the risk factor. Hence ϕ_n^k denotes the firm's sensitivity to the kth market risk factor and ψ_n denotes the firm's sensitivity to the firm's risk factor. Moreover we assume that the market risk factors are normally distributed with expectation 0 and covariance matrix Ω and the firm's risk factor has the standard normal distribution.

Usually it is like this, that one portfolio consists of a lot of contracts. Because we have to know the correlation between each two contracts, the

number of computations grows large very fast. This lengthens the calculation time. To simplify calculations, and to make them more efficient we will introduce selection. The idea is to select contracts according to country, industry, etc. It gives us sufficient granularity and enables making our computations easier.
In the aftermath of this, we will introduce structures which we call *the risk buckets*. Each risk bucket includes contracts, which are similar in the sense of some feature. We assume that contracts which are in the same risk bucket have the same expected risk and sensitivity to the unexpected risk. This gives us instead of consideration of each two contracts opportunity to consider risk buckets.

Let G denote the set of the risk buckets. A bucket g will be one of them consisting of some firms. Then in accordance to the above assumptions we can write:

$$A_{n,t} = \delta_g + \sum_{k=1}^{K} (\phi_g^k \theta_{k,t}) + \psi_n \varepsilon_{n,t} \quad n \in g \tag{3.18}$$

Let us remind that the main aim of these considerations is getting the joint probability of default for two companies. To this end, we will keep the assumption that the company defaults when the asset value of company is smaller than the amount of debts. Hence in our case

$$\mathbb{P}(D_{n,t} = 1) = \mathbb{P}(A_{n,t} < B_{n,t}), \tag{3.19}$$

where $A_{n,t}$ is expressed by (3.18), and $B_{n,t}$ denotes the amount of liabilities at time t. Further

$$A_{n,t} < B_{n,t} \Leftrightarrow \delta_g + \sum_{k=1}^{K} (\phi_g^k \theta_{k,t}) + \psi_n \varepsilon_{n,t} < B_{n,t} \Leftrightarrow \sum_{k=1}^{K} (\phi_g^k \theta_{k,t}) + \psi_{n,t} \varepsilon_{n,t} < B_{n,t} - \delta_g$$

According to our above assumption we know that δ_g denotes the expected value of the risk so it is deterministic. Thus, without loss of generality, we can introduce a constant $C_{n,t} = B_{n,t} - \delta_g$ and it automatically gives conclusion that default occurs when

$$\sum_{k=1}^{K} (\phi_g^k \theta_{k,t}) + \psi_n \varepsilon_{n,t} < C_{n,t} \tag{3.20}$$

Determination of the default probability is a rather difficult issue. Due to that, we will define a new random variable denoted by $Z_{g,t}$ and expressed by formula

$$Z_{g,t} = \sum_{k=1}^{K} (\phi_g^k \theta_{k,t}), \tag{3.21}$$

where $t = 0, \ldots, T-1$. Let us notice that this notation is very efficient for us. First, $Z_{g,t}$ expresses the market risk factor for bucket g and is the same for all contracts in the bucket g. Second, $Z_{g,t}$ is the sum of standard normally distributed random variables, so it has a normal distribution with expectation 0 and a variance given by a formula:

$$Var(Z_{g,t}) = Var(\sum_{k=1}^{K}(\phi_g^k \theta_{k,t})) = \sum_{k=1}^{K}\sum_{k'=1}^{K'}(Var(\phi_g^k \theta_{k,t}) + \sigma_{kk'} Var(\phi_g^k \theta_{k,t}) \cdot$$

$$\cdot Var(\phi_g^{k'} \theta_{k',t}) + Var(\phi_g^{k'} \theta_{k',t})) = \sum_{k=1}^{K}\sum_{k'=1}^{K'}((\phi_g^k)^2 Var(\theta_{k,t}) + \sigma_{kk'} \phi_g^k \phi_g^{k'} \cdot$$

$$\cdot Var(\theta_{k,t}) Var(\theta_{k',t}) + (\phi_g^{k'})^2 Var(\theta_{k',t})).$$

Shortly,

$$Z_{g,t} \sim N\left(0, \sqrt{Var(Z_{g,t})}\right). \tag{3.22}$$

We know that dividing normally distributed random variable minus its expectation value by its standard deviation we will result in a random variable which has a standard normal distribution. Hence:

$$\frac{Z_{g,t}}{\sqrt{Var(Z_{g,t})}} \sim N(0,1). \tag{3.23}$$

Let us denote this fraction by $X_{g,t}$.
We can also denote $\sqrt{Var(Z_{g,t})}$ by w_g. Then we will get:

$$Z_{g,t} = w_g X_{g,t} = \sum_{k=1}^{K} \phi_g^k \theta_{k,t},$$

and coming back to formula (3.18):

$$A_{n,t} = \delta_g + w_g X_{g,t} + \psi_n \varepsilon_{n,t}. \tag{3.24}$$

Because of that the probability of default is given by

$$\mathbb{P}(\mathrm{D}_{n,t} = 1) = \mathbb{P}(w_g X_{g,t} + \psi_n \varepsilon_{n,t} < C_{n,t}). \tag{3.25}$$

For facilitation of the notation and computations we will denote $w_g X_{g,t} + \psi_n \varepsilon_{n,t}$ by $u_{n,t}$. Knowing that $X_{g,t} \sim N(0,1)$ we can designate distribution of $u_{n,t}$. Seeing that $u_{n,t}$ is the sum of normally distributed random variables, we can conclude that it is normally distributed as well. Moreover, we assume that the firm risk factor and the market risk factor are independent, what means also that the covariance between them is equal 0. Hence $u_{n,t}$ has a normal distribution with expected value equal to 0 and a variance given by

$$Var(u_{n,t}) = Var(\delta_g + w_g X_{g,t} + \psi_n \varepsilon_{n,t}) = w_g^2 Var(X_{g,t}) + \psi_n^2 Var(\varepsilon_{n,t}) = w_g^2 + \psi_n^2,$$

and finally

$$u_{n,t} \sim N\left(0, \sqrt{w_g^2 + \psi_n^2}\right) \tag{3.26}$$

where $u_{n,t}$ is the random variable expressing the market risk and the firm risk.

Using basic knowledge from probability theory and particularly from properties of the normal distribution we will write

$$\phi(u_{n,t}) = \frac{1}{\sqrt{2\pi(w_g^2 + \psi_n^2)}} \exp\left(-\frac{1}{2}\frac{u_{n,t}^2}{(w_g^2 + \psi_n^2)}\right), \tag{3.27}$$

$$\Phi(u_{n,t}) = \int_{-\infty}^{C_{n,t}} \frac{1}{\sqrt{2\pi(w_g^2 + \psi_n^2)}} \exp\left(-\frac{1}{2}\frac{u_{n,t}^2}{(w_g^2 + \psi_n^2)}\right) \tag{3.28}$$

and in the next section we will use these formulas for further considerations.

3.3.2 Default threshold

In the previous section, we considered the market asset value of a firm. We expressed this as a sum of corresponding risk factors and we obtained the distribution of this random variable. This means that we are ready to consider the joint probability of default for two companies.

We know already that the probability of default for one company is expressed by formula:

$$\mathbb{P}(\mathrm{D}_{n,t} = 1) = \mathbb{P}(A_{n,t} < B_{n,t}) = \mathbb{P}(w_g X_{g,t} + \psi_n \varepsilon_{n,t} < C_{n,t}) = \mathbb{P}(u_{n,t} < C_{n,t}),$$

where $u_{n,t}$, $t = 0, \ldots, T-1$ has a normal distribution. Knowledge which we have so far gives us the opportunity to say something about $C_{n,t}$ as well. Let us notice that till now this value was unknown. Unfortunately, we can not give an exact value of $C_{n,t}$, but we can express the threshold value by a different variable which is already known. Thanks to the normal distribution of $u_{n,t}$ and standardization of the random variable, we get that

$$\frac{u_{n,t}}{\sqrt{w_g^2 + \psi_n^2}} \sim N(0, 1),$$

and directly

$$\mathbb{P}(u_{n,t} < C_{n,t}) = \mathbb{P}\left(\frac{u_{n,t}}{\sqrt{w_g^2+\psi_n^2}} < \frac{C_{n,t}}{\sqrt{w_g^2+\psi_n^2}}\right) = \Phi\left(\frac{C_{n,t}}{\sqrt{w_g^2+\psi_n^2}}\right),$$

where Φ is the cumulative function of the standard normal distribution.

Existence of an inverse cumulative function to a cumulative function of a continuous distribution guarantees us opportunity to consider inversion of cumulative function of standard normal distribution. Therefore we can write that

$$C_{n,t} = \Phi^{-1}(P\mathrm{D}_{n,t})\sqrt{w_g^2+\psi_n^2},$$

and this is an expression of the default threshold which will be using from now. It gives us directly a new default probability formula

$$\mathbb{P}(\mathrm{D}_{n,t}=1) = \mathbb{P}\left(u_{n,t} < \Phi^{-1}(P\mathrm{D}_{n,t})\sqrt{w_g^2+\psi_n^2}\right).$$

In the future, the above two formulas will be a basis to construct joint default probabilities.

3.3.3 Joint probability of default

In the beginning of this section let make a remark about the probability of default for two firms. We know that every company is going to default with certain probability and it is obvious that probability can be the same for both companies or totally different for each of them.

If probability is the same for companies then situation is very easy, because the joint probability is equal probability of default for one company. We will consider the second situation, when the probabilities are different. For us this situation is more interesting, because situation when companies are going to default with the same probability are rather rare on the real market.

Let us take company n and m with probability of default at time $t = 0,\ldots,T-1$ $\mathbb{P}\mathrm{D}_{n,t}$ and $\mathbb{P}\mathrm{D}_{m,t}$ respectively. Moreover for simplicity computations we will keep the assumption that these two firms default on their contract in the same time interval.

In the previous section we showed that probability of default for one company is equal probability of event that D_i is equal 1 and is expressed by formula

$$\mathbb{P}\mathrm{D}_{n,t} = \mathbb{P}(\mathrm{D}_{n,t}=1) = \mathbb{P}\left(u_{n,t} < \Phi^{-1}(\mathbb{P}\mathrm{D}_{n,t})\sqrt{w_{g_1}^2+\psi_n^2}\right),$$

where g_1 denotes the bucket which contains nth company. The same we can write for company m

$$\mathbb{P}\mathrm{D}_{m,t} = \mathbb{P}(\mathrm{D}_{m,t} = 1) = \mathbb{P}\left(u_{m,t} < \Phi^{-1}(\mathbb{P}\mathrm{D}_{m,t})\sqrt{w_{g_2}^2 + \psi_m^2}\right)$$

where g_2 denotes the bucket which contains mth company. This approach lets us consider joint probability of default as probability of event that $\mathrm{D}_{n,t}$ is equal 1 and $\mathrm{D}_{m,t}$ as well. According to this we have:

$$\mathbb{P}(\mathrm{D}_{n,t}\mathrm{D}_{m,t}) = \mathbb{P}(\mathrm{D}_{n,t} = 1, \mathrm{D}_{m,t} = 1) =$$

$$= \mathbb{P}\left(u_{n,t} < \Phi^{-1}(\mathbb{P}\mathrm{D}_{n,t})\sqrt{w_{g_1}^2 + \psi_n^2}, u_{m,t} < \Phi^{-1}(\mathbb{P}\mathrm{D}_{m,t})\sqrt{w_{g_2}^2 + \psi_{m,t}^2}\right)$$

Now consider two random variable X_1 and X_2 - both of them normal distributed with the expected values μ_1, μ_2 and the standard deviations σ_1, σ_2 respectively. We know that in general case the density function of bivariate normal distribution is expressed by formula

$$f(x_1, x_2) = \frac{1}{2\pi}|\Sigma|^{-\frac{1}{2}} \exp\left(-\frac{1}{2}(X^T\Sigma^{-1}X)\right)$$

where

$$X = \begin{bmatrix} x_1 - \mu_1 \\ x_2 - \mu_2 \end{bmatrix}$$

and covariance matrix Σ as follow

$$\Sigma = \begin{bmatrix} \sigma_1^2 & \sigma_1\sigma_2\rho_{12} \\ \sigma_1\sigma_2\rho_{12} & \sigma_2^2 \end{bmatrix}$$

Try to use this to solve our problem.

Generally it is not true that if we have two random variables with the normal distribution then the distribution of random vector is normal distributed as well. But in this case it will be like this and it is a result of follow lemma:

Lemma 3.1 *The vector $X = (X_1, X_2, \ldots, X_k)$ is $N_k(\mu, \Sigma)$-distributed if and only if a^TX is $N_1(a^T\mu, a^T\Sigma a)$-distributed for every $a \in \mathbb{R}^k$.*

We know that random variables $u_{n,t}$ and $u_{m,t}$ are dependent (because market risk factors X_{g_1} and X_{g_2} are dependent). But on the other hand they are linear combination of random variables, which are independent. And this guarantees fact that every time we can find the vector a, which satisfies the

condition of above lemma.
Thus, the vector $(D_{n,t}, D_{m,t})$ has bivariate normal distribution.

We see that applying above formula requires the computation of some values. We know already that $u_{n,t}$ and $u_{m,t}$ are standard normal distributed with expected values 0 and variations $w_{g_1}^2 + \psi_n^2$ and $w_{g_2}^2 + \psi_m^2$ respectively. It gives us directly a vector

$$U = \begin{bmatrix} u_{n,t} \\ u_{m,t} \end{bmatrix}$$

To get a covariance matrix we need variation of random variables and covariation. The variation we already have, so look at covariance.

The covariance represents the co-movement of the two variables and is defined as

$$Cov(u_{n,t}, u_{m,t}) = \mathbb{E}(u_{n,t} - \mathbb{E}(u_{n,t}))(u_{m,t} - \mathbb{E}(u_{m,t}))$$

The expectation values of variables equal 0 and independence between corresponding risk factors lead us to

$$Cov(u_{n,t}, u_{m,t}) = \mathbb{E}(u_{n,t}u_{m,t}) = \mathbb{E}((w_{g_1}X_{g_1,t}+\varepsilon_{n,t}\psi_n)(w_{g_2}X_{g_2,t}+\varepsilon_{n,t}\psi_n)) = w_{g_1}w_{g_2}\mathbb{E}(X_{g_1,t}X_{g_2,t})$$

As we assumed in the beginning of our consideration the firm risk factors are not related each other and they are not related with market risk factors as well. Hence, the correlation coefficient, which expressed relationship between them will depend only on market risk factors and will be defined by $\rho_{g_1g_2} = \mathbb{E}(X_{g_1,t}X_{g_2,t})$. So finally the covariance matrix is as follow

$$\Sigma = \begin{bmatrix} w_{g_1}^2 + \psi_n^2 & w_{g_1}w_{g_2}\rho_{g_1g_2} \\ w_{g_1}w_{g_2}\rho_{g_1g_2} & w_{g_2}^2 + \psi_m^2 \end{bmatrix}$$

As we see we got a 'values' which characterizes bivariate normal distribution. Now we make some computation which let us to get final formula of density function. Using mostly properties of matrix in the computations we will get

$$|\Sigma| = (w_{g_1}^2 + \psi_n^2)(w_{g_2}^2 + \psi_m^2) - w_{g_1}^2 w_{g_2}^2 \rho_{g_1g_2}^2$$

where $|\Sigma|$ denotes determinant of the covariance matrix Σ.

$$|\Sigma|^{-\frac{1}{2}} = \frac{1}{\sqrt{(w_{g_1}^2 + \psi_n^2)(w_{g_2}^2 + \psi_m^2) - w_{g_1}^2 w_{g_2}^2 \rho_{g_1g_2}^2}}$$

$$\Sigma^{-1} = \begin{bmatrix} \frac{w_{g_2}^2+\psi_m^2}{\sqrt{(w_{g_1}^2+\psi_n^2)(w_{g_2}^2+\psi_m^2)-w_{g_1}^2 w_{g_2}^2\rho_{g_1g_2}^2}} & -\frac{w_{g_1}w_{g_2}\rho_{g_1g_2}}{\sqrt{(w_{g_1}^2+\psi_n^2)(w_{g_2}^2+\psi_m^2)-w_{g_1}^2 w_{g_2}^2\rho_{g_1g_2}^2}} \\ -\frac{w_{g_1}w_{g_2}\rho_{g_1g_2}}{\sqrt{(w_{g_1}^2+\psi_n^2)(w_{g_2}^2+\psi_m^2)-w_{g_1}^2 w_{g_2}^2\rho_{g_1g_2}^2}} & \frac{w_{g_1}^2+\psi_n^2}{\sqrt{(w_{g_1}^2+\psi_n^2)(w_{g_2}^2+\psi_m^2)-w_{g_1}^2 w_{g_2}^2\rho_{g_1g_2}^2}} \end{bmatrix}$$

Getting a nice formula of the density function is not easily solvable problem in this case. In theory it should be possible by deriving an integral of this function, but in practice it will be very difficult and not necessary. Presently, a lot of corresponding programs is available (for instance SPSS, Matlab, etc), which enable getting the results without manual computations.

Chapter 4

The Basel II Capital Accord

4.1 Introduction

In June 2004 the Basel Committee introduced a New Accord called Basel II. Similarly to the Basel I accord, the Basel II settles regulations concerning banking. The document includes methods of measuring risk and calculating capital requirements. However, the Basel Committee does not force banks to use exactly these methods. Contrary to Basel I, the New Accord gives banks freedom of choice. They can adapt analytical methods of computing the amount of required capital and the advancement level of it to their own needs. Hence, the banks have an opportunity to reduce their economic capital and regulatory capital through efficient data management and reporting.

The Basel II is based on a "three pillars" concept:

- minimum capital requirements – introduces methods and rules of computing required capital,
- supervisory review – determines rights and duties of banking supervisors,
- market discipline – sets the rules of reporting the information concerning risk taken by the bank.

As we mentioned in the previous chapters, the required capital is strictly related to risk, in particular, to credit risk. With respect to the risk, Basel II extended old methods presented by the Basel I, and introduced new one. Hence, according to the Basel Committee directives, banks can apply two approaches to calculate credit risk: the standardized approach and the *Internal Rating Based* approach (IRB).

The first one, based largely on the current accord, is its slight modification, what still means that capital requirements are equal 8% of the outstanding amount of money. Banks use this method because application of methods of determine the $\mathbb{P}D's$ based on ratings is rather impossible in many countries. The reason is that only few borrowers possess ratings which are convenient for local markets and give banks favorable risk weights.

The Internal Rating Based approach gives banks more possibilities. In general, banks are ought to calculate borrowers' probability of default using internal measures. In particular, banks can also estimate the loss given default and the exposure at default using theirs own methods. Then, we speak about the *Advanced Internal Ratings Based* model. Anyway, both approaches lead banks to the same – to get an estimate of the capital requirements, with the exception that the Advanced IRB is more adapted to the bank. In this case the LGD and the EAD are based on the historical data, which gives a better estimate. However, in accordance with the basic IRB, the LGD and the EAD are given by Basel Committee. Hence, they are constant and they do not depend on the bank.

According to the Basel II, one of the tool which can be used by the financial institutions to determine the probability of default and the capital requirements is an *Asymptotic Risk Factor Model*. The ASRFM is based on the Merton model and by acceptance of certain assumptions, it gives the estimate of above components.

4.2 The Basel II approach on the contract level

4.2.1 The Asymptotic Risk Factor

The Asymptotic Risk Factor Model gives the opportunity to calculate economic capital requirements using risk weight formulas. Similar to Basel I, we distinguish two risk types: the market risk and the risk which is specified for the company. The idea behind the ASRFM is that the market risk is completely diversified and the portfolio becomes more fine-grained which means that large individual exposures have smaller shares in the exposure of the whole portfolio. Hence, we assume that the bank's portfolio consists of a large number of contracts with small exposure. Moreover, because of total diversification of the market risk, we assume that the economic capital depends only on the contract and the company, not on the portfolio which includes this contract.

For the sake of expressing the market asset value as the sum of normally distributed market risk factor and firm's risk factor multiplied by corresponding sensitivities, the market asset value has also the normal distribution. However, in comparison to the model presented by Basel I, the ASFRM requires something more than only the normal distribution of the market asset value. According to the model it should be a standard normal distributed. Because of that, the model introduced new formula, which is as follows:

$$A_{n,t} = \sqrt{\rho_n} X_t + \sqrt{1-\rho_n}\varepsilon_{n,t}, \tag{4.1}$$

where X_t denotes the market risk factor at time t, $\varepsilon_{n,t}$ denotes the firm's risk factor for company n at time t, and t as previously, means the tth year during the contract time $t = 0, \ldots, T-1$.
Directly, it follows from this equation that the assets of firms n and m are multivariate Gaussian distributed (a similar proof is presented in Chapter 3) and the assets of two firms are correlated, with the linear correlation coefficient

$$\mathbb{E}(A_{n,t}A_{m,t}) = \sqrt{\rho_n \rho_m}.$$

Moreover, the correlation between the asset's return $A_{n,t}$ and the market risk factor X_t is equal to $\sqrt{\rho_n}$, therefore $\sqrt{\rho_n}$ is interpreted as the sensitivity to the systematic risk.

The same as in the Basel I approach, the ASRFM is based on the single asset model of Merton. Hence, the firm's asset value is expressed as the sum of liability and the amount of debt, and the probability of default is equal to the probability that the firm's asset value is less than the amount of debts. We remember from Chapter 3 that

$$D_{n,t} = \begin{cases} 1 \text{ if } A_{n,t} \leq \Phi^{-1}(\mathbb{P}\mathrm{D}_{n,t}), \\ 0 \text{ if } A_{n,t} \geq \Phi^{-1}(\mathbb{P}\mathrm{D}_{n,t}). \end{cases} \tag{4.2}$$

where $\mathrm{D}_{n,t}$ denotes the default at time t and $\mathbb{P}\mathrm{D}_{n,t}$ the probability of default. According to the definition, $\mathbb{P}\mathrm{D}_{n,t}$ is an unconditional probability. It is specify for the customer and based on 'the quality of the customer'. The unconditional probability is used mainly to obtain the loss distribution. In the case of small number of contracts it it possible, however using the unconditional probability and unconditional loss distribution when we have a lot of contracts is not efficient for the sake of difficulty of computations. Therefore, along with the unconditional probability, the ASRFM introduces a conditional probability. This probability is characterized by dependence on the market risk factor. We calculate it knowing the outcome of the systematic risk factor at time t. Further, it will enable determination of expected distribution.

$$
\begin{aligned}
\mathbb{P}(\mathrm{D}_{n,t}=1|X_t=x) &= \mathbb{P}(A_{n,t} \leq \Phi^{-1}(\mathbb{P}\mathrm{D}_{n,t})|X_t=x) \\
&= \mathbb{P}(\sqrt{\rho_n}X_t + \sqrt{1-\rho_n}\varepsilon_{n,t} \leq \Phi^{-1}(\mathbb{P}\mathrm{D}_{n,t})|X_t=x) \\
&= \mathbb{P}\left(\varepsilon_{n,t} \leq \frac{\Phi^{-1}(\mathbb{P}\mathrm{D}_{n,t}) - \sqrt{\rho_n}X_t}{\sqrt{1-\rho_n}}\Big|X_t=x\right) \qquad (4.3) \\
&= \Phi\left(\frac{\Phi^{-1}(\mathbb{P}\mathrm{D}_{n,t}) - \sqrt{\rho_n}x}{\sqrt{1-\rho_n}}\right)
\end{aligned}
$$

Usually this probability is called *The Stress Probability* and expresses the probability of the total loss. Moreover, the conditional probability can be interpreted as assuming various *scenarios* for the economy, determining the probability of a given portfolio loss under each scenario, and then weighting each scenario by its likelihood.

We already know that the economic capital is kept to cover only the unexpected losses. Hence, if we would like to have the probability of the unexpected loss we have to subtract the probability of the expected loss, which is expressed by the unconditional probability of default. Hence:

$$
\mathbb{P}\mathrm{D}_{\text{the unexpected loss}} = \mathbb{P}\mathrm{D}_{\text{the stress}} - \mathbb{P}\mathrm{D}_{\text{the expected loss}}.
$$

Because we know that the unexpected losses are strictly linked with the loss given default and the exposure at default, hence, to hedge the contract with α's certainty the nth bank has to keep at time t the economic capital as follows:

$$
\begin{aligned}
ECAP_{n,t} &= LGD_n \cdot EAD_{n,t} \cdot \mathbb{P}D_{\textit{the unexpected loss}} \\
&= LGD_n \cdot EAD_{n,t} \cdot \left(\Phi\left(\frac{\Phi^{-1}(\mathbb{P}\mathrm{D}_{n,t}) - \sqrt{\rho_n}x}{\sqrt{1-\rho_n}}\right) - \mathbb{P}D_{n,t}\right). \qquad (4.4)
\end{aligned}
$$

This formula holds for all x – realizations of X_t.

4.3 The Basel II approach on the portfolio level

The idea of the Basel II for portfolios, similar to the individual contract is largely based on the Merton model.
Similarly, as in the previous sections, we will consider the portfolio loss and later we will introduce the formula expressing the required capital. We will present the computations separately for two types of the portfolio.

4.3.1 Heterogenous portfolio

A *heterogenous portfolio* includes contracts with all different characteristics for each of them: the exposures $EXP_{n,t}$, the probabilities of default $\mathbb{P}\text{D}_{n,t}$, the asset correlations ρ_n, and the loss given defaults LGD_n.
Assume that we have this kind of portfolio with N contracts. Each of these contracts has its share in the whole portfolio. This means that when a contract goes into default, then its default will trigger the loss proportional to the share. The most convenient tool to express the share of each contract loss in the loss of whole portfolio is using the weight. But what is the smartest way of determining these weights?

We already know that the loss is related to the exposure. The exposure denotes the remaining outstanding amount, hence the greater exposure, the greater loss. Because of that, taking the share of the contract's exposure in the exposure of portfolio as the weights seems to be logical. So let us define the exposure weight at time t as

$$w_{n,t} = \frac{EXP_{n,t}}{\sum_{n=1}^{N} EXP_{n,t}} \qquad n = 1, \ldots, N \quad t = 0, \ldots, T-1 \tag{4.5}$$

Considering the Basel I approach, we said that the capital requirements are expressed by αth percentile of the loss distribution. The same approach is applied in the ASFRM and now we will discuss the details.

Let us take into consideration the heterogenous portfolio with N contracts, with the exposures $EXP_{n,t}$, the asset correlations ρ_n, the probabilities of default $PD_{n,t}$, the losses given default LGD_n and the weights $w_{n,t}$. Then the portfolio loss per monetary unit of exposure (for instance dollars, euros) at time t is given by a formula

$$L_t^p := \sum_{n=1}^{N} EXP_{n,t} \cdot LGD_n \cdot D_{n,t}.$$

If we denote $LGD_{n,t} \cdot D_{n,t}$ as a random variable $Z_{n,t}$ then

$$Z_{n,t} = \begin{cases} 0, \text{ when the default does not occur at time } t, \\ LGD_{n,t}, \text{ otherwise.} \end{cases} \tag{4.6}$$

For the sake of the construction of $Z_{n,t}$ and properties of the LGD, we can assume:

(A.1) *$Z_{n,t}$ to belong to the interval $[0, 1]$ and conditionally on X_t,to be independent for all $n = 1, \ldots, N$.*

With respect to this portfolio we can also consider the *portfolio loss ratio* R_t^p expressing the ratio of the total portfolio loss to total portfolio exposure

$$R_t^p = \frac{L_t^p}{\sum_{n=1}^N EXP_{n,t}} = \frac{\sum_{n=1}^N EXP_{n,t} \cdot LGD_n \cdot D_{n,t}}{\sum_{n=1}^N EXP_{n,t}} = \frac{\sum_{n=1}^N EXP_{n,t} \cdot Z_{n,t}}{\sum_{n=1}^N EXP_{n,t}},$$

for $Z_{n,t}$ given by Formula (4.6). We provide it, because as we will see further, the ASFRM proposes to determine a distribution of the portfolio loss ratio, instead of the loss distribution.

In accordance with the ASFRM, we consider the portfolio only with contracts characterized by small amount of exposure. However, the sequence of exposure amounts should not converge to zero too quickly. Because of that, we will pose some restrictions concerning the exposure. This is necessary to guarantee that the market risk will be completely diversified. Thus, we assume:

(A.2) *$EXP_{n,t}$ forms a sequence of positive numbers such that $\sum_{n=1}^N EXP_{n,t} \uparrow \infty$ for $N \to \infty$ and for $\tau > 0$*

$$\frac{EAD_{N,t}}{\sum_{n=1}^N EXP_{n,t}} = O\left(N^{-(\frac{1}{2}+\tau)}\right). \quad (4.7)$$

Assumptions (A.1) and (A.2) are weak; they hold in the real world and for us are very important. Due to these assumptions, we are sure that the share of the greatest exposures shrinks to 0, as the number of exposures in the portfolio increases. In that case, it can be shown that

Proposition 4.1 *If we assume (A.1) and (A.2), then for $N \to \infty$*

$$R_t^p - \mathbb{E}[R_t^p | X_t = x] \to 0, \text{ almost surely.} \quad (4.8)$$

Proof: To prove Proposition 4.1 we will use the special version of the *Strong Law of Large Numbers* presented in the *"Oxford Studies in Probability"* by Valentin V. Petrov in 1995 (Theorem 6.7):

Theorem 4.1 *If $a_N \uparrow \infty$ and $\sum_{N=1}^{\infty} \frac{Var(Y_N)}{a_N^2} < \infty$ then*

$$\frac{1}{a_N}\left(\sum_{n=1}^N Y_n - \mathbb{E}\left(\sum_{n=1}^N Y_n\right)\right) \to 0, \text{ almost surely.} \quad (4.9)$$

Let $Y_{n,t} \equiv Z_{n,t} \cdot EXP_{n,t}$ and $a_{N,t} \equiv \sum_{n=1}^{N} EXP_{n,t}$ for $t = 0, \dots, T-1$ and $n = 1, \dots, N$. Hence:

$$\sum_{N=1}^{\infty} \frac{Var(Y_{N,t})}{a_{N,t}^2} = \sum_{N=1}^{\infty} \frac{Var(Z_{N,t} \cdot EXP_{N,t})}{\left(\sum_{n=1}^{N} EXP_{n,t}\right)^2} \tag{4.10}$$

Because $EXP_{n,t}$ at time t is given for each contract n, we have:

$$\sum_{N=1}^{\infty} \frac{Var(Y_{N,t})}{a_{N,t}^2} = \sum_{N=1}^{\infty} \left(\frac{EXP_{N,t}}{\sum_{n=1}^{N} EXP_{n,t}}\right)^2 \cdot Var(Z_{N,t}). \tag{4.11}$$

For all realization of X_t x, a conditional independence implies that

$$\sum_{N=1}^{\infty} \frac{Var(Y_{N,t}|X_t = x)}{a_{N,t}^2} = \sum_{N=1}^{\infty} \left(\frac{EXP_{N,t}}{\sum_{n=1}^{N} EXP_{n,t}}\right)^2 \cdot Var(Z_{N,t}|X_t = x).$$

To apply Theorem (4.1) we have to show that this sum is finite. In accordance to assumption about $Z_{n,t}$, we know that for $t = 0, \dots, T-1$ and $n = 1, \dots, N$ it belongs to $[0, 1]$. Thus, $Var(Z_{N,t}|X_t = x)$ is finite for any $X_t = x$. In that case, for $t = 0, \dots, T-1$ there exists the constant M_t such that

$$Var(Z_{N,t}|X_t = x) < M_t.$$

Moreover, using Assumption (A.2) we have that

$$\frac{EXP_{N,t}}{\sum_{n=1}^{N} EXP_{n,t}} = O\left(N^{-(\frac{1}{2}+\tau)}\right) \text{ for } \tau > 0,$$

and directly

$$\left(\frac{EXP_{N,t}}{\sum_{n=1}^{N} EXP_{n,t}}\right)^2 = \left(O\left(N^{(-\frac{1}{2}+\tau)}\right)\right)^2 = \left(O(1) \cdot N^{(-\frac{1}{2}+\tau)}\right)^2 =$$

$$= O(1) \cdot N^{(1+2\tau)} = O\left(N^{(1+2\tau)}\right).$$

The lemma presented below and fact that

$$R_t^p = \frac{\sum_{n=1}^{N} Y_n}{a_N}, \text{ for } t = 0, \dots, T-1$$

finishes the explanation why the assumption of theorem holds, what means that $R_t^p - \mathbb{E}[R_t^p|X_t = x] \to 0$ for $t = 0, \dots, T-1$.

Lemma 4.1 *If* $\{b_N\}$ *is a sequence of positive real numbers such that* $\{b_N\}$ *is* $O(N^{-\varsigma})$ *for some* $\varsigma > 1$, *then* $\sum_{N=1}^{\infty} b_N < \infty$.[1]

■

Intuitively, this proposition says that shrinking the exposure sizes of the single contracts cause total diversification of the market risk. In the limit, the loss ratio converges to the function depending on the market risk factor. This is very useful, because we would like to know the distribution of portfolio loss ratio.

According to Proposition (4.1) we can consider the conditional distribution of $\mathbb{E}[R_t^p|X_t = x]$, and then automatically we have the unconditional distribution. Now it is natural, that we would like to know something about the variance. It is reasonable that we may expect getting the variance of R_t^p by computing the variance of $\mathbb{E}[R_t^p|X_t = x]$. But for us it is more important to obtain knowledge about the percentiles of the unconditional distribution, because it expressed the capital requirements. We said that the capital requirements, enabling covering α of the unexpected losses, are expressed by the αth percentile. The ASRFM is using definition of percentile as follows:

$$q_\alpha(Y) = \inf\{y : \mathbb{P}(Y \leq y) \geq \alpha\} \tag{4.12}$$

The first step to show that the α percentile of R_t^p can be turn into the α percentile of $\mathbb{E}[R_t^p|X_t = x]$ for $t = 0, \dots, T$, is to prove a proposition below.

Proposition 4.2 *If (A.1) and (A.2) hold, then for* $t = 0, \dots, T$

$$\lim_{N\to\infty} \Big(Var(R_t^p) - Var(\mathbb{E}[R_t^p|X_t]) \Big) = 0.$$

Proof: Similar like the proof of Proposition (4.1), this proof will be based on Theorem (4.1). Moreover we will use the lemma below:

Lemma 4.2 *Let* $\{b_N\}$ *and* $\{c_N\}$ *be sequences of real numbers such that* $a_N = \sum_{n=1}^{N} b_n \uparrow \infty$ *and* $c_N \to 0$. *Then* $\frac{1}{a_N}\sum_{n=1}^{N} b_n c_n \to 0$.[2]

Let us take $b_{n,t} \equiv EXP_{n,t}$ and $c_{n,t} \equiv \frac{EXP_{n,t}}{\sum_{i=1}^{n} EXP_{i,t}}$ for $t = 0, \dots, T-1$. Then, similar as in the previous proof, Assumption (A.2) gives us that $a_{N,t} \uparrow \infty$ and $c_{N,t} \to 0$ if $N \to \infty$. Hence, according to the lemma

$$\frac{1}{\sum_{n=1}^{N} EXP_{n,t}} \sum_{n=1}^{N} \frac{(EXP_{n,t})^2}{\sum_{i=1}^{n} EXP_{i,t}} \longrightarrow 0. \tag{4.13}$$

[1] Knopp, Konrad, *Infinite Sequences and Series*, New York: Dover Publications, 1956 (a corollary of Theorem 3.5.2)

[2] Petrov, Valentin V., *Limit Theorems of Probability Theory* , n.4 'Oxford Studies in Probability', Oxford University Press (1995), Lemma 6.10

Using the standard property of the conditional variance we get

$$Var(R_t^p) - Var(\mathbb{E}[R_t^p|X_t]) = \mathbb{E}(Var[R_t^p|X_t]) =$$

$$\frac{\sum_{n=1}^{N}(EXP_{n,t})^2 \cdot \mathbb{E}(Var[Z_{n,t}|X_t])}{\left(\sum_{n=1}^{N} EXP_{n,t}\right)^2}$$

Under Assumption (A.1) it exists for $t = 0, \ldots, T-1$ the constant M_t such as

$$\mathbb{E}(Var[Z_{n,t}|X_t]) < M_t$$

and then

$$Var(R_t^p) - Var(\mathbb{E}[R_t^p|X_t]) < M_t \cdot \frac{\sum_{n=1}^{N}(EXP_{n,t})^2}{\left(\sum_{n=1}^{N} EXP_{n,t}\right)^2} =$$

$$= \frac{M_t}{\sum_{n=1}^{N} EXP_{n,t}} \cdot \frac{\sum_{n=1}^{N}(EXP_{n,t})^2}{\sum_{i=1}^{n} EXP_{i,t}} < \frac{M_t}{\sum_{n=1}^{N} EXP_{n,t}} \cdot \frac{\sum_{n=1}^{N}(EXP_{n,t})^2}{\sum_{i=1}^{n} EXP_{i,t}} \longrightarrow 0$$

based on (4.13). ■

Therefore, we see that we can approximate the unconditional distribution of the loss by the conditional distribution. This is very convenient for us because it is easier to get the conditional distribution than the unconditional.

In particular, we will be interested in the approximation of the α percentile from the unconditional distribution by the percentile from the conditional distribution. If this is possible then we can easily get the capital requirements for the heterogenous portfolio.

We will start from the proposition below:

Proposition 4.3 *If assumptions (A.1) and (A.2) hold then for any $\delta > 0$ and*

$$\lim_{N\to\infty} F_N(q_\alpha(\mathbb{E}[R_t^p|X_t]) - \delta) \in [0, \alpha]$$

$$\lim_{N\to\infty} F_N(q_\alpha(\mathbb{E}[R_t^p|X_t]) + \delta) \in [\alpha, 1]$$

F_N denotes the sequence of the cumulative distribution functions of the distribution of the R_t^p. The literal interpretation of this proposition is that the αth percentile of $\mathbb{E}[R_{p,t}|X]$ covers almost whole distribution of the loss.

Proof: Due to the previous proposition and the fact that almost sure convergence implies convergence in probability we have that for all x and $\epsilon > 0$

$$\lim_{N\to\infty} \mathbb{P}(|R_t^p - \mathbb{E}(R_t^p|X_t)| \leq \epsilon \; |X_t = x) = 1. \tag{4.14}$$

Let F_N be a cumulative density function of R_t^p, then (4.14) implies

$$\lim_{N\to\infty} \big(F_N(\mathbb{E}(R_t^p|X_t) + \epsilon\ |X_t = x) - F_N(\mathbb{E}(R_t^p|X_t) - \epsilon\ |X_t = x)\big) = 1.$$

A cumulative density function is bounded in $[0, 1]$, hence we get

$$\lim_{N\to\infty} F_N(\mathbb{E}[R_t^p|X_t] + \epsilon\ |X_t = x) = 1,$$
$$\lim_{N\to\infty} F_N(\mathbb{E}[R_t^p|X_t] - \epsilon\ |X_t = x) = 0.$$

Let S_N^+ denote the set of realizations x of X_t such that $\mathbb{E}[R_t^p|X_t = x]$ is less than or equal to its αth quantile value, i.e.,

$$S_N^+ := \Big\{x : \mathbb{E}[R_t^p|X_t = x] \le q_\alpha(\mathbb{E}[R_t^p|X_t])\Big\}.$$

Now,

$$\mathbb{P}(X_t \in S_N^+) = \mathbb{P}\big(\mathbb{E}[R_t^p|X_t] \le q_\alpha(\mathbb{E}[R_t^p|X_t])\big) \ge \alpha.$$

By the total probability theorem and the above we obtain

$$\begin{aligned} F_N(q_\alpha(\mathbb{E}[R_t^p|X_t]) + \epsilon) &= F_N(q_\alpha(\mathbb{E}[R_t^p|X_t]) + \epsilon\ |X_t \in S_N^+)\mathbb{P}(X_t \in S_N^+) \\ &+ F_n(\alpha_q(E[R_t^p|X_t]) + \epsilon\ |X \notin S_n^+)P(X \notin S_n^+) \\ &\ge F_N(q_\alpha(\mathbb{E}[R_t^p|X_t]) + \epsilon\ |Xt \in S_N^+)\mathbb{P}(X_t \in S_N^+) \\ &\ge F_N(q_\alpha(\mathbb{E}[R_t^p|X_t]) + \epsilon\ |X_t \in S_N^+)\ \alpha. \end{aligned} \tag{4.15}$$

A cumulative distribution is increasing and bounded in $[0, 1]$, hence for all $x \in S_N^+$ the following holds

$$1 \ge \lim_{N\to\infty} F_N(q_\alpha(\mathbb{E}[R_t^p|X_t]) + \epsilon\ |X_t = x) \ge \lim_{N\to\infty} F_N(\mathbb{E}[R_t^p|X_t] + \epsilon\ |X_t = x) = 1$$

and we get

$$\lim_{N\to\infty} F_N(q_\alpha(\mathbb{E}[R_t^p|X_t]) + \epsilon\ |X_t \in S_N^+) = 1$$

Hence:

$$\lim_{N\to\infty} F_N(q_\alpha(\mathbb{E}[R_t^p|X]) + \delta) \in [q, 1].$$

Similar we can define set S_N^- as follows:

$$S_n^- = \Big\{x : \mathbb{E}[R_t^p|X_t = x] \le q_\alpha\big(\mathbb{E}[R_t^p|X_t = x]\big)\Big\}.$$

Applying analogically approach like in the case of S_N^+ we get:

$$\lim_{N\to\infty} F_N(q_\alpha(\mathbb{E}[R_t^p|X]) - \delta) \in [0, q].$$

■

The above proposition has a very important advantage. Due to this, we can substitute the percentile of R_t^p by the percentile of $\mathbb{E}[R_t^p|X_t]$. Moreover, if we assume some additional restrictions, then the percentile of $\mathbb{E}[R_t^p|X_t]$ for all $t = 0, \ldots, T$ is expressed in a simple and desirable form. So, let us assume that:

(A.3) *the market risk factor X_t is one-dimensional for $t = 0, \ldots, T-1$*

(A.4) *there exists an open interval I containing $q_\alpha(X_t)$ and the number of contracts in the portfolio p $N_0 \in \mathbb{R}$ such that for all $N > N_0$:*

- **(a)** $\mathbb{E}[R_t^p|X_t = x]$ *is nondecreasing on* I,
- **(b)** $\inf_{x\in I} \mathbb{E}[R_t^p|X_t = x] \geq \sup_{x\leq \inf I} \mathbb{E}[R_t^p|X_t = x]$,
- **(c)** $\sup_{x\in I} \mathbb{E}[R_t^p|X_t = x] \leq \inf_{x\geq \inf I} \mathbb{E}[R_t^p|X_t = x]$.

To give an explanation: the first assumption guarantees that $q_\alpha(X_t)$ is determined uniquely, the assumptions (b) and (c) are needed to ensure that the neighborhood of q_α is associated with the neighborhood of the percentile of $\mathbb{E}[R_t^p|X_t]$.

Proposition 4.4 *If (A.3) and (A.4) hold, then for $N > N_0$*

$$q_\alpha(\mathbb{E}[R_t^p|X_t]) = \mathbb{E}[R_t^p|\, q_\alpha(X_t)]. \tag{4.16}$$

Proof: To prove the proposition we will use Assumption (A.3). Let $N > N_0$ be fixed.

$$X_t \leq q_\alpha(X_t) \;\longrightarrow\; \mathbb{E}[R_t^p|X_t] \leq \mathbb{E}[R_t^p|q_\alpha(X_t)] \tag{4.17}$$

Hence,

$$\mathbb{P}(\mathbb{E}[R_t^p|X_t] \leq \mathbb{E}[R_t^p|q_\alpha(X_t)]) \geq \mathbb{P}(X_t \leq q_\alpha(X_t)) \geq \alpha.$$

Similarly we can consider reverse implication in Equation (4.17) and then we will get finally

$$\inf\{y : \mathbb{P}(\mathbb{E}[R_t^p|X_t] \leq y) \geq \alpha\} = \mathbb{E}[R_t^p|q_\alpha(X_t)].$$

Note, that the left side of this equation is exactly the definition of the αth percentile so the proposition is proved. ■

To get the final formula we have to consider the continuity. We have to avoid the discontinuity at the percentile, hence, for the certainty we will provide additional constraints:

(A.5) *There is an open interval I containing $q_\alpha(X_t)$ and for this interval the following conditions hold:*

(a) *the cumulative distribution function of the market risk factor X_t is increasing and continuous*

(b) *there exist $\underline{\kappa}, \overline{\kappa} \in \mathbb{R}$ such that*

$$0 < \underline{\kappa} < \mathbb{E}[R_t^p | X_t = x] < \overline{\kappa} < \infty, \qquad \textit{for } N > N_0.$$

Due to this assumptions and previous ones as well we can use proposition:

Proposition 4.5 *If (A.1) to (A.5) hold then for $N \to \infty$*

$$\mathbb{P}(R_t^p \leq \mathbb{E}[R_t^p | q_\alpha(X_t)]) \longrightarrow \alpha, \tag{4.18}$$

$$|q_\alpha(R_t^p) - \mathbb{E}[R_t^p | q_\alpha(X_t)]| \longrightarrow 0.^3 \tag{4.19}$$

Using the above proposition and Formula (4.16) we can write that

$$\lim_{N\to\infty} q_\alpha(R_t^p) = q_\alpha(\mathbb{E}[R_t^p | X_t]).$$

It gives us an opportunity to use the expected distribution of the loss ratio instead of the unexpected distribution in the computations.

Continuing, we can write based on Proposition (4.4) and the definition of the loss fraction that:

$$\lim_{N\to\infty} q_\alpha(R_t^p) = \mathbb{E}[R_t^p | q_\alpha(X_t)] = \frac{\sum_{n=1}^{N} EXP_{n,t} \cdot \mathbb{E}[Z_{n,t} | X_t = \Phi^{-1}(\alpha)]}{\sum_{N}^{n=1} EXP_{n,t}}.$$

Further, since the LGD's are assumed to be known in the advance and deterministic

$$\lim_{N\to\infty} q_\alpha(R_t^p) = \frac{\sum_{n=1}^{N} EXP_{n,t} \cdot LGD_n \cdot \mathbb{P}(D_{n,t} = 1 | X_t = \Phi^{-1}(\alpha))}{\sum_{n=1}^{N} EXP_{n,t}}.$$

Inserting Formulas (4.3) and (4.5) we will get finally

$$\lim_{N\to\infty} q_\alpha(R_t^p) = \sum_{n=1}^{N} w_{n,t} \cdot LGD_n \cdot \Phi\left(\frac{\Phi^{-1}(\mathbb{P}D_{n,t}) - \sqrt{\rho_n}\Phi^{-1}(\alpha)}{\sqrt{1-\rho_n}}\right).$$

[3] the proposition and proof we can find in M.Gordy. A risk-factor foundation for risk-based capital rules. *Journal of Banking and Finance*, 24:119-142,2000

Let us remark that we considered everything provided that the customers depend on the same unique risk factor and any exposure has not the significant share in the portfolio. Hence as the number of the customers in the portfolio $N \to \infty$, the α-percentile of the resulting portfolio loss distribution approaches the asymptotic value

$$q_\alpha(R_{p,t}) = \sum_{n=1}^{N} w_{n,t} \cdot LGD_n \cdot \Phi\left(\frac{\Phi^{-1}(\mathbb{P}D_{n,t}) - \sqrt{\rho_n}\Phi^{-1}(\alpha)}{\sqrt{1-\rho_n}}\right),$$

and similarly like in the approach on the contract level (Formula (4.4)) the economic capital for nth company at time t is expressed by formula

$$ECAP_{n,t} = \sum_{n=1}^{N} w_{n,t} \cdot LGD_n \cdot \left(\Phi\left(\frac{\Phi^{-1}(\mathbb{P}\mathrm{D}_{n,t}) - \sqrt{\rho_n}\Phi^{-1}(\alpha)}{\sqrt{1-\rho_n}}\right) - \mathbb{P}D_{n,t}\right).$$

4.3.2 Homogeneous portfolio

The portfolios of financial institutions have different sizes and the size of portfolio has significant influence on the efficiency of computations. Let us imagine a portfolio with 1 million contracts. Then, the consideration of each contract separately is time-consuming. To increase the efficiency we can apply the same approach as Basel I proposed. We can split the whole portfolio into smaller parts so-called *subportfolios*.

Each subportfolio contains some number of the contracts characterized by the same exposure, the asset correlation, the probability of default, and the loss given default. If we assume that in the range of one portfolio we have S subportfolios, then the loss of each subportfolio is expressed by the formula

$$L_t^s = LGD_s \cdot w_{s,t} \cdot \sum_{n=1}^{N_s} D_{n,t},$$

where s denotes one of the subportfolios, N_s denotes the number of contracts in the sth subportfolio, and

$$D_{n,t} = \begin{cases} 1, & \text{when } A_{n,t} \le \Phi^{-1}(\mathbb{P}D_{n,t}), \\ 0, & \text{otherwise} \end{cases}$$

Because at time $t = 0, \ldots, T-1$ all contracts depend on the same market risk X_t, the total loss of the portfolio is equal to

$$L_t^p = \sum_{s=1}^{S} L_t^s = \sum_{s=1}^{S} LGD_s \cdot w_{s,t} \cdot \sum_{n=1}^{N_s} D_{n,t}.$$

In the case of one subportfolio, $\sum_{n=1}^{N_s} D_{n,t}$ denotes exactly the number of contracts included in this subportfolio which defaulted at time t.

Assume like above that nth customer goes into default at time t with the probability $\mathbb{P}D_{n,t}$ and that we have k defaults for $k \leq N_s$. Then, the probability of having exactly k defaults is the average of the conditional probabilities of k defaults, averaged over the possible realizations of X_t and weighted with the probability density function $\phi(x)$.

$$\mathbb{P}\left(\sum_{n=1}^{N_s} D_{n,t} = k\right) = \int_{-\infty}^{\infty} \mathbb{P}\left[\sum_{n=1}^{N_s} D_{n,t} = k | X_t = x\right] \phi(x)dx. \tag{4.20}$$

We mentioned before that the number of defaults is a binomially distributed random variable, so the conditional probability is expressed as follows:

$$\mathbb{P}\left[\sum_{n=1}^{N_s} D_{n,t} = k | X_t = x\right] =$$

$$= \binom{N_s}{k} (\mathbb{P}[D_{n,t} = 1 | X_t = x])^k (1 - \mathbb{P}[D_{n,t} = 1 | X_t = x])^{N_s - k}$$

We already know that according to the ASFRM model the individual conditional default probability $\mathbb{P}[D_{n,t} = 1 | X_t = x]$ is given by the formula

$$\mathbb{P}\left[D_{n,t} = 1 | X_t = x\right] = \Phi\left(\frac{\Phi^{-1}(\mathbb{P}\mathrm{D}_{n,t}) - \sqrt{\rho_n} x}{\sqrt{1 - \rho_n}}\right)$$

Therefore substituting this into Equation (4.20) yields

$$\mathbb{P}\left(\sum_{n=1}^{N_s} D_{n,t} = k\right) = \int_{-\infty}^{\infty} \binom{N_s}{k} \left(\Phi\left(\frac{\Phi^{-1}(\mathbb{P}\mathrm{D}_{n,t}) - \sqrt{\rho_n} x}{\sqrt{1 - \rho_n}}\right)\right)^k$$

$$\cdot \left(1 - \Phi\left(\frac{\Phi^{-1}(\mathbb{P}\mathrm{D}_{n,t}) - \sqrt{\rho_n} x}{\sqrt{1 - \rho_n}}\right)\right)^{N_s - k} \phi(x)dx,$$

and finally the distribution of the number of defaults is as follows

$$\mathbb{P}\left(\sum_{n=1}^{N_s} D_{n,t} \leq l\right) = \sum_{k=0}^{l} \int_{-\infty}^{\infty} \binom{N_s}{k} \left(\Phi\left(\frac{\Phi^{-1}(\mathbb{P}\mathrm{D}_{n,t}) - \sqrt{\rho_n} x}{\sqrt{1 - \rho_n}}\right)\right)^k$$

$$\cdot \left(1 - \Phi\left(\frac{\Phi^{-1}(\mathbb{P}\mathrm{D}_{n,t}) - \sqrt{\rho_n} x}{\sqrt{1 - \rho_n}}\right)\right)^{N_s - k} \phi(x)dx$$

Above formula and everything which we said till this time works for subportfolios with finite number of contracts. Of course, the subportfolios with infinite number of contracts never occur. However, for completeness of the model we will consider this case as well.

Let us assume that we have a subportfolio with very large N_s. Unfortunately, we are not able to compute exact values, hence we will make the approximation for large subportfolio. Because of that we will consider the fraction of customers who defaulted.

Conditional on the realization x of X_t for $t = 0, \dots, T$ the individual defaults occur independently from each other. Hence, in a very large subportfolio, the Law of Large Numbers ensures that the fraction is equal to the individual default probability:

$$\mathbb{P}\left[\frac{\sum_{n=1}^{N_s} D_{n,t}}{N_s} = \mathbb{P}D_{n,t}\Big| X_t = x\right] = 1. \tag{4.21}$$

Applying the same approach as above we have:

$$\mathbb{P}\left[\frac{\sum_{n=1}^{N_s} D_{n,t}}{N_s} \leq l\right] = \mathbb{E}\left(\mathbb{P}\left[\frac{\sum_{n=1}^{N_s} D_{n,t}}{N_s} \leq l \big| X_t = x\right]\right) =$$

$$= \int_{-\infty}^{\infty} \mathbb{P}\left[\frac{\sum_{n=1}^{N_s} D_{n,t}}{N_s} \leq l \big| X_t = x\right] \phi(x)dx.$$

Knowing (4.21) we can write

$$\begin{aligned}\mathbb{P}\left[\frac{\sum_{n=1}^{N_s} D_{n,t}}{N_s} \leq l\right] &= \int_{-\infty}^{\infty} \mathbb{P}[\mathbb{P}D_{n,t} \leq l | X_t = x]\phi(x)dx \\ &= \int_{-\infty}^{\infty} I_{\{\mathbb{P}D_{n,t} \leq l\}}\phi(x)dx = \int_{-\overline{x}}^{\infty} \phi(x)dx = \Phi(\overline{x}), \quad (4.22)\end{aligned}$$

where $\overline{x}$ is taken such that

$$\mathbb{P}D_{n,t}(-x) = l \text{ for } x = \overline{x},$$

$$\mathbb{P}D_{n,t}(x) \leq l \text{ for } x > \overline{x}.$$

The above formula and the application of the formula of the probability of default given by the ASFRM, enables the computation of the value of $\overline{x}$. Hence:

$$\overline{x} = \frac{\sqrt{1-\rho_n}\Phi^{-1}(l) - \Phi^{-1}(\mathbb{P}D_{n,t})}{\sqrt{\rho_n}}.$$

Inserting to Formula (4.22) we have that:

$$\mathbb{P}\left[\frac{\sum_{n=1}^{N_s} D_{n,t}}{N_s} \leq l\right] = \Phi\left(\frac{\sqrt{1-\rho_n}\Phi^{-1}(l) - \Phi^{-1}(\mathbb{P}D_{n,t})}{\sqrt{\rho_n}}\right)$$

If now the αth percentile of the loss distribution is denoted by l_α, then:

$$\alpha = \Phi\left(\frac{\sqrt{1-\rho_n}\Phi^{-1}(l_\alpha) - \Phi^{-1}(\mathbb{P}D_{n,t})}{\sqrt{\rho_n}}\right),$$

and directly

$$l_\alpha = \Phi\left(\frac{\Phi^{-1}(\mathbb{P}D_{n,t}) + \sqrt{\rho_n}\Phi^{-1}(\alpha)}{\sqrt{1-\rho_n}}\right).$$

Hence if we assume that the number of the customers in each subportfolio $N_1, \ldots, N_S \to \infty$, then the αth percentile of the portfolio loss distribution is given by

$$q_\alpha(L_t^p) = \sum_{n=1}^{N} EXP_{n,t} \cdot LGD_n \cdot \Phi\left(\frac{\Phi^{-1}(\mathbb{P}D_{n,t}) + \sqrt{\rho_n}\Phi^{-1}(\alpha)}{\sqrt{1-\rho_n}}\right),$$

and the economic capital at time t:

$$ECAP_{n,t} = \sum_{n=1}^{N} EXP_{n,t} \cdot LGD_n \cdot \left(\Phi\left(\frac{\Phi^{-1}(\mathbb{P}D_{n,t}) + \sqrt{\rho_n}\Phi^{-1}(\alpha)}{\sqrt{1-\rho_n}}\right) - \mathbb{P}D_{n,t}\right).$$

Chapter 5

Internal Return Capacity – IRC Risk adjusted Return on Capital – RaRoC

The measure which is used in the quantification of risk is the *Risk-adjusted Return on Capital* denoted by RaRoC. According to definition presented by the Basel Committee, the RaRoC is a risk-adjusted profitability measurement and management framework for measuring risk-adjusted financial performance for providing a consistent view of profitability across business.

Development of the RaRoC methodology began in the late 1970s. The first steps in this field were made by a group at Bankers Trust. Their original interest was to measure the risk of the bank's credit portfolio, as well as the number of the bank's depositors and other debt holders to a specified probability of loss. Since then, a number of other large banks have developed the RaRoC with the aim, in most cases, of quantifying the amount of equity capital necessary to support all of their operating activities.

The key principle of the RaRoC is adjusting for risk so that the rate of return reflects the risk on a given facility. Hence, the RaRoC percentage is the risk adjusted return as a percentage of the capital requirements. Generally we can write:

$$RaRoC_{n,t} = \frac{Cash\ Flow_{n,t}}{Capital\ Requirements_{n,t}} = \frac{CF_{n,t}}{ECAP_{n,t}} \tag{5.1}$$

We already know that the Basel Committee provided two main methods of computing the capital requirements. In the accordance to Basel I, the capital requirements are defined as 8% of the outstanding amount of money and according to Basel II, they are expressed by the economic capital (the formulas using in the computations were presented in the previous chapters). In aftermath to this, we can also distinguish next to the RaRoC, measure which was introduced by Basel I and is called *Internal Return Capacity* (IRC). Because of the model specification and its assumption, IRC is less risk-adjusted, but the method of computations remains the same like in the RaRoC case. Hence generally:

$$IRC_{n,t} = \frac{Cash\ Flow_{n,t}}{Capital\ Requirements_{n,t}} = \frac{CF_{n,t}}{RECAP_{n,t}} \tag{5.2}$$

In relation to the Cash Flow, Basel II does not propose any changes in the computation, so the definition from Chapter 2 holds in both cases. As a reminder, by Cash Flow we understand the net income diminished by all costs, and the losses in case of default occurrences. We gave the specification of the above components in the second chapter as well.

The IRC as well as the RaRoC give a information about contracts. Mainly, they are the measures introduced to determine the risk. In that case, the IRC and the RaRoC should give the information about the risk which was related with the realized contract, and which will be taken by bank in connection with the acceptance of new contract. When considering the distinction between the *realized risk* and the *expected risk*, this naturally leads to the *realized* RaRoC and the *expected* RaRoC. In the case of IRC we rather do not distinguish *realized* IRC and *expected* IRC. However, for purposes of this thesis we will be consider both of them.

5.1 The Realized IRC and RaRoC

The *realized* RaRoC and *realized* IRC are the measures reflected the past. More precisely, we use them to describe the contracts from the past. Because these contracts are realized, we know everything about them. We have the knowledge about all parameters, such as the costs, the net income, etc, and about the course of contract. To put it differently, we know whether it defaulted or not. Hence, the computations of the IRC and the RaRoC are mainly based on summing up all discounted components and dividing by sum of the capital requirements over the years.
If we denote the income by I, the costs by C and the loss by L then the RaRoC for ith contract:

$$RaRoC_n = \frac{\sum_{t=1}^{T} \frac{CF_{n,t}}{(1+r_D)^t}}{\sum_{t=0}^{T-1} \frac{ECAP_{n,t}}{(1+r_D)^t}} = \frac{\sum_{t=1}^{T} \left(\frac{I_{n,t}}{(1+r_D)^t} - \frac{C_{n,t-1}}{(1+r_D)^{t-1}} - \frac{L_{n,t-1}}{(1+r_D)^{t-1}} \cdot D_{n,t-1} \right)}{\sum_{t=0}^{T-1} \frac{ECAP_{n,t}}{(1+r_D)^t}},$$

and the IRC for ith contract:

$$IRC_n = \frac{\sum_{t=1}^{T} \frac{CF_{n,t}}{(1+r_D)^t}}{\sum_{t=0}^{T-1} \frac{RECAP_{n,t}}{(1+r_D)^t}} = \frac{\sum_{t=1}^{T} \left(\frac{I_{n,t}}{(1+r_D)^t} - \frac{C_{n,t-1}}{(1+r_D)^{t-1}} - \frac{L_{n,t-1}}{(1+r_D)^{t-1}} \cdot D_{n,t-1} \right)}{\sum_{t=0}^{T-1} \frac{CR_{n,t}}{(1+r_D)^t}},$$

where T denotes the contract time in years and r_D is an *annual discounted rate.*

Let us remark that value of the income, the costs, etc. discounted to time 0 are called a *present value* of the income, the costs, etc.. Usually, the present value is denote by prefix PV. Hence, to simplify the notation we can write:

$$RaRoC_n = \frac{\sum_{t=1}^{T} PV(CF_{n,t})}{\sum_{t=0}^{T-1} PV(CR_{n,t})} \qquad IRC_n = \frac{\sum_{t=1}^{T} PV(CF_{n,t})}{\sum_{t=0}^{T-1} PV(CR_{n,t})}.$$

5.2 The Expected IRC and RaRoC

The state of the financial market changes very often. Because of that it is very difficult to predict the economy and the future of contracts. Despite that, financial companies would like to have some knowledge about a contract before making the decision about acceptance or rejection. As long as the IRC and the RaRoC are used, this knowledge is expressed by the expected IRC and the expected RaRoC.

Distinct from the realized IRC, RaRoC, the expected IRC, RaRoC give us information about future contracts. We do not know parameters which describe the contract. Hence, we try to estimate them in some way. To this end, we compute the IRC and the RaRoC over the years and we get the excepted value of IRC , RaRoC multiplied by probability of default and a *survival probability*. The survival probability for the nth contract in the tth year is defined as $1 - \mathbb{P}D_{n,t}$.
More precisely, we assume a certain number of scenarios for the contract. Because the contract can default in each year within the contract term or finish normally without default, the number of possible scenarios is equal to the contract term plus one. In connection with this, we can provide a *survived probability*, which expresses probability that the customer survives till tth year and then defaults in the tth year, or survives till the end of the contract term. Hence, the survived probability is expressed by formula:

$$(1 - \mathbb{P}D_{n,1}) \cdot (1 - \mathbb{P}D_{n,2}) \cdot \ldots \cdot (1 - \mathbb{P}D_{n,t-1}) \cdot \mathbb{P}D_{n,t}.$$

As long as we would keep the assumption that the probability of default for single contract is the same in each year, we can write:

$$(1 - \mathbb{P}D_n)^{t-1} \cdot \mathbb{P}D_n.$$

We mentioned above that the expected IRC, RaRoC are indeed the expected value of IRC, RaRoC. Hence, summarizing this section we can provide the corresponding formulas, which are as follows:

$$\mathbb{E}(RaRoC_n) = \sum_{t=0}^{T} RaRoC_{n,t} \cdot (1 - \mathbb{P}D_{n,0}) \cdot \ldots \cdot (1 - \mathbb{P}D_{n,t-1}) \cdot \mathbb{P}D_{n,t}.$$

$$\mathbb{E}(IRC_n) = \sum_{t=0}^{T} IRC_{n,t} \cdot (1 - \mathbb{P}D_{n,0}) \cdot \ldots \cdot (1 - \mathbb{P}D_{n,t-1}) \cdot \mathbb{P}D_{n,t}.$$

The formulas presented above enable to consider the IRC and RaRoC on the single contract level, but it also enables computations on the portfolio level. The idea is to take as the nominator the sum of the cash flows over the contracts included in this portfolio. Hence, for the realized IRC, RaRoC as well as for the expected IRC, RaRoC, we aggregate the cash flows of all contracts within each year into total cash flow for the portfolio within each year and apply the same approach as in the case of one contract. To get the total cash flow of the portfolio within the portfolio term, we sum up the present values of the cash flows over years.
The methodology of computing of the capital requirements was presented in details in Chapters 3 and 4. Therefore, we can directly conclude that:

$$RaRoC_p = \frac{\sum_{t=1}^{T} PV(\sum_{n=1}^{N} CF_{n,t})}{\sum_{t=0}^{T-1} PV(ECAP_{p,t})} \qquad IRC_p = \frac{\sum_{t=1}^{T} PV(\sum_{n=1}^{N} CF_{n,t})}{\sum_{t=0}^{T-1} PV(RECAP_{p,t})},$$

where in accordance to our notation p means the portfolio with N contracts.

Chapter 6

Comparison of Model Performance between Basel I and Basel II

In the previous chapters we have presented models, which are directly used to compute the capital requirements, and indirectly to compute the IRC and the RaRoC. However, everything that was presented till now enabled us to get a theoretical overview. We have shown the framework of these models, considered the mathematical and economical basis, and detailed the assumptions that need to hold. Now, we will summarize the previous chapters by comparing the models and support what we said by studying an example.

For the example, we will choose four companies with different ratings: AAA, AA+, AA, and A. On account of the different ratings, these companies are characterized by different confidence levels and cost of funds. We said before that the higher rating, the greater the confidence level and the smaller the cost of funds. Moreover, we consider the situation that each company accepts the same contract. Hence, the conditions of this contract presented in the table below, are the same for each company.

Company	**AAA**	**AA+**	**AA**	**A**
Confidence Level	99,99%	98,08%	97,29%	92,51%
Net Principal Amount	1000	1000	1000	1000
Active Rate	5%	5%	5%	5%
Costs of Funds	2%	2,5%	3%	3,5%
Costs	0,7%	0,7%	0,7%	0,7%
LGD	20%	20%	20%	20%
Discounted Rate	4%	4%	4%	4%
Term	4 years	4 years	4 years	4 years

Further, for each company we consider the variable PD to be in the range $(0\%, 25\%)$.

According to definitions, the IRC and the RaRoC reflect the risk level. They are generally determined as the fraction of the cash flow and the capital requirements. Let us emphasize, that the IRC is less-risk adjusted.
We said that both accords, the Basel I and the Basel II accord, provide the same method of calculating the cash flow. Thus, it is obvious that as long as the input data are the same, the results are the same for all companies as well.

Therefore, the difference between the IRC and the RaRoC is the aftermath of the differences between the methods of computing the capital requirements.

The models, which are presented in this thesis, are commonly used to derive the capital requirements. For Basel I as well as Basel II, we showed how to compute the amount of the required capital on the contract level and on the portfolio level as well. The approach for both accords is mainly the same. The models consider the default and the loss as the random variables, determine the loss distribution and express the capital requirements as the αth percentile of this distribution. However, they do this in a different way, in particular, on the portfolio level.

Both models distinguish two types of risk: the market risk and the firm-specific risk, and they express the market asset value as the weighted average of the market risk factor and the firm's risk factor. By the assumption that these factors are normally distributed, it is guaranteed that the market asset value is normally distributed. But the ASRFM (provided by Basel II) requires, additionally, that the market asset value should have standard normal distribution.
Next, they model the probability of default based on the Merton model, which assumes that the default occurs when the market asset value is smaller than the total debts. However, in the further computations, the model provided by Basel I uses the unconditional probability, whereas the ASFRM uses the conditional probability. Remember that the conditional probability is computed provided that we know the value of the market risk factor. In other words, when we know the situation of economy.

We said repeatedly that the capital requirements are kept to cover the unexpected loss. Depending on the ratings of the financial institutions, companies are obliged to hedge the contract to some extent, determined by the confidence level. The general rule is that the lower rating, the smaller the confidence level (but this is observed only in the Basel II accord). According to the ASFRM model, the capital requirements depend on the quality of the financial institution, and this is reflected in the ECAP's formula.

In comparison to Basel II, Basel I presents a poor assessment of the risk incurred. Due to lack of granularity within risk levels, the same contracts from different customers are treated the same. Hence, the model makes the capital requirements on the contract level conditional only on the contract's condition. On the portfolio level, Basel I continues with the same approach. Hence, on the portfolio level as well as on the contract level, Basel I requires hedging the contract with 8% of the outstanding amount of money.

Distinct from Basel I, Basel II proposed some changes on the portfolio level. First of all, it distinguishes between heterogenous and homogeneous portfolios. The methods of computations are adjusted to these two types of portfolios. Moreover, instead of the loss distribution, for the heterogenous portfolio it considers the portfolio loss fraction distribution. However, for the homogeneous portfolio the loss distribution is defined as the Bernoulli distribution with k successes (k contracts went to default) with the conditional probability of default. The interpretation of the ECAP's formula remains without changes.

In relation to the IRC and the RaRoC, Basel I and Basel II keep the same methods of computation. Thus, the difference is caused only by the difference in calculating the capital requirements.

Summarizing, we can conclude that the Basel Committee rightly made a decision about substituting the Basel I accord with the Basel II accord. First of all, Basel I almost did not take the risk into account, even though it is very substantial. Second, no distinction was made between the customers, despite the fact that their risk profiles were significantly different. Besides, risk was viewed on the contract level and not on the portfolio level. The risk-reducing effects were only taken very marginally into account. Third, with reference to the portfolio, Basel I did not take the correlation between the assets into consideration. Due to all these shortcomings, the results given by the Basel I model are not fully correct. Mainly, an accuracy of estimation depends on the single contract and its characteristics.

In comparison to Basel I, Basel II presents a slightly different approach. It takes into account everything that was omitted in the Basel I Accord. Due to that, the ASFRM and all models recommended by Basel II give better estimate. It is very important for correctness of model, that the accuracy of the estimation depends on the model, not only on the single contract.
By more possibilities for banks, they can adopt models to their needs. It guarantees better results. Looking at that from the economic point of view, it is more profitable for banks. Banks can save more money, but from the other side they keep enough to cover unexpected losses.

Chapter 7

Conclusions

The Basel I and the Basel II are international agreements regulating general rules in the banking. In this thesis we have compared both of these documents, more precisely, the models which were or are commonly used. Generally, these models give us the opportunity to compute the IRC and the RaRoC, which are the tools to verify the decision about acceptation or rejection the contract.

Models were built up from the most simple case provided by the Basel I to the complex case of a heterogenous portfolio with different exposure sizes. The last one was consistent with the Basel II convention and recommended by the Basel Committee. As we said, the Basel II does not require from banks using the same model. The banks can choose the model, which is the most adjusted to their needs.

For the sake of taken assumptions these models are not the most general ones. Namely, all models are one-factor models, what means that the market risk factor, which is taken into account is only one. However, it is also possible to consider the multi-factor models. The idea behind them is to replace the random variable expressing the market risk factor by the random vector. Then the dimension of the vector is equal to the number of the market risk factors, and they are the coordinates of this vector. This idea has very important advantage - it enables the application of the slightly modified approach applied in the one-factor model. Therefore, we consider the default and the loss as random variables, determine the loss distribution and express the capital requirements as the α percentile of this distribution. The method of computing the IRC and the RaRoC remains the same as well.

Due to comparison of the Basel I and the Basel II accord we could draw the conclusions. Among them, the most important conclusion says that indeed the Basel II presents the better approach. But even though it does not make Basel II perfect. Basel II is better than Basel I, but still it bares several disadvantages. The assumptions taken by the models are more restrictive, and because of that the models are less general. Hence, it will be developing by Basel Committee. But as long as correctness of Basel II is sufficient for all financial institutions, it will be commonly using by them.

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