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Braid Floer homology ☆

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Abstract

Area-preserving diffeomorphisms of a 2-disc can be regarded as time-1 maps of (non-autonomous) Hamiltonian flows on $\mathbb{R}/\mathbb{Z} \times \mathbb{D}^2$. The periodic flow-lines define braid (conjugacy) classes, up to full twists. We examine the dynamics relative to such braid classes and define a new invariant for such classes, the BRAID FLOER HOMOLOGY. This refinement of Floer homology, originally used for the Arnol'd Conjecture, yields a Morse-type forcing theory for periodic points of area-preserving diffeomorphisms of the 2-disc based on braiding.

Contributions of this paper include (1) a monotonicity lemma for the behavior of the nonlinear Cauchy–Riemann equations with respect to algebraic lengths of braids; (2) establishment of the topological invariance of the resulting braid Floer homology; (3) a shift theorem describing the effect of twisting braids in terms of shifting the braid Floer homology; (4) computation of examples; and (5) a forcing theorem for the dynamics of Hamiltonian disc maps based on braid Floer homology.

MSC: 37B30; 57R58; 37J05

Keywords: Floer homology; Braid; Symplectomorphism; Hamiltonian dynamics

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1. Motivation

The interplay between dynamical systems and algebraic topology is traceable from the earliest days of the qualitative theory: it is no coincidence that Poincaré's investigations of invariant manifolds and (what we now know as) homology were roughly coincident. Morse theory, in particular, provides a nearly perfect mirror in which qualitative dynamics and algebraic topology reflect each other. Said by Smale to be the most significant single contribution to mathematics by an American mathematician, Morse theory gives a relationship between the dynamics of a gradient flow on a space X and the topology of this space. This relationship is often expressed as a homology theory [27,36]. One counts (nondegenerate) zeroes of $-\nabla f$ on a closed manifold M (with $\mathbb{Z}_2$ coefficients), grades them according to the dimension of the associated unstable manifold, then constructs a boundary operator based on counting heteroclinic connections. Careful but straightforward analysis shows that this boundary operator yields a chain complex whose corresponding (Morse) homology $\mathrm{HM}_*(f)$ is isomorphic to $H_*(M;\mathbb{Z}_2)$, the (singular, mod-2) homology of M, a topological invariant.

Morse's original work established the finite-dimensional theory and pushed the tools to apply to the gradient flow of the energy function on the loop space of a Riemannian manifold, thus using closed geodesics as the basic objects to be counted. The problem of extending Morse theory to a fully infinite-dimensional setting with a strongly indefinite functional remained open until, inspired by the work of Conley and Zehnder on the Arnol'd Conjecture, Floer established the theory that now bears his name.

Floer homology considers a formal gradient flow and studies its set of bounded flowlines. Floer's initial work studied the elliptic nonlinear Cauchy–Riemann equations, which occur as a formal L^2-gradient flow of a (strongly indefinite) Hamiltonian action. The key idea is that no locally defined flow is needed: generically, the space of bounded flow-lines has the structure of an invariant set of a gradient flow. As in the construction of Morse homology one builds a complex by grading the critical points via the Fredholm index and constructs a boundary operator by counting heteroclinic flowlines between points with difference one in index. The homology of this complex — Floer homology HF_* — satisfies a continuation principle and remains unchanged under suitable (large) perturbations. Floer homology and its descendants have found use in the solution of the Arnol'd Conjecture [13], in instanton homology [14], elliptic systems [5], heat flows [32], strongly indefinite functionals on Hilbert spaces [1], contact topology and symplectic field theory [11], symplectic homology [12,28,37], and invariants of knots, links, and 3-manifolds [29].

The disconnect between practitioners of Floer theory and applied mathematicians is substantial, in large part due to the lack of algorithms for computing what is in every respect a truly infinite-dimensional construction. We suspect and are convinced that better insights into the computability of Floer homology will be advantageous for its applicability. It is that long-term goal that motivates this paper.

The intent of this paper is to use Floer homology to define a new invariant for braids and use the invariant to study the dynamics of time-dependent Hamiltonians on a 2-disc. We build a relative homology, for purposes of creating a dynamical forcing theory in the spirit of the Sharkovski theorem for 1-dimensional maps or Nielsen theory for 2-dimensional homeomorphisms [23]. Given one or more periodic orbits of a time-periodic Hamiltonian system on a disc, which other periodic orbits are forced to exist? Our answer, in the form of a Floer homology, is independent of the Hamiltonian. We define the Floer theory, demonstrate topological invariance, and connect the theory to that of braids.

Any attempt to establish Floer theory as a tool for applied dynamical systems must address issues of computability. This paper serves as a foundation for what we predict will be a computational Floer theory — a highly desirable but challenging goal. By combining the results of this paper with a spatially-discretized braid index from [18], we hope to demonstrate and implement algorithms for the computation of braid Floer homology. We are encouraged by the potential use of the Garside normal form to this end: Section 14.1 outlines this program. The goal in this paper is to present the construction of Floer homology in a more or less self-contained manner and based on elementary principles valid in the simple case of the 2-disc.

2. Statement of results

2.1. Background and notation

Recall that a smooth orientable 2-manifold M with area form ω is an example of a symplectic manifold, and an area-preserving diffeomorphism between two such surfaces is an example of a symplectomorphism. Symplectomorphisms of (M,ω) form the group $\mathrm{Symp}(M,\omega)$ with respect to composition. The standard unit 2-disc in the plane $\mathbb{D}^2 = \{x \in \mathbb{R}^2 \mid |x| \le 1\}$ with area form $\omega_0 = dp \wedge dq$ is the canonical example, with the area-preserving diffeomorphisms as $\mathrm{Symp}(\mathbb{D}^2, \omega_0)$.

Hamiltonian systems on a symplectic manifold are defined as follows. Let $X_H(t,\cdot)$ be the 1-parameter family of vector fields given via the relation $\iota_{X_H}\omega = -dH$, where $H(t,\cdot): M \to \mathbb{R}$ is a smooth family of functions, or Hamiltonians, with the property that H is constant on ∂M. This boundary condition is equivalent to $i^*\iota_{X_H}\omega = 0$ where $i: \partial M \to M$ is the inclusion. As a consequence $X_H(t,x) \in T_x\partial M$ for $x \in \partial M$, and the differential equation

$$\frac{dx(t)}{dt} = X_H(t,x(t)), \quad x(0) = x, \tag{2.1.1}$$

defines diffeomorphisms $\psi_{t,H}: M \to M$ via $\psi_{t,H}(x) \stackrel{\text{def}}{=} x(t)$ with ∂M invariant. Since ω is closed it holds that $\psi_{t,H}^*\omega = \omega$ for any t, which implies that $\psi_{t,H} \in \mathrm{Symp}(M,\omega)$. Symplectomorphisms of the form $\psi_{t,H}$ are called Hamiltonian, and the subgroup of such symplectomorphisms is denoted $\mathrm{Ham}(M,\omega)$.

The dynamics of Hamiltonian maps are closely connected to the topology of the domain. Any map of $\mathbb{D}^2$ has at least one fixed point, via the Brouwer fixed point theorem. The content of the Arnol'd Conjecture is that the number of fixed points of a generic Hamiltonian map of a (closed) (M,ω) is at least $\sum_k \dim H_k(M;\mathbb{R})$, the sum of the Betti numbers of M. Periodic points are more delicate still. A general result by Franks [15] states that an area-preserving map of the 2-disc has either one or infinitely many periodic points (the former case being that of a unique fixed point, e.g., irrational rotation about the center). For a large class of closed symplectic manifolds (M,ω) a similar result was proved by Salamon and Zehnder [35] under appropriate non-degeneracy conditions; recent results by Hingston for tori [21] and Ginzburg for closed, symplectically aspherical manifolds show that any Hamiltonian symplectomorphism has infinitely many geometrically distinct periodic points [20]. These latter results hold without non-degeneracy conditions. In [9] extensions of Franks' Theorem are proved using Floer homology techniques.

In this article we develop a more detailed Morse-type theory for periodic points utilizing the fact that area-preserving diffeomorphisms of the 2-disc are Hamiltonian and that orbits of non-autonomous Hamiltonian systems form links in $\mathbb{R}/\mathbb{Z} \times \mathbb{D}^2$. The central problem can be phrased as follows. Let $A_m = \{y^1, \cdots, y^m\} \subset \mathbb{D}^2$ be a discrete invariant set for a Hamiltonian

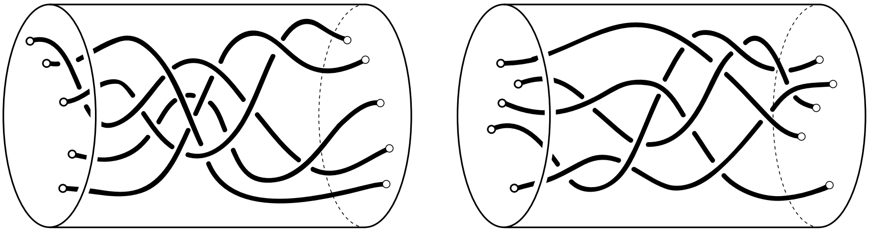

Fig. 2.1.1. A pair of braids on five strands are in the same connected component of ΛC_5.

diffeomorphism $f = \psi_{1,H}$: are there additional periodic points? We build a forcing theory based on a new invariant for braids, defined via Floer homology, to answer this question. The new invariant for braids using Floer homology will be referred to as BRAID FLOER HOMOLOGY.

2.1.1. Closed braids and Hamiltonian systems

We focus our attention on the unit disc $\mathbb{D}^2$, keeping track of a set A_m of m distinguished points. Extensions to discs with b holes, e.g. the annulus, are easily made within the framework of the theory as presented here.

2.1. Proposition. *(See Boyland [8, Lemma 1(b)].) For every area-preserving diffeomorphism f of the disc $\mathbb{D}^2$, with $f(A_m) = A_m$, there exists a Hamiltonian isotopy $\psi_{t,H}$ on $\mathbb{D}^2$ such that $f = \psi_{1,H}$. In particular* $\mathrm{Symp}(\mathbb{D}^2, \omega_0) = \mathrm{Ham}(\mathbb{D}^2, \omega_0)$.

We note that (contrary to the typical case in the literature) $\partial\mathbb{D}^2$ is assumed invariant, but not pointwise fixed. As a consequence of the proof, the Hamiltonian $H(t, x)$ can be assumed to be C^∞, 1-periodic in t, and vanishing on $\partial\mathbb{D}^2$. We denote this class of Hamiltonians by $\mathscr{H}$ (see Section 3.2 for the precise definition).

Periodic orbits of a map of $\mathbb{D}^2$ are described in terms of configuration spaces and braids. The CONFIGURATION SPACE $C_m(\mathbb{D}^2)$ is the space of all subsets of $\mathbb{D}^2$ of cardinality m. Let $x \in \mathbb{D}^2$ be an m-periodic point, i.e. $f^m(x) = x$, with $m \geq 1$ the minimal period. Then the set $A_m = \{x, f(x), \cdots, f^{m-1}(x)\}$ satisfies $f(A_m) = \{f(x), f^2(x), \cdots, f^m(x) = x\} = A_m$, and a periodic point is thus represented by a point $A_m \in C_m(\mathbb{D}^2)$. More generally, any invariant set A_m of f of cardinality m is a point in $C_m(\mathbb{D}^2)$. The free loop space ΛC_m of continuous maps $\mathrm{x} : \mathbb{R}/\mathbb{Z} \to C_m(\mathbb{D}^2)$ captures the manner in which periodic orbits 'braid' themselves in the Hamiltonian flow on $\mathbb{R}/\mathbb{Z}^1 \times \mathbb{D}^2$, as in Fig. 2.1.1. If we represent a periodic point, or more generally a discrete invariant set, by a configuration $A_m \in C_m(\mathbb{D}^2)$, then $t \mapsto \psi_{t,H}(A_m)$ is (smooth) loop contained in ΛC_m. Geometrically, this path yields a closed braid in $\mathbb{S}^1 \times \mathbb{D}^2$. This justifies referring to ΛC_m as the space of closed geometric braids on $\mathbb{R}/\mathbb{Z} \times \mathbb{D}^2$. Recall that the classical BRAID GROUP $\mathcal{B}_m$ on m strands is $\pi_1(C_m(\mathbb{D}^2))$ (pointed homotopy classes of loops). Modulo conjugacy, the elements of $\mathcal{B}_m$ label the connected components of ΛC_m. Although elements of ΛC_m are not themselves elements of $\mathcal{B}_m$, we abuse notation and refer to such as braids.

There is a natural homomorphism $\mathcal{B}_m \to S_m$, which associates a braid with a permutation. A loop x therefore defines a permutation $\sigma(\mathrm{x})$. The cycles of the permutation correspond to the components of x and a finite number of components makes a SUB-BRAID. The order of the cycle determines the number of 'strands' of the component. To build a forcing theory, we work with RELATIVE BRAIDS — braids split into 'free' and 'skeletal' sub-braids. The configuration space $C_{n,m}(\mathbb{D}^2)$ of 2-COLORED CONFIGURATIONS is the space of (ordered) pairs of disjoint subsets

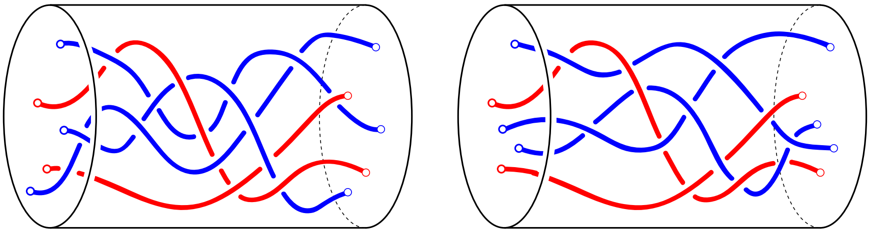

Fig. 2.1.2. Shown are two relative braids in the fiber [X] rel Y: the fixed strands Y are called the skeleton (in red). (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

of $\mathbb{D}^2$ of cardinality n and m respectively. Denote by $\Lambda C_{n,m}$ the loops (braids) in ΛC_{n+m} which decompose in sub-braids of n and m strands respectively. Such loops are denoted by X rel Y and are called RELATIVE BRAIDS, with the sub-braids X $\in \Lambda C_n$ and Y $\in \Lambda C_m$; its RELATIVE BRAID CLASS [X rel Y] is the connected component in $\Lambda C_{n,m}$. A RELATIVE BRAID CLASS FIBER [X] rel Y is defined to be the subset of elements X$'$ rel Y $\in$ [X rel Y]. The braid class fiber represents all possible free braids which stay in the braid class, keeping the SKELETON Y fixed: see Fig. 2.1.2. The group of pointed homotopy classes $\pi_1(C_{n,m}(\mathbb{D}^2))$ is called the 2-color braid group $\mathcal{B}_{n,m}$. In Section 3.1 we will give detailed definitions of these concepts.

2.1.2. *The variational approach*

Fix a Hamiltonian $H \in \mathscr{H}$, then the Hamiltonian action of smooth functions $x : \mathbb{R}/\mathbb{Z} \to \mathbb{D}^2$ is given by

$$\mathscr{L}_H(x) = \int_0^1 \theta(x(t)) = \int_0^1 \alpha_0(x_t(t))dt - \int_0^1 H(t, x(t))dt, \quad t \in \mathbb{S}^1, \tag{2.1.2}$$

where $\theta = \alpha_0 - H dt$, $\alpha_0 = p dq$ and $d\alpha_0 = \omega_0$. Now $x(t) = \psi_{t,H}(x)$, $t \in \mathbb{R}$, is a 1-periodic orbit of the Hamiltonian flow (2.1.1) if $\mathscr{L}_H$ is stationary at $x(t)$.

Let $A_m \subset \mathbb{D}^2$ be an invariant set for $f = \psi_{1,H}$ and Y $\in \Lambda C_m$ is given by Y$(t) = \psi_{t,H}(A_m)$. The components of Y correspond to periodic solutions of Eq. (2.1.1) of integer period and therefore Y may be regarded as a 'set of periodic solutions' of Eq. (2.1.1). We will assign Floer homology to relative braid class fibers [X] rel Y. In order to provide intuition consider the special case of relative braids in $\Lambda C_{1,m}$. In this case X$(t) = \{x(t)\}$ is closed loop in $\Lambda C_1 = \Lambda \mathbb{D}^2$ and X 'avoids' the pre-existing solutions given by Y. Closed loops in $\Lambda \mathbb{D}^2$ that are solutions of the Hamilton equations can be found as critical points of the action $\mathscr{L}_H$ on the fiber [X] rel Y. The topology of the fiber [X] rel Y may force critical points of $\mathscr{L}_H$ in the fiber.

We investigate critical points of $\mathscr{L}_H$ on [X] rel Y via nonlinear Cauchy–Riemann equations given as an L^2-gradient flow of $\mathscr{L}_H$. We choose the standard compatible almost-complex structures J on $(\mathbb{D}^2, \omega_0)$, which is defined by the properties $J^2 = -\mathrm{Id}$, $\omega_0(J\cdot, J\cdot) = \omega_0(\cdot,\cdot)$ and $\langle\cdot,\cdot\rangle_g = \omega_0(\cdot, J\cdot)$, which is an inner product on $\mathbb{D}^2$. For functions $u : \mathbb{R} \times \mathbb{R}/\mathbb{Z} \to \mathbb{D}^2$ the nonlinear CAUCHY–RIEMANN EQUATIONS are defined as

$$\big(\partial_{J,H}(u)\big)(s,t) \stackrel{\text{def}}{=} \frac{\partial u(s,t)}{\partial s} - J\frac{\partial u(s,t)}{\partial t} - \nabla_g H(t, u(s,t)), \tag{2.1.3}$$

where ∇_g is the gradient with respect to the metric $\langle\cdot,\cdot\rangle_g$. Stationary, or s-independent solutions $u(s,t) = x(t)$ satisfy Eq. (2.1.1) since $J\nabla H = X_H$. In Section 4.1 we will provide a more

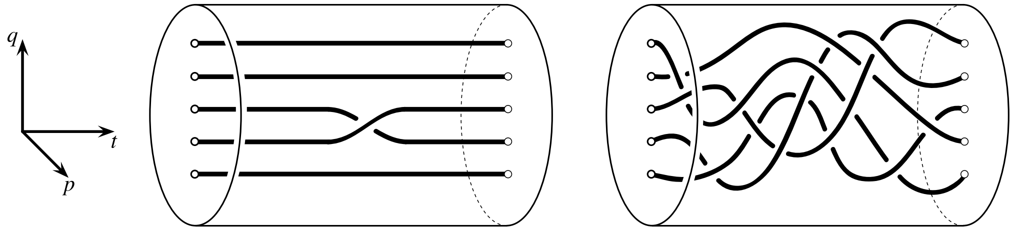

Fig. 2.2.1. The standard positive generator σ_i for the braid group $\mathcal{B}_n$ sends the ith strand over the $(i+1)$st. Pictured in $\mathcal{B}_5$ is (left) σ_2 and (right) the braid $\sigma_4^{-1}\sigma_3\sigma_1\sigma_3\sigma_2^{-1}\sigma_1\sigma_2\sigma_3^{-1}\sigma_1\sigma_4^{-1}\sigma_2\sigma_3\sigma_4^{-1}\sigma_1\sigma_2^{-1}$. This braid has length (algebraic crossing number) equal to $+3$.

detailed account on the Cauchy–Riemann equations for relative braids X rel $\mathrm{Y} \in \Lambda C_{n,m}$, with $n \geq 1$.

2.2. *Result 1: Monotonicity*

There is a crucial link between bounded solutions of the Cauchy–Riemann equations and algebraic–topological properties of the associated braid classes. For $\mathrm{X} \in \Lambda C_n$, the associated braid can be represented as a CONJUGACY CLASS in the braid group $\mathcal{B}_n$, using the standard generators $\{\sigma_i\}_{i=1}^{n-1}$. The LENGTH of the braid word (the sum of the exponents of the σ_i's) is well-defined and is a braid invariant. Geometrically, this length is the total CROSSING NUMBER $\mathrm{Cross}(\mathrm{X})$ of the braid: the algebraic sum of the number of crossings of strands in the standard planar projection (see Fig. 2.2.1).

Let us return to the case of braid class fibers $[\mathrm{X}]$ rel $\mathrm{Y} \in \Lambda C_{1,m}$ as discussed in Section 2.1.2. The algebraic sum of the crossings in any X rel Y, $\mathrm{Cross}(\mathrm{X} \text{ rel } \mathrm{Y})$, is the same for all relative braids in $[\mathrm{X}]$ rel Y. Let u be a bounded solution of the Cauchy–Riemann equations (2.1.3). Define $\mathrm{U}(s) = u(s,\cdot)$ as a 1-parameter family of closed loops in $\Lambda\mathbb{D}^2$. It follows from the analysis of Eq. (2.1.3) that for almost all $s \in \mathbb{R}$, $\mathrm{U}(s)$ rel $\mathrm{Y} \in \Lambda C_{1,m}$. However, $\mathrm{Cross}(\mathrm{U}(s) \text{ rel } \mathrm{Y})$ need not be constant in s. More precisely, the following monotonicity principle holds (see Section 5.2 for details).

2.2. Monotonicity Principle. *Suppose* $\mathrm{U}(s)$ rel $\mathrm{Y} \in \Lambda C_{1,m}$, *for* $s = s_0 \pm \epsilon$. *If there exists an* $s \in [s_0 - \epsilon, s_0 + \epsilon]$, *such that* $\mathrm{U}(s)$ rel $\mathrm{Y} \notin \Lambda C_{1,m}$, *then*

$$\mathrm{Cross}(\mathrm{U}(s_0 - \epsilon) \text{ rel } \mathrm{Y}) > \mathrm{Cross}(\mathrm{U}(s_0 + \epsilon) \text{ rel } \mathrm{Y}). \tag{2.2.1}$$

Otherwise, $\mathrm{Cross}(\mathrm{U}(s) \text{ rel } \mathrm{Y})$ *is constant for* $s \in [s_0 - \epsilon, s_0 + \epsilon]$.

This consequence of the maximum principle follows from the positivity of intersections of J-holomorphic curves in almost-complex 4-manifolds [25,24]; other expressions of this principle arise in applications to heat equations in one space dimension, see e.g. [3,4].

The monotonicity principle leads to an isolation property for certain relative braid classes which makes a Morse-theoretic approach viable. Denote by $\mathscr{M}([\mathrm{X}] \text{ rel } \mathrm{Y})$ the set of bounded solutions $u(s,t)$ of the Cauchy–Riemann equations for which $\mathrm{U}(s)$ rel $\mathrm{Y} \in [\mathrm{X}]$ rel Y for all $s \in \mathbb{R}$. For a Floer homology construction compactness of this space is required. However, the invariant set in a braid class fiber is non-compact in general. In order to obtain compactness we consider relative braid classes that satisfy the following topological property: loosely speaking, for any

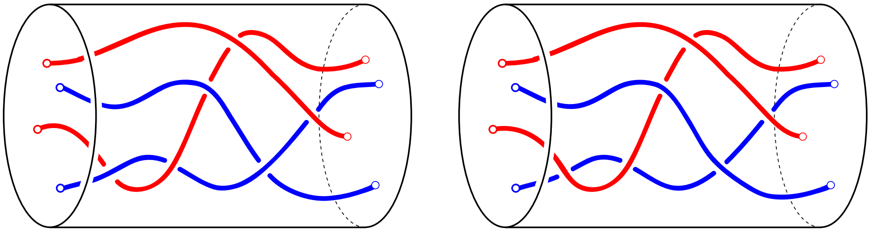

Fig. 2.2.2. Proper (left) and improper (right) relative braid classes. On the right, the free strands can collapse onto the skeleton (in red). (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

representative in the fiber $[\mathrm{X}]$ rel Y, X cannot collapse onto strands in Y, nor can X collapse onto the boundary $\partial\mathbb{D}^2$. Such braid classes are called PROPER: see Fig. 2.2.2. From elliptic regularity of the Cauchy–Riemann equations and the Monotonicity Principle we obtain compactness: for a proper relative braid class, the set of bounded solutions $\mathscr{M}([\mathrm{X}]$ rel $\mathrm{Y})$ is compact in the topology of uniform convergence on compact subsets in $\mathbb{R}^2$.

2.3. Compactness Theorem. *Let* $[\mathrm{X}]$ rel Y *be a proper relative braid class. Then, in the topology of uniform convergence on compact subsets in* $\mathbb{R}^2$*, the set of bounded solutions* $\mathscr{M}([\mathrm{X}]$ rel $\mathrm{Y})$ *is compact.*

The proof is a combination of Proposition 4.1 and Proposition 6.2. In Sections 4 and 5 we will develop this theory in detail and state the appropriate results for arbitrary proper relative braids in $\Lambda C_{n,m}$.

2.3. Result 2: Braid Floer homology

The above proposition is used to define a Floer homology (cf. [13]) for proper relative braid classes. The concept of properness is a topological property that extends to any $n \geq 1$. For a proper relative braid class fiber, define $\mathscr{N} = [\mathrm{X}]$ rel Y. By the Compactness Theorem 2.3 the set of bounded solutions of the Cauchy–Riemann equations in $\mathscr{N}$ is compact and $|\mathrm{U}| < 1$, for all U rel $\mathrm{Y} \in \mathscr{M}(\mathscr{N})$. In order to define Floer homology for $\mathscr{N}$, the system needs to be embedded into a generic family of systems. The usual approach is to establish that for 'generic' choices of Hamiltonians the critical points of the action are non-degenerate and the sets of orbits connecting stationary braids are finite dimensional manifolds $\mathscr{M}_{\mathrm{X}_-,\mathrm{X}_+}(\mathscr{N})$.

The Fredholm theory for the Cauchy–Riemann equations yields an index function μ on stationary points of the action $\mathscr{L}_H$ and $\dim \mathscr{M}_{\mathrm{X}_-,\mathrm{X}_+}([\mathrm{X}]$ rel $\mathrm{Y}) = \mu(\mathrm{X}_-) - \mu(\mathrm{X}_+)$. Following Floer [13] we can define a chain complex $C_k([\mathrm{X}]$ rel $\mathrm{Y}) = \bigoplus \mathbb{Z}_2\langle \mathrm{X}\rangle$, where the direct sum is taken over critical points of index k. The boundary operator $\partial_k : C_k \to C_{k-1}$ is the linear operator generated by counting orbits (modulo 2) between critical points of index difference one. The structure of the space of bounded solutions reveals that (C_*, ∂_*) is a chain complex. The homology of this chain complex is denoted $\mathrm{HF}_*([\mathrm{X}]$ rel $\mathrm{Y}; J, H)$. This is finite dimensional for all k and nontrivial for only finitely many values of k. Independence of choices is our first major result. Any almost complex structure J and any Hamiltonian H for which Y is stationary, i.e. $\mathrm{Y}(t) = \psi_{t,H}(A_m) \in \Lambda C_m$, yields the same Floer homology — $\mathrm{HF}_*([\mathrm{X}]$ rel $\mathrm{Y}; J, H)$ is independent of H. Homotopies of Y within the braid class also leave the Floer homology invariant. To be more precise, for $\mathrm{Y}, \mathrm{Y}' \in [\mathrm{Y}]$, the Floer homology groups for the relative braid class fibers $[\mathrm{X}]$ rel Y and $[\mathrm{X}']$ rel Y' of $[\mathrm{X}$ rel $\mathrm{Y}]$

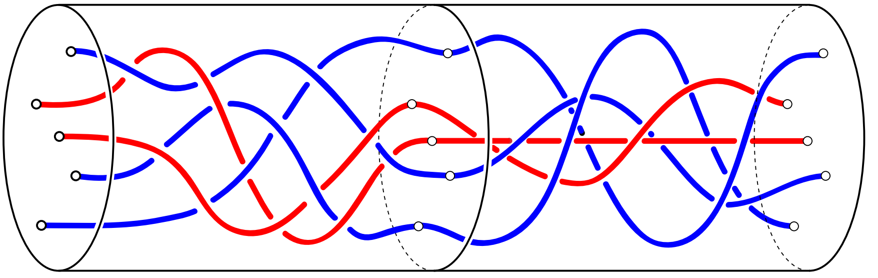

Fig. 2.4.1. The addition of a full positive twist (on right) to a proper relative braid class (on left) shifts the Floer homology up by degree $2n$ (skeletal strands in red). (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

are isomorphic, $\mathrm{HF}_k([\mathrm{X}]\ \mathrm{rel}\ \mathrm{Y}; J, H) \cong \mathrm{HF}_k([\mathrm{X}']\ \mathrm{rel}\ \mathrm{Y}'; J', H')$. The Floer homology groups $\{FH_*([\mathrm{X}]\ \mathrm{rel}\ \mathrm{Y}; J, H)\}$ are an inverse system. This allows us to assign the Floer homology to the entire product class $[\mathrm{X}\ \mathrm{rel}\ \mathrm{Y}]$ (within the admissible class of Hamiltonians, i.e., Hamiltonians in $\mathscr{H}$ with a stationary skeleton $\mathrm{Y}' \in [\mathrm{Y}]$), thereby establishing a true braid invariant.

2.4. Braid Floer Homology Theorem. *The braid Floer homology of a proper relative braid class,*

$$\mathrm{HB}_*([\mathrm{X}\ \mathrm{rel}\ \mathrm{Y}]) \stackrel{\mathrm{def}}{=} \varprojlim \mathrm{HF}_*([\mathrm{X}]\ \mathrm{rel}\ \mathrm{Y}; J, H), \tag{2.3.1}$$

is a function of the braid class $[\mathrm{X}\ \mathrm{rel}\ \mathrm{Y}]$ *alone, independent of choices for* J, H *and the representative skeleton* Y. *The braid Floer homology groups* $\mathrm{HB}_k([\mathrm{X}\ \mathrm{rel}\ \mathrm{Y}])$ *are finite dimensional for all* k *and nontrivial for only finitely many values of* k.

The relative braid classes $[\mathrm{X}\ \mathrm{rel}\ \mathrm{Y}]$ are in one-to-one correspondence with the conjugacy class of the 2-color braid group $\mathcal{B}_{n,m}$. Therefore, braid Floer homology is an invariant for proper conjugacy classes in $\mathcal{B}_{n,m}$.

2.4. Result 3: Shifts & twists

The braid Floer homology HB_* entwines topological braid data with dynamical information about braid-constrained Hamiltonian systems. One example of the braid-theoretic content of HB_* comes from an examination of twists. Recall that the braid group $\mathcal{B}_n$ has as its group operation concatenation of braids. This does not extend to a well-defined product on conjugacy classes; however, $\mathcal{B}_n$ has a nontrivial center $Z(\mathcal{B}_n) \cong \mathbb{Z}$ generated by $\square = \Delta^2$, the FULL TWIST on n strands. Thus, products with full twists are well-defined on conjugacy classes. These full twists have a well-defined impact on the braid Floer homology (Section 12). Twists shift the grading: see Fig. 2.4.1.

2.5. Shift Theorem. *Let* $[\mathrm{X}\ \mathrm{rel}\ \mathrm{Y}]$ *denote a proper relative braid class in* $\Lambda C_{n,m}$. *Then*

$$\mathrm{HB}_*\big([(\mathrm{X}\ \mathrm{rel}\ \mathrm{Y}) \cdot \Delta^2]\big) \cong \mathrm{HB}_{*-2n}([\mathrm{X}\ \mathrm{rel}\ \mathrm{Y}]),$$

where Δ^2 *is a full twist on* $n+m$ *strands.*

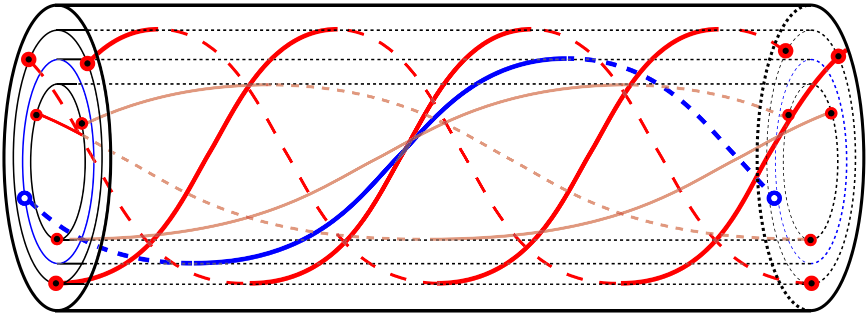

Fig. 2.4.2. The braid class of Example 2.6 has one free strand surrounded by a pair of skeletal braids whose rotation numbers are bound from above and below (skeletal strands in red). (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

In any Floer theory, computable examples are nontrivial. We compute examples in Section 13, including the following.

2.6. Example. Consider a skeleton Y consisting of two braid components $\{\mathrm{Y}^1, \mathrm{Y}^2\}$, with Y^1 and Y^2 defined by (see Fig. 2.4.2)

$$\mathrm{Y}^1 = \left\{ r_1 e^{\frac{2\pi n}{m} it}, \cdots, r_1 e^{\frac{2\pi n}{m} i(t-m+1)} \right\}, \quad \mathrm{Y}^2 = \left\{ r_2 e^{\frac{2\pi n'}{m'} it}, \cdots, r_2 e^{\frac{2\pi n'}{m'} i(t-m'+1)} \right\},$$

where $0 < r_1 < r_2 \le 1$, and (n, m) and (n', m') are relatively prime integer pairs with $n, n' \in \mathbb{Z}$, $n \neq 0$, $m \geq 2$, and $m' > 0$. A free strand is given by $\mathrm{X} = \{x^1\}$, with $x^1(t) = re^{2\pi \ell it}$, for $r_1 < r < r_2$ and some $\ell \in \mathbb{Z}$, with either $n/m < \ell < n'/m'$ or $n/m > \ell > n'/m'$, depending on the ratios of n/m and n'/m'. The relative braid class $[\mathrm{X} \text{ rel } \mathrm{Y}]$ is defined via the representative X rel Y. The associated braid class is proper, and $\mathrm{HB}_*([\mathrm{X} \text{ rel } \mathrm{Y}])$ is non-zero in exactly two dimensions: 2ℓ and $2\ell \pm 1$, see Section 13.

This example agrees with a similar computation of a (finite-dimensional) Conley index of positive braid classes in [18]. Indeed, we believe that the braid Floer homology agrees with that index on positive braid classes. We anticipate using Theorem 2.5 combined with Garside's Theorem on normal forms for braids as a means of algorithmically computing HB_*, see e.g. Section 14.1.

2.5. Result 4: Forcing

Braid Floer homology HB_* contains information about the existence of periodic points or invariant sets A_m of area-preserving diffeomorphisms f. Recall that an invariant set $A_m = \{y^1, \cdots, y^m\}$ for f determines a braid class $[\mathrm{Y}]$ via $\mathrm{Y}(t) = \psi_{t,H}(A_m) = \big(\psi_{t,H}(y^1), \cdots, \psi_{t,H}(y^n)\big)$. The representation as an element $\beta(\mathrm{Y})$ in the braid group $\mathcal{B}_n$, given by a choice of a Hamiltonian H, is uniquely determined modulo full twists.

2.7. Braid Forcing Theorem. *Let $f \in \mathrm{Symp}(\mathbb{D}^2, \omega_0)$ have an invariant set A_m representing the m-strand braid class* $[\mathrm{Y}]$. *Then, for any proper relative braid class* $[\mathrm{X} \text{ rel } \mathrm{Y}]$ *in* $\Lambda C_{n,m}$ *for which* $\mathrm{HB}_*([\mathrm{X} \text{ rel } \mathrm{Y}]) \neq 0$, *there exists an invariant set A'_n for f such that the union $A_m \cup A'_n$ represents the relative braid class* $[\mathrm{X} \text{ rel } \mathrm{Y}]$.

Since the braid Floer homology is an invariant of conjugacy classes on $\mathcal{B}_{n,m}$ the same result holds for symplectomorphisms $f \in \mathrm{Symp}(M, \omega)$, with $M \cong \mathbb{D}^2$.

2.8. Example. Consider the braid class [X rel Y] defined in Example 2.6. For any area-preserving diffeomorphism f of the (closed) disc with invariant set represented (up to full twists) by the braid class [Y], there exist infinitely many distinct periodic points. To prove this statement we invoke the invariant HB_*, computed in Example 2.6, and use Theorem 2.5. In particular, this result implies that if f has a fixed point at the boundary and a periodic point of period larger than two in the interior, then f has periodic points of arbitrary period, and thus infinitely many periodic points. In Section 13 we give more details: the main results are presented in Theorem 13.4.

3. Background: configuration spaces and braids

In the following section we give the necessary background on braids and introduce some new concepts needed to build a forcing theory of periodic solutions of Hamiltonian systems.

3.1. Configuration spaces and braid classes

Configuration spaces provide a natural setting to introduce braids. The CONFIGURATION SPACE $C_n(\mathbb{D}^2)$ is the space of subsets of $\mathbb{D}^2$ of cardinality n. An equivalent way to define $C_n(\mathbb{D}^2)$ is via configuration spaces of ordered sets of points. The CONFIGURATION SPACE OF FINITE ORDERED SETS is defined as $F_n(\mathbb{D}^2) = \{(x^1, \cdots x^n) \in \mathbb{D}^2 \times \cdots \times \mathbb{D}^2 \mid x^i \neq x^j, \forall i \neq j\}$, with the metric topology inherited from $\mathbb{D}^2 \times \cdots \times \mathbb{D}^2$. For $n = 1$, $F_1(\mathbb{D}^2) = \mathbb{D}^2$ and for $n > 1$, $F_n(\mathbb{D}^2)$ is a smooth, non-compact, aspherical, symplectic manifold of dimension $2n$ and the symplectic structure is given by $\bar{\omega}_0 = \omega_0 \oplus \cdots \oplus \omega_0$. The group of permutation S_n acts freely, smoothly and evenly as a right action on the manifold $F_n(\mathbb{D}^2)$; let $\sigma \in S_n$, then

$$\Big((x^1, \cdots, x^n), \sigma\Big) \mapsto (x^1, \cdots, x^n) \cdot \sigma, \quad (x^1, \cdots, x^n) \cdot \sigma = \big(x^{\sigma(1)}, \cdots, x^{\sigma(n)}\big).$$ [1]

The configuration space is given as $C_n(\mathbb{D}^2) = F_n(\mathbb{D}^2)/S_n$ — the orbit space —, and $C_n(\mathbb{D}^2)$ is a smooth, open (for $n > 1$), aspherical, symplectic manifold of dimension $2n$. The mapping that sends points in $F_n(\mathbb{D}^2)$ to their orbits,

$$\varpi_n : F_n(\mathbb{D}^2) \to C_n(D^2), \quad (x^1, \cdots, x^n) \mapsto [(x^1, \cdots, x^n) \cdot S_n],$$

is an ($n!$-sheeted) covering map. For the analytic theory of braids in this paper it is convenient to work with ordered sets.

3.1. Definition. The SPACE OF CLOSED BRAIDS ON n STRANDS, denoted ΓF_n, consists of continuous mappings $\mathrm{x}(t) = \big(x^1(t), \cdots, x^n(t)\big) : [0, 1] \to \mathbb{D}^2 \times \cdots \times \mathbb{D}^2$, which satisfy

(i) for some permutation $\sigma \in S_n$, $x^k(1) = x^{\sigma(k)}(0)$, for all $k = 1, \cdots, n$;
(ii) for any pair $k \neq k'$, it holds that $x^k(t) \neq x^{k'}(t)$, for all $t \in [0, 1]$.

On ΓF_n we consider the strong metric topology of $C^0\big([0, 1]; \mathbb{D}^2 \times \cdots \times \mathbb{D}^2\big)$.

[1] The action of σ is defined by $\sigma\big((1, \cdots, n)\big) = \big(\sigma(1), \cdots, \sigma(n)\big)$.

The elements in ΓF_n are the lifts of based loops $\gamma : [0,1] \to C_n(\mathbb{D}^2)$. Choose a point $\mathrm{x}_0 \in F_n(\mathbb{D}^2)$, then for any based loop $\gamma : [0,1] \to C_n(\mathbb{D}^2)$ with $\gamma(0) = \mathrm{x}_0$ and $\gamma(1) \in \varpi_n(\mathrm{x}_0)$, there exists a unique lift $\mathrm{x} : [0,1] \to F_n(\mathbb{D}^2)$ with $\mathrm{x}(0) = \mathrm{x}_0$ and $\mathrm{x}(1) = \mathrm{x}(0) \cdot \sigma$, for some permutation $\sigma \in S_n$. The same holds for free loops in the free loop space ΛC_n of continuous mappings $\gamma : \mathbb{R}/\mathbb{Z} \to C_n(\mathbb{D}^2)$ by choosing a base point. For any $\mathrm{x} \in \Gamma F_n$ we define the $\mathbb{R}$-extension by $\underline{\mathrm{x}}(t+\ell) = \mathrm{x}(t) \cdot \sigma^\ell$. It follows that $\underline{\mathrm{x}}$ is $n!$-periodic, since $\sigma^{n!} = \mathrm{id}$, and the extension $\underline{\mathrm{x}}$ defines a free loop in ΛF_n, denoted by $n! \cdot \mathrm{x} : \mathbb{R}/n!\mathbb{Z} \to F_n(\mathbb{D}^2)$, with $n! \cdot \mathrm{x}|_{[0,1]} = \mathrm{x}$.

The path connected components of ΓF_n are called braid classes. The BRAID CLASS of x is the path connected component $[\mathrm{x}]$ in ΓF_n. Since we are dealing with ordered sets of points in $\mathbb{D}^2$ a braid class $[\mathrm{x}]$ determines a unique permutation σ for all $\mathrm{x}' \in [\mathrm{x}]$. It is useful to point out that shift along a braid defines a closed loop in $[\mathrm{x}]$: $h(s)(t) = \underline{\mathrm{x}}(t+s)$, $s \in [0,1]$.

The C^0-closure of $[\mathrm{x}]$ is denoted by $\mathrm{cl}([\mathrm{x}])$ and the braids in $\mathrm{cl}([\mathrm{x}])$ are mappings that satisfy Condition (i) in Definition 3.1, but not necessarily Condition (ii). We define the SINGULAR BRAIDS as $\Sigma[\mathrm{x}] = \mathrm{cl}([\mathrm{x}]) \setminus [\mathrm{x}]$ and the COLLAPSED SINGULAR BRAIDS as

$$\Sigma^-[\mathrm{x}] = \left\{\mathrm{x} \in \Sigma[\mathrm{x}] \;:\; x^k(t) = x^{k'}(t),\ \forall t \in \mathbb{R},\ \text{for at least one pair } k \neq k'\right\}.$$

Singular braids in general are denoted by $\Sigma = \mathrm{cl}(\Gamma F_n) \setminus \Gamma F_n$, and similarly for Σ^-. We also define the set of PROPERLY SINGULAR BRAIDS as

$$\Sigma^+[\mathrm{x}] = \left\{\mathrm{x} \in \Sigma[\mathrm{x}] \;:\; \exists 1 \le k, k' \le n \text{ and } t_0, t_1 \in \mathbb{R} \text{ s.t. } x^k(t_0) = x^{k'}(t_0) \text{ and } x^k(t_1) \neq x^{k'}(t_1)\right\}.$$

Singular braids in Σ^+ may have collapsed strands, but there are at least two strands that intersect and are not collapsed.

Let $\mathrm{x} \in \Gamma F_n$ and $\mathrm{y} \in \Gamma F_m$, and define mapping $\Gamma F_n \times \Gamma F_m \to \mathrm{cl}(\Gamma F_{n+m})$ by

$$(\mathrm{x}, \mathrm{y}) \mapsto \mathrm{x} \text{ rel } \mathrm{y} = (x^1, \cdots, x^n, y^1, \cdots, y^m).$$

The space of relative closed braids is defined by

$$\Gamma F_{n,m} = \left\{\mathrm{x} \text{ rel } \mathrm{y} \in \Gamma F_{n+m}\right\}.$$

As before, $[\mathrm{x} \text{ rel } \mathrm{y}]$ denotes the path connected component of x rel y in $\Gamma F_{n,m}$ and is called the RELATIVE BRAID CLASS of x rel y. The components x and y of a relative braid x rel y are referred to as the FREE STRANDS and the SKELETON, respectively. For a given skeleton $\mathrm{y} \in \Gamma F_m$ define the RELATIVE BRAID CLASS FIBER $[\mathrm{x}]$ rel y as the set of relative braids x' rel $\mathrm{y} \in [\mathrm{x} \text{ rel } \mathrm{y}]$. The subset of $\Gamma F_{n,m}$ of relative braids with skeleton y is denoted by ΓF_n rel y. Under projection onto the first component of a relative braid, a fiber can be regarded as a subset of $[\mathrm{x}]$. We will abuse notation and sometimes write $[\mathrm{x}]$ rel $\mathrm{y} \subset [\mathrm{x}]$.

In this paper we distinguish two types of relative braid classes; proper and improper classes. These are purely topological properties of braid classes.

3.2. Definition. A relative braid class $[\mathrm{x} \text{ rel } \mathrm{y}]$ is PROPER if for any braid class fiber $[\mathrm{x}]$ rel y it holds that

(i) x' rel $\mathrm{y} \in \mathrm{cl}([\mathrm{x}] \text{ rel } \mathrm{y})$ implies that $|x^k(t)| \not\equiv 1$, for all k;
(ii) $\mathrm{cl}([\mathrm{x}] \text{ rel } \mathrm{y}) \cap \Sigma[\mathrm{x} \text{ rel } \mathrm{y}] \subset \Sigma^+[\mathrm{x} \text{ rel } \mathrm{y}]$.

Braid classes for which either (i) or (ii) is not satisfied are called IMPROPER.

All the elements in the boundary of a proper relative braid class are thus properly singular braids. The compactness and isolation of the invariant set in [X] rel Y are consequences of the properness of the relative braid class, see Theorem 2.3, Lemma 3.4 and Section 6.

3.2. The Hamiltonian action

In order to consider Hamiltonian systems on the loop spaces ΓF_n we introduce the class $\mathscr{H}(\mathbb{D}^2)$ of Hamiltonians, characterized by the following hypotheses:

(h1) $H \in C^\infty(\mathbb{R} \times \mathbb{D}^2; \mathbb{R})$;
(h2) $H(t+1, x) = H(t, x)$ for all $t \in \mathbb{R}$ and all $x \in \mathbb{D}^2$;
(h3) $H(t, x) = 0$ for all $x \in \partial\mathbb{D}^2$ and for all $t \in \mathbb{R}$.

For $H \in \mathscr{H}$ define the 1-form $\theta = pdq - Hdt$ on $\mathbb{R} \times \mathbb{D}^2$. Consider differentiable functions $x : [0, 1] \to \mathbb{D}^2$, with $x_t(t) = \frac{dx(t)}{dt} \in T_{x(t)}\mathbb{D}^2$ continuous. This implies in particular that $x_t(t_0) \in T_{x(t_0)}\partial\mathbb{D}^2$, whenever $x(t_0) \in \partial\mathbb{D}^2$. The action of $x(t)$ is defined by

$$\int_0^1 \theta(x(t)) = \int_0^1 \alpha_0(x_t(t))dt - \int_0^1 H(t, x(t))dt. \tag{3.2.1}$$

For $\mathrm{X} \in [\mathrm{X}]$ we define the action by

$$\mathscr{L}_H(\mathrm{X}) = \sum_k \int_0^1 \theta(x^k(t)).$$

With respect to smooth variations ξ we have:

$$\begin{aligned} d\mathscr{L}_H(\mathrm{X})\xi &= -\sum_k \int_0^1 \omega_0\big(x_t^k(t), \xi^k(t)\big)dt - \int_0^1 dH(t, x^k(t))\xi^k(t)dt \\ &= -\sum_k \int_0^1 \omega_0\Big(x_t^k(t) - X_{t,H}(t, x^k(t)), \xi^k(t)\Big)dt, \end{aligned}$$

where $\xi^k \in C^\infty(\mathbb{R})$ satisfy $\xi^k(t+1) = \xi^{\sigma(k)}(t)$ for all t. Setting the first variation equal to zero gives the variational principle for Eq. (2.1.1). From the variational principle we define the notion of a critical, or stationary braid for the Hamilton action.

3.3. Definition. A braid $\mathrm{X} \in [\mathrm{X}]$ is CRITICAL, or STATIONARY for the Hamilton equations if the components x^k satisfy

$$x_t^k = X_H(t, x^k), \text{ with boundary conditions } x^k(t+1) = x^{\sigma(k)}(t),$$

for all t and all $k = 1, \cdots, n$. The braid X is said to satisfy the Hamilton equations.

Let $\mathrm{Y} \in \Gamma F_m$ satisfy the Hamilton equations and consider proper relative braid classes $[\mathrm{X} \text{ rel } \mathrm{Y}]$ in $\Gamma F_{n,m}$. The braid Y is called a stationary skeleton. A relative braid X rel Y satisfies the Hamilton equations if also X satisfies the Hamilton equations. In other words X rel Y is stationary in the sense of Definition 3.3. For a fiber $[\mathrm{X}]$ rel Y we denote the stationary relative braids by $\mathrm{Crit}_H([\mathrm{X}] \text{ rel } \mathrm{Y})$.

3.4. Lemma. *Let* Y *be a stationary skeleton and let* $[\mathrm{X} \text{ rel } \mathrm{Y}]$ *be a proper relative braid class. Then set* $\mathrm{Crit}_H([\mathrm{X}] \text{ rel } \mathrm{Y}) \subset [\mathrm{X}] \text{ rel } \mathrm{Y}$ *is compact (with respect to the C^r topology, for any $r \geq 0$).*

Proof. Since $|x^k| \leq 1$, for all k, it follows from Eq. (2.1.1) and the assumptions on H, that $|x_t^k| = |\nabla H(t, x^k)| \leq C$. Therefore, $\|\mathrm{X}\|_{W^{1,\infty}} \leq C$, for all $\mathrm{X} \text{ rel } \mathrm{Y} \in \mathrm{Crit}_H([\mathrm{X}] \text{ rel } \mathrm{Y})$. Let X_n rel Y be a sequence in $\mathrm{Crit}_H([\mathrm{X}] \text{ rel } \mathrm{Y})$, then via the compact embeddings (Arzela–Ascoli) there is a subsequence, again denoted by X_n rel Y, such that the components x_n^k converge in C^0 to a limit $x^k \in C^0([0,1], \mathbb{D}^2)$, for all k. Using the equation we obtain the convergence in C^1 and the components x^k satisfy the Hamilton equations (2.1.1). The C^r-convergence is achieved by differentiating Eq. (2.1.1) repeatedly.

Since $[\mathrm{X} \text{ rel } \mathrm{Y}]$ is proper, x_n^k cannot converge to a strand $y^{k'}$, nor does the limit satisfy $|x^k(t)| \equiv 1$. If a limiting component x^k satisfies $x^k(t_0) = y^{k'}(t_0)$, or $|x^k(t_0)| = 1$, for some t_0 and some k', then we obtain a contradiction with the uniqueness of the initial value problem for Eq. (2.1.1). Therefore the boundary conditions are satisfied and $\mathrm{X}(t) \text{ rel } \mathrm{Y}(t) \in F_{n+m}(\mathbb{D}^2)$ for all t, which implies that $\mathrm{X} \text{ rel } \mathrm{Y} \in [\mathrm{X}] \text{ rel } \mathrm{Y}$, completing the proof. □

3.5. Remark. The critical braids in $\mathrm{Crit}_H([\mathrm{X}] \text{ rel } \mathrm{Y})$, for a proper relative braid class, have the property that $|x^k(t)| < 1$, for all t and all k. We say that such a braid is supported in $\mathrm{int}(\mathbb{D}^2)$.

4. Background: Cauchy–Riemann equations and a priori estimates

4.1. Almost complex structures

The standard almost complex matrix J_0 is defined by the relation $\langle \cdot, \cdot \rangle = \omega_0(\cdot, J_0 \cdot)$, where $\langle \cdot, \cdot \rangle$ is the standard inner product defined by $dp \otimes dp + dq \otimes dq$, and J_0 defines an almost complex structure on $(\mathbb{D}^2, \omega_0)$ which corresponds to complex multiplication with i in $\mathbb{C}$. In general an almost complex structure on $(\mathbb{D}^2, \omega_0)$ is a mapping $J : T\mathbb{D}^2 \to T\mathbb{D}^2$, with the property that $J^2 = -\mathrm{id}$. An almost complex structure is compatible with ω_0 if $\omega_0(\cdot, J\cdot)$ defines a metric on $\mathbb{D}^2$, and $g = \omega_0(\cdot, J\cdot)$ is J-invariant. The space of t-families of almost complex structures is denoted by $\mathscr{J} = \mathscr{J}(\mathbb{R}/\mathbb{Z} \times \mathbb{D}^2)$.

In terms of the standard inner product $\langle \cdot, \cdot \rangle$ the metric g is given by $\langle \xi, \eta \rangle_g = g(\xi, \eta) = \langle -J_0 J\xi, \eta \rangle$, where $-J_0 J$ is a positive definite symmetric matrix function. With respect to the metric g it holds that $J\nabla_g H = X_H$.

4.2. A priori estimates

In order to study 1-periodic solutions of Eq. (2.1.1) the variational method due to Floer and Gromov explores the perturbed nonlinear Cauchy–Riemann equations (2.1.3) which can be rewritten as

$$\partial_{J,H}(u) = u_s - J\big[u_t - X_H(t,u)\big] = 0,$$

for functions $u : \mathbb{R} \times \mathbb{R} \to \mathbb{D}^2$ (short hand notation). We consider almost complex structures J that do not depend on (t, x). When we consider continuation properties of the Floer homology we will consider $J = J(s)$, but in most of the paper J is a constant complex structure. When J does depend on s, we may always choose the path such that J is constant for $|s|$ sufficiently large.

In this section we derive a priori estimates for functions satisfying the periodicity condition $u(s, t + \ell) = u(s, t)$, $\ell \in \mathbb{N}$. We start with global bounds on all derivatives. These bounds follow from local regularity combined with the a priori bound $u(s, t) \in \mathbb{D}^2$.

4.1. Proposition. *Let $u : \mathbb{R} \times \mathbb{R}/\ell\mathbb{Z} \xrightarrow{C^1} \mathbb{D}^2$ be a solution of the Cauchy–Riemann equations $\partial_{J,H}(u) = 0$, then there is a uniform constant $C = C(\alpha, J, H)$, independent of u, such that*

$$|\partial^\alpha u(s,t)| \le C(\alpha, J, H), \quad \forall (s,t) \in \mathbb{R} \times \mathbb{R}/\ell\mathbb{Z}, \tag{4.2.1}$$

where $\alpha = (\alpha_s, \alpha_t)$ is any multi-index and $\partial^\alpha u = \partial_s^{\alpha_s} \partial_t^{\alpha_t} u$.

4.2. Remark. Consider almost complex structures $J = J(s)$ and Hamiltonians $H = H(s, t, x)$ with the property that $J(s)$ and $H(s, t, x)$ are independent of s for $|s| \ge R$, then the same a priori estimates hold for the associated non-autonomous Cauchy–Riemann equations

$$u_s - J(s)u_t - \nabla_{g_s} H(s, t, u) = 0,$$

where $g_s(\cdot, \cdot) = \omega_0(\cdot, J(s)\cdot)$.

Proof. In this proof, the constant C changes from line to line. Define the operators

$$\partial_J = \frac{\partial}{\partial s} - J\frac{\partial}{\partial t}, \quad \bar{\partial}_J = \frac{\partial}{\partial s} + J\frac{\partial}{\partial t}.$$

Eq. (2.1.3) can now be written as $\partial_J u = \nabla_g H(t, u) \stackrel{\text{def}}{=} f(s, t)$. By the hypotheses on H and the fact that $|u(s, t)| \le 1$, we have that $f(s, t) \in L^\infty(\mathbb{R}^2)$ and $\|f\|_{L^\infty} \le C(J, H)$. This follows from the fact that the solutions u can be regarded as functions on $\mathbb{R}^2$ via periodic extension in t. From the interior regularity estimates due to Douglis and Nirenberg for elliptic systems [10], and in particular the operators ∂_J and $\bar{\partial}_J$ we have the following L^p-estimates for functions $u \in W_0^{k+1,p}(B_1(0))$, $1 < p < \infty$ and $k \ge 0$:

$$\|u\|_{W_0^{k+1,p}} \le C(k, p, J)\|\partial_J u\|_{W^{k,p}}, \quad \|u\|_{W_0^{k+1,p}} \le C(k, p, J)\|\bar{\partial}_J u\|_{W^{k,p}}, \tag{4.2.2}$$

which also follow from L^p-estimates for the Laplacian $\Delta = \bar{\partial}_J \partial_J = \partial_J \bar{\partial}_J$, cf. [19,33]. Using a partition of unity we derive the standard interior regularity estimates for the Cauchy–Riemann operator from the above interior estimates, e.g. [33]. Let $K \subset\subset G \subset \mathbb{R}^2$, with K, G compact domains, then

$$\|u\|_{W^{1,p}(K)} \le C(p, J, K, G)\Big(\|\partial_J u\|_{L^p(G)} + \|u\|_{L^p(G)}\Big), \tag{4.2.3}$$

for $1 < p < \infty$. Indeed, let $\epsilon < \text{dist}(K, \partial G)$, then the compact set K can be covered by balls $B_{\epsilon/2}(x_0)$ for finitely many $x_0 \in K$. Furthermore, let $\{\omega_{\epsilon,x_0}\}_{x_0}$ be a partition of unity of

$\bigcup_{x_0} B_{\epsilon/2}(x_0) \supset K$ subordinate to $\{B_\epsilon(x_0)\}_{x_0}$. Recall that $\|fg\|_{L^p} \le \|f\|_{L^\infty}\|g\|_{L^p}$, which yields the basic estimate on $B_\epsilon(x_0)$:

$$\begin{aligned}\|\omega_{\epsilon,x_0}u\|_{W^{1,p}} &= \|\omega_{\epsilon,x_0}u\|_{W_0^{1,p}} \le C\|\partial_J(\omega_{\epsilon,x_0}u)\|_{L^p}\\ &\le C\|\omega_{\epsilon,x_0}\partial_J u\|_{L^p} + C\|u\partial_J\omega_{\epsilon,x_0}\|_{L^p} \le C\|\partial_J u\|_{L^p(G)} + C\|u\|_{L^p(G)}.\end{aligned}$$

Since $\{\omega_{\epsilon,x_0}\}_{x_0}$ is a partition of unity it follows that

$$\|u\|_{W^{1,p}(K)} = \Big\|\sum_{x_0}\omega_{\epsilon,x_0}u\Big\|_{W^{1,p}(K)} \le \sum_{x_0}\|\omega_{\epsilon,x_0}u\|_{W^{1,p}(B_\epsilon(x_0))},$$

which proves (4.2.3). For an extensive survey of compactness for J-holomorphic curves see [26].

We apply these basic regularity estimates to the nonlinear Cauchy–Riemann equation (2.1.3) to obtain a priori estimates on the Hölder norm. We consider nested sets $G^j_{S,T} = [S-j, S+j+1] \times [T-j, T+j+1]$ for $j = 0, 1, 2$. Note that S and T just represent shifts in the variables. Since the estimates are autonomous, while S and T are arbitrary, the estimates will be independent of S and T, and we will drop them from the notation. Choose $K = G^1 \subset\subset G^2 = G$. It holds that $\|f\|_{L^p(G^2)} \le C(p,H)$, since $f \in L^\infty$. Similarly, $\|u\|_{L^p(G^2)} \le C(p)$ by the assumption that $|u(s,t)| \le 1$. Therefore,

$$\begin{aligned}\|u\|_{W^{1,p}(G^1)} &\le C(p,J)\Big(\|\partial_J u\|_{L^p(G^2)} + \|u\|_{L^p(G^2)}\Big)\\ &= C(p,J)\Big(\|f\|_{L^p(G^2)} + \|u\|_{L^p(G^2)}\Big) \le C(p,J,H).\end{aligned}$$

In order to further bootstrap the regularity of solutions we argue as follows. Recall that $\|fg\|_{W^{1,p}} \le C\big(\|f\|_{W^{1,p}}\|g\|_{L^\infty} + \|g\|_{W^{1,p}}\|f\|_{L^\infty}\big)$. As before, on balls $B_\epsilon(x_0)$ we obtain

$$\begin{aligned}\|\omega_{\epsilon,x_0}u\|_{W^{2,p}} &= \|\omega_{\epsilon,x_0}u\|_{W_0^{2,p}} \le C\|\partial_J(\omega_{\epsilon,x_0}u)\|_{W^{1,p}}\\ &\le C\|\omega_{\epsilon,x_0}\partial_J u\|_{W^{1,p}} + C\|u\partial_J\omega_{\epsilon,x_0}\|_{W^{1,p}}\\ &\le C\|\partial_J u\|_{L^\infty(G)} + C\|\partial_J u\|_{W^{1,p}(G)} + C\|u\|_{L^\infty(G)} + C\|u\|_{W^{1,p}(G)}.\end{aligned}$$

Since $\{\omega_{\epsilon,x_0}\}_{x_0}$ is a partition of unity we obtain the estimate

$$\|u\|_{W^{2,p}(K)} \le C(p,J,K,G)\Big(\|\partial_J u\|_{W^{1,p}(G)} + \|u\|_{W^{1,p}(G)} + \|\partial_J u\|_{L^\infty(G)} + \|u\|_{L^\infty(G)}\Big),$$

for compact domains $K \subset\subset G$, and $1 < p < \infty$. Now choose $K = G^0 \subset\subset G^1 = G$. In order to estimate the term $\|\partial_J u\|_{W^{1,p}(G^1)}$ in the above inequality, observe that $\partial_J u = \nabla_g H(t,u)$. Then, by the $W^{1,p}$-interior estimates, the components

$$f_s(s,t) \overset{\text{def}}{=} (\nabla_g H(t,u))_s = d_{t,u}\nabla_g H(t,u)(0,u_s),$$

$$f_t(s,t) \overset{\text{def}}{=} (\nabla_g H(t,u))_t = d_{t,u}\nabla_g H(t,u)(1,u_t),$$

both lie in $L^p(G^1)$ and thus also $Df = (f_s, f_t)$ lies in $L^p(G^1)$. Moreover, $\|Df\|_{L^p(G^1)} \leq C(p, J, H)$, hence $\|\partial_J u\|_{L^p(G^1)} \leq C(p, J, H)$. From the $W^{2,p}$-interior estimates it then follows that

$$\|u\|_{W^{2,p}(G^0)} \leq C(p, J)\Big(\|\partial_J u\|_{L^p(G^1)} + \|u\|_{W^{1,p}(G^1)} + \|f\|_{L^\infty(G^1)} + \|u\|_{L^\infty(G^1)}\Big) \leq C(p, J, H).$$

Additional regularity is obtained from the Sobolev embeddings [2, Ch. 5], $W^{2,p}(G^0) \to C^1(G^0)$ for $p > 2$, and thus $\|u\|_{C^1([S,S+1]\times[T,T+1])} \leq C(J, H)$. Since the estimate is independent of S and T we obtain the a priori estimate

$$|u_s(s,t)| + |u_t(s,t)| \leq C(J, H). \tag{4.2.4}$$

In order to get bounds on the higher derivative we differentiate the equation. The derivatives satisfy the equations

$$\partial_J u_s = \big(\nabla_g H(t,u)\big)_s, \quad \text{and} \quad \partial_J u_t = \big(\nabla_g H(t,u)\big)_t.$$

By the a priori estimates on u, u_s and u_t, the right hand sides of the above equations are uniformly bounded and we can apply the previous estimates to the equations for u_s and u_t. This yields a priori bounds on the second derivatives $D^2 u$. By iterating this process we obtain a priori bounds on all derivatives of any order, which completes the proof. □

4.3. Corollary. *Let* $u : \mathbb{R} \times \mathbb{R}/\ell\mathbb{Z} \xrightarrow{C^1} \mathbb{D}^2$ *be a solution of the Cauchy–Riemann equations* $\partial_{J,H}(u) = 0$, *then there is a uniform constant* $C = C(\ell, J, H)$ *such that*

$$\int_{\mathbb{R}} \int_0^\ell |u_s|_g^2 dt ds \leq 2C, \tag{4.2.5}$$

and the action satisfies $\left|\int_0^\ell \theta(u(s,t))dt\right| \leq C$.

Proof. Due to the a priori bounds on the derivatives of u it holds that $\left|\int_0^\ell \theta(u(s,t))dt\right| \leq C(\ell, J, H)$ and since

$$\frac{d}{ds}\int_0^\ell \theta(u(s,t))dt = -\int_0^\ell |u_s|_g^2 dt \leq 0,$$

it follows that the limits $\lim_{s\to\pm\infty}\int_0^\ell \theta(u(s,t))dt = c_\pm$ exist and are a priori bounded by the same constant $C(\ell, J, H)$. Finally, for any $T, T' > 0$,

$$\int_{-T}^{T'} \int_0^\ell |u_s|_g^2 dt ds = \int_0^\ell \theta(u(T',t))dt - \int_0^\ell \theta(u(-T,t))dt.$$

By the uniform boundedness of the action along all orbits u we obtain the estimate $\int_{\mathbb{R}}\int_0^{\ell} |u_s|_g^2 dtds \le 2C(\ell, J, H)$, which completes the proof. □

4.3. Compactness for parametric families

Consider the non-autonomous Cauchy–Riemann equations:

$$u_s - J(s)u_t - \nabla_{g_s} H(s,t,u) = 0, \tag{4.3.1}$$

where $s \mapsto J(s,\cdot)$ is a smooth path in $\mathscr{J}$ and $s \mapsto H(s,\cdot,\cdot)$ is a smooth path in $\mathscr{H}$. Both paths are assumed to have the property that the limits as $s \to \pm\infty$ exist. The path of metrics $s \mapsto g_s$ is defined via the relation $g_s(\cdot,\cdot) = \omega_0(\cdot, J(s,\cdot)\cdot)$. Assume that $|H_s| \le \kappa(s) \to 0$ as $s \to \pm\infty$ uniformly in $(t,x) \in \mathbb{R}/\mathbb{Z} \times \mathbb{D}^2$, with $\kappa \in L^1(\mathbb{R})$. For the equation $\partial_J v = f(s,t)$, the analogue of Proposition 4.1 holds via the L^∞-estimates on the right hand side, see [33,34]. We sketch the main idea.

Define $s \mapsto \mathscr{L}_H(s,x)$ as the action path with Hamiltonian path $s \mapsto H(s,\cdot,\cdot)$. The first variation with respect to s can be computed as before:

$$\begin{aligned}
\frac{d}{ds}\mathscr{L}_H(s, \mathrm{U}(s,\cdot)) &= \frac{\partial \mathscr{L}_H}{\partial s} + \sum_{k=1}^{n} \int_0^1 \omega_0\big(u_t^k - X_H(t,u^k), u_s^k\big)dt \\
&= \frac{\partial \mathscr{L}_H}{\partial s} - \sum_{k=1}^{n} \int_0^1 |u_t^k - X_H(t,u^k)|_{g_s}^2 dt \\
&= \frac{\partial \mathscr{L}_H}{\partial s} - \int_0^1 |\mathrm{U}_s|_{g_s}^2 dt.
\end{aligned}$$

The partial derivative with respect to s is given by

$$\frac{\partial \mathscr{L}_H}{\partial s} = \sum_{k=1}^{n} \int_0^1 \frac{\partial H}{\partial s}\Big(s,t,u^k(s,t)\Big)dt,$$

and $\left|\frac{\partial \mathscr{L}_H}{\partial s}\right| \le C\kappa(s)$. It now follows from the inequality $\frac{d}{ds}\mathscr{L}_H(s,\mathrm{U}(s,\cdot)) - \frac{\partial}{\partial s}\mathscr{L}_H \le 0$ and the assumption $\kappa \in L^1(\mathbb{R})$ that the limits $\lim_{s\to\pm\infty} \mathscr{L}_H(s,\mathrm{U}(s,\cdot)) = c_\pm$ exist. Since $\kappa \in L^1(\mathbb{R})$ we also obtain the integral bound $\int_{\mathbb{R}}\int_0^1 |\mathrm{U}_s|_{g_s}^2 dtds \le C(J,H)$. This non-autonomous Cauchy–Riemann equation will be used to establish continuation for Floer homology.

5. Crossing numbers and the monotonicity principle

5.1. The crossing number

We begin with an important property of the (linear) Cauchy–Riemann equations in dimension two. We consider Eq. (2.1.3), or more generally Eq. (4.3.1), and local solutions of the form

$u : G \subset \mathbb{R}^2 \to \mathbb{R}^2$, where $G = [a, a'] \times [b, b']$. For two local solutions $u, u' : G \to \mathbb{R}^2$ of (2.1.3) assume that

$$u(s,t) \neq u'(s,t), \text{ for all } (s,t) \in \partial G.$$

Intersections of u and u', where $u(s_0, t_0) = u'(s_0, t_0)$ for some $(s_0, t_0) \in G$, have constrained evolutions. Consider the difference function $w(s,t) = u(s,t) - u'(s,t)$. By the assumptions on u and u' we have that $w|_{\partial G} \neq 0$, and intersections are given by $w(s_0, t_0) = 0$. The following lemma is a special feature of the Cauchy–Riemann equations in dimension two and is a manifestation of the well-known positivity of intersection of J-holomorphic curves in almost complex 4-manifolds [25,24].

5.1. Lemma. *Let u, u' and G be as defined above. Assume that $w(s_0, t_0) = 0$ for some $(s_0, t_0) \in G$. Then (s_0, t_0) is an isolated zero and* $\deg(w, G, 0) < 0$.

Proof. Taylor expand: $\nabla_g H(t, u') = \nabla_g H(t, u) + R_1(t, u, u' - u)(u' - u)$, where R_1 is smooth. Substitution yields

$$w_s - J(s)w_t - A(s,t)w = 0, \quad w(s_0, t_0) = 0,$$

where $A(s,t) = R_1(t, u, -w)$ is smooth on G. Define complex coordinates $z = s - s_0 + i(t - t_0)$ and identify the target space $\mathbb{R}^2$ with $\mathbb{C}$. Then by [22, Appendix A.6], there exist a $\delta > 0$, sufficiently small, a disc $D_\delta = \{z \mid |z| \leq \delta\}$, a holomorphic map $h : D_\delta \to \mathbb{C}$, and a continuous mapping $\Phi : D_\delta \to \mathrm{GL}_{\mathbb{R}}(\mathbb{R}^2)$ such that

$$\det \Phi(z) > 0, \qquad J(z)\Phi(z) = \Phi(z)i, \qquad w(z) = \Phi(z)\bar{h}(z),$$

for all $z \in D_\delta$. Clearly, Φ can be represented by a real 2×2 matrix function of invertible matrices.

Since $w = \Phi\bar{h}$ and Φ is invertible, it holds that the condition $w(z_0) = 0$ implies that $\bar{h}(z_0) = h(z_0) = 0$. The analyticity of h then implies that either z_0 is an isolated zero in D_δ, or $h \equiv 0$ on D_δ. If the latter holds, then also $w \equiv 0$ on D_δ. If we repeat the above arguments we conclude that $w \equiv 0$ on G (cf. analytic continuation), in contradiction with the boundary conditions. Therefore, all zeroes of w in G are isolated, and there are finitely many zeroes $z_i \in \mathrm{int}(G)$.

For the degree we have for small $\epsilon_i > 0$ that, since $\det \Phi(z) > 0$,

$$\deg(w, G, 0) = \sum_{i=1}^{m} \deg(w, B_{\epsilon_i}(z_i), 0) = \sum_{i=1}^{m} \deg(\bar{h}, B_{\epsilon_i}(z_i), 0) = -\sum_{i=1}^{m} \deg(h, B_{\epsilon_i}(z_i), 0),$$

and for an analytic function with an isolated zero z_i it holds that $\deg(h, B_{\epsilon_i}(z_i), 0) = n_i \geq 1$; thus $\deg(w, G, 0) < 0$. $\square$

For a curve $\Gamma : I \to \mathbb{R}^2 \setminus \{(0,0)\}$, with I a bounded interval, one can define the WINDING NUMBER about the origin by

$$W(\Gamma, 0) \stackrel{\text{def}}{=} \frac{1}{2\pi} \int_I \Gamma^* \alpha = \frac{1}{2\pi} \int_\Gamma \alpha,$$

for $\alpha = (-qdp + pdq)/(p^2 + q^2)$. In particular, for curves $w(s, \cdot) : [b, b'] \to \mathbb{R}^2 \setminus \{(0, 0)\}$ and for $s = a, a'$ the (local) winding number is

$$W(w(s,\cdot),0) \stackrel{\text{def}}{=} \frac{1}{2\pi} \int_{[b,b']} w^*\alpha = \frac{1}{2\pi} \int_w \alpha.$$

We denote these winding numbers by $W_a^{[b,b']}(w)$ and $W_{a'}^{[b,b']}(w)$ respectively. In the case that $[b, b'] = [0, 1]$ we simply write $W_a(w) \stackrel{\text{def}}{=} W_a^{[0,1]}(w)$. Similarly, we have winding numbers for the curves $w(\cdot, t) : [a, a'] \to \mathbb{R}^2 \setminus \{(0, 0)\}$ and for $t = b, b'$, which we denote by $W^b_{[a,a']}(w)$ and $W^{b'}_{[a,a']}(w)$ respectively. These local winding numbers are related to the degree of the map $w : G \to \mathbb{R}^2$. The following lemma is a direct consequence of the definitions above.

5.2. Lemma. *Let $u, u' : G \to \mathbb{R}^2$ be solutions of Eq. (4.3.1). Let w and G be as above, with $w(s_0, t_0) = 0$ and $w|_{\partial G} \neq 0$. Then*

$$\left[W_{a'}^{[b,b']}(w) - W_a^{[b,b']}(w)\right] - \left[W^{b'}_{[a,a']}(w) - W^b_{[a,a']}(w)\right] = \deg(w, G, 0). \tag{5.1.1}$$

In particular, for each zero $(s_0, t_0) \in \operatorname{int}(G)$, there exists an $\epsilon_0 > 0$ such that

$$W_{s_0+\epsilon}^{[b,b']}(w) - W_{s_0-\epsilon}^{[b,b']}(w) < W^{b'}_{[s_0-\epsilon,s_0+\epsilon]}(w) - W^b_{[s_0-\epsilon,s_0+\epsilon]}(w),$$

for all $0 < \epsilon \leq \epsilon_0$.

From Lemma 5.2 we can also derive the following a priori estimate for solutions of the Cauchy–Riemann equations.

5.3. Proposition. *Let $u : G \to \mathbb{D}^2$ be a solution of Eq. (4.3.1), then either*

$$|u(s,t)| = 1, \qquad \text{or} \qquad |u(s,t)| < 1,$$

for all $(s, t) \in G$.

Proof. By assumption, the boundary of the disc is invariant for X_H and thus consists of solutions $x(t)$ with $|x(t)| = 1$. Assume that $u(s_0, t_0) = x(t_0)$ for some (s_0, t_0) and some boundary trajectory $x(t)$. For convenience, we write $u'(s, t) = x(t)$ and we consider the difference $w(s, t) = u'(s, t) - u(s, t) = x(t) - u(s, t)$. By the arguments presented in the proof of Lemma 5.1, we know that either all zeroes of w are isolated, or $w \equiv 0$. In the latter case $u \equiv x$, hence $|u(s, t)| \equiv 1$.

Consider the remaining possibility, namely that (s_0, t_0) is an isolated zero of w, which leads to a contradiction. Indeed, choose a (small) rectangle $G = [\sigma, \sigma'] \times [\tau, \tau']$ containing (s_0, t_0), such that $w|_{\partial G} \neq 0$. With $\gamma = \partial G$ positively oriented, we derive from Lemma 5.2 that

$$W(w(\gamma), 0) = \deg(w, G, 0) \leq -1.$$

The latter is due to the assumption that G contains a zero. Consider on the other hand the loops $u(\gamma)$ and $u'(\gamma)$. By assumption $(u' - u)(\gamma) = w(\gamma) \neq 0$ and $|(u' - w)(\gamma)| = |u(\gamma)| \leq |u'(\gamma)| = 1$.

If we now apply the 'Dog-on-a-Leash' Lemma[2] from the theory of winding numbers [16], we conclude that $-1 \geq W(w(\gamma), 0) = W(u'(\gamma), 0) = 0$, which contradicts the assumption that u touches $\partial\mathbb{D}^2$. Hence $|u(s,t)| < 1$ for all (s,t). $\square$

5.2. *The monotonicity principle*

On the level of comparing two local solutions of Eq. (2.1.3), the winding number behaves like a discrete Lyapunov function with respect to the time variable s. This can be further formalized for solutions of the Cauchy–Riemann equations. For a closed braid $\mathrm{x} \in \Gamma F_n$ and an arbitrary lift $\mathrm{x} = (x^1, \cdots, x^n)$, one defines the TOTAL CROSSING NUMBER

$$\mathrm{Cross}(\mathrm{x}) \stackrel{\text{def}}{=} \sum_{k \neq k'} W\big(x^k - x^{k'}, 0\big) = 2 \sum_{\substack{\{k,k'\} \\ k \neq k'}} W\big(x^k - x^{k'}, 0\big), \tag{5.2.1}$$

where the second sum is over all unordered pairs $\{k, k'\}$, using the fact that the winding number is invariant under the inversion $(p,q) \to (-p,-q)$. Here we use the abbreviated notation $W = W^{[0,1]}$ (there is no s-dependence). The number $\mathrm{Cross}(\mathrm{x})$ is equal to the total linking/self-linking number of all components in a closed braid x. The local winding number as introduced above is not necessarily an integer. However, for closed curves the winding number is integer valued. It is clear that the number $\mathrm{Cross}(\mathrm{x})$ as defined above is also an integer, one interpretation of which is via the associated braid diagrams as the ALGEBRAIC CROSSING NUMBER:

5.4. Lemma. *The number* $\mathrm{Cross}(\mathrm{x})$ *is an integer, and*

$$\mathrm{Cross}(\mathrm{x}) = \#\{\textit{positive crossings}\} - \#\{\textit{negative crossings}\}.$$

This is a braid class invariant; i.e., $\mathrm{Cross}(\mathrm{x}) = \mathrm{Cross}(\mathrm{x}')$ *for all* $\mathrm{x}, \mathrm{x}' \in [\mathrm{x}]$. *See Fig. 2.2.1 for the geometric meaning of positive and negative crossings.*

This result is standard and a proof is left to the reader.

Let $\mathrm{u}(s,t) = \big(u^1(s,t), \ldots, u^k(s,t)\big) : \mathbb{R} \times [0,1] \to \mathbb{D}^2 \times \cdots \times \mathbb{D}^2$ with the periodicity conditions

$$u^k(s,1) = u^{\sigma(k)}(s,0), \quad \forall s, \text{ and } \forall k = 1, \ldots, n, \tag{5.2.2}$$

for some permutation $\sigma \in S_n$. This implies that $\mathrm{u}(s,\cdot) \in \mathrm{cl}(\Gamma F_n)$ for all $s \in \mathbb{R}$.

5.5. Definition. A smooth mapping $\mathrm{u}(s,t) = (u^1(s,t), \ldots, u^k(s,t)) : \mathbb{R} \times [0,1] \to \mathbb{D}^2 \times \cdots \times \mathbb{D}^2$, satisfying the periodicity condition (5.2.2), is said to satisfy the Cauchy–Riemann equations if $\partial_{J,H}(u^k) = 0$ for all $k = 1, \ldots, n$. The space of such mappings will be denoted by $\mathscr{M}\big(\mathrm{cl}(\Gamma F_n)\big)$.

[2] Roughly, if two closed planar paths $\Gamma(t)$ (dog) and $\Gamma'(t)$ (walker) satisfy $|\Gamma'(t) - \Gamma(t)| < |\Gamma'(t) - 0|$ (i.e., the leash is shorter than the walkers distance to the origin) then $W(\Gamma, 0) = W(\Gamma', 0)$.

For $\mathrm{U} \in \mathscr{M}\big(\mathrm{cl}(\Gamma F_n)\big)$, note that $\mathrm{U}(s,\cdot)$ is not necessarily in ΓF_n for all s. If $\mathrm{U}(s,\cdot) \in \Gamma F_n$, then the crossing number $\mathrm{Cross}(\mathrm{U}(s,\cdot))$ is well-defined.

5.6. Definition. A smooth mapping $\mathrm{U} : \mathbb{R} \times [0,1] \to F_n(\mathbb{D}^2)$ has the property that $\mathrm{U}(s,\cdot) \in \Gamma F_n$ for all $s \in \mathbb{R}$. The space of such mappings that satisfy the Cauchy–Riemann equations is denoted by $\mathscr{M}(\Gamma F_n)$.

Using the representation of the crossing number for a braid in terms of winding numbers, we can prove a Lyapunov property, which is the crucial step in setting up a Floer theory for braid classes.

5.7. Proposition *(Monotonicity Principle). Let $\mathrm{U} \in \mathscr{M}\big(\mathrm{cl}(\Gamma F_n)\big)$. If $\mathrm{U}(s_0,\cdot) \in \Sigma$, then either there exists an $\epsilon_0 > 0$ such that $\mathrm{U}(s_0 \pm \epsilon,\cdot) \in \Gamma F_n$, for all $0 < \epsilon \le \epsilon_0$, and*

$$\mathrm{Cross}(\mathrm{U}(s_0 - \epsilon,\cdot)) > \mathrm{Cross}(\mathrm{U}(s_0 + \epsilon,\cdot))$$

or $\mathrm{U} \in \Sigma^-$, i.e. $u^k \equiv u^{k'}$ for at least one pair $k \neq k'$.

Proof. We use the notation from the proof of Lemma 5.4. Since $\mathrm{U}(s_0,\cdot) \in \Sigma$, there are $k \neq k'$ and $t_0 \in [0,1]$ such that $u^k(s_0,t_0) = u^{k'}(s_0,t_0)$ for at least one class π_j. As in the proof of Lemma 5.4 we define $w_{\pi_j}(s,t) = u^k(s,t) - u^{k'}(s,t)$ for some representative $\{k,k'\} \in \pi_j$. From the proof of Lemma 5.1 we know that (s_0,t_0) is either isolated, or $u^k \equiv u^{k'}$. In the case that (s_0,t_0) is an isolated zero there exists an $\epsilon_0 > 0$, such that (s_0,t_0) is the only zero in $[s_0-\epsilon, s_0+\epsilon] \times [t_0-\epsilon, t_0+\epsilon]$, for all $0 < \epsilon \le \epsilon_0$. Furthermore, by isolation we can choose ϵ_0 sufficiently small to ensure that $\mathrm{Cross}(\mathrm{U}(s,\cdot))$ is well-defined for all $0 < |s - s_0| \le \epsilon_0$. By periodicity it holds that $w_{\pi_j}(s, t+2|\pi_j|) = w_{\pi_j}(s,t)$, for all $(s,t) \in \mathbb{R}^2$, and therefore $W^{t_0-\epsilon+2|\pi_j|}_{[s_0-\epsilon,s_0+\epsilon]}(w_{\pi_j}) = W^{t_0-\epsilon}_{[s_0-\epsilon,s_0+\epsilon]}(w_{\pi_j})$. From Lemma 5.2 it then follows that we have strict inequality

$$W^{[t_0-\epsilon,t_0-\epsilon+2|\pi_j|]}_{s_0-\epsilon}(w_{\pi_j}) > W^{[t_0-\epsilon,t_0-\epsilon+2|\pi_j|]}_{s_0+\epsilon}(w_{\pi_j}). \tag{5.2.3}$$

Such terms make up the expression for $\mathrm{Cross}(\mathrm{U}(s,\cdot))$ in Eq. (5.2.1), hence we obtain the desired inequality (since we have (5.2.3) with non-strict inequality for all other classes $\pi_{j'}$). □

6. Isolation and compactness

6.1. Spaces of bounded solutions

Let Y be a skeleton for a Hamiltonian systems generated by a Hamiltonian H and let $[\mathrm{X}\ \mathrm{rel}\ \mathrm{Y}] \subset \Gamma F_{n,m}$ be a proper relative braid class. Following Floer [13] we define the set of bounded solutions, or moduli spaces of the Cauchy–Riemann equation inside a proper relative braid class fiber $[\mathrm{X}]\ \mathrm{rel}\ \mathrm{Y}$ by

$$\mathscr{M}\big([\mathrm{X}]\ \mathrm{rel}\ \mathrm{Y}; J, H\big) \stackrel{\mathrm{def}}{=} \Big\{\mathrm{U}\ \mathrm{rel}\ \mathrm{Y} \in \mathscr{M}(\Gamma F_n) \mid \mathrm{U}(s,\cdot)\ \mathrm{rel}\ \mathrm{Y} \in [\mathrm{X}]\ \mathrm{rel}\ \mathrm{Y},\ \forall\, s \in \mathbb{R}\Big\}.$$

Note that since Y is stationary, U rel Y may be regarded as a path in $\Gamma F_{n,m}$. We are also interested in the paths traversed (as a function of s) by these bounded solutions in phase space. Hence we define

$$\mathscr{S}\big([\mathrm{X}] \text{ rel } \mathrm{Y}; J, H\big) \stackrel{\text{def}}{=} \Big\{\mathrm{X} \text{ rel } \mathrm{Y} = \mathrm{U}(0,\cdot) \text{ rel } \mathrm{Y} \mid \mathrm{U} \text{ rel } \mathrm{Y} \in \mathscr{M}\big([\mathrm{X}] \text{ rel } \mathrm{Y}; J, H\big)\Big\}.$$

If there is no ambiguity about the relative braid class, or J and H, we write $\mathscr{M}$ and $\mathscr{S}$. The space $\mathscr{M}$ can be equipped with the topology induced by compact-open C^r topology of mappings $\mathbb{R}\times[0,1]\to\mathbb{D}^2\times\cdots\times\mathbb{D}^2$, for any $r\geq 0$. The topology on $\mathscr{S}$ is induced by the compact-open C^r topology (strong C^r topology) in $C^r([0,1];\mathbb{D}^2\times\cdots\times\mathbb{D}^2)$.

6.2. *Isolation*

From Proposition 5.3 we derive that for proper braid classes, bounded solutions in $\mathscr{M}([\mathrm{X}] \text{ rel } \mathrm{Y})$ have modulus less than one, i.e. for every U rel $\mathrm{Y}\in\mathscr{M}([\mathrm{X}] \text{ rel } \mathrm{Y})$ it holds that $|\mathrm{U}(s,t)|<1$, since braids in $[\mathrm{X}]$ rel Y cannot be contracted onto the boundary.

6.1. Corollary. *For any* U rel $\mathrm{Y}\in\mathscr{M}([\mathrm{X}] \text{ rel } \mathrm{Y})$*, it holds that* $|\mathrm{U}(s,t)|<1$*, for all* $(s,t)\in\mathbb{R}\times\mathbb{R}/\mathbb{Z}$*.*

This implies that solutions in $\mathscr{M}$ are supported in $\mathrm{int}(\mathbb{D}^2)$. Next we combine this property with the monotonicity principle and the compactness of bounded solutions of the Cauchy–Riemann equations to formulate the main compactness statement for $\mathscr{M}([\mathrm{X}] \text{ rel } \mathrm{Y})$.

6.3. *Compactness*

The following compactness statement is the culmination of the previous properties and results for solutions of the Cauchy–Riemann equations on proper relative braid classes.

6.2. Proposition. *For any fiber* $[\mathrm{X}]$ rel Y *of a proper relative braid class* $[\mathrm{X} \text{ rel } \mathrm{Y}]$*, the set* $\mathscr{M}\big([\mathrm{X}] \text{ rel } \mathrm{Y}\big)$ *is compact and* $|\mathrm{U}(s,t)|<1$*, for any* U rel $\mathrm{Y}\in\mathscr{M}$*.*

Proof. Let $\{\mathrm{U}_i \text{ rel } \mathrm{Y}\}\subset\mathscr{M}\big([\mathrm{X}] \text{ rel } \mathrm{Y}\big)$ be a sequence of solutions. The extensions $n!\cdot\mathrm{U}_i$ are smooth mappings, with $n!\cdot\mathrm{U}_i\in\Lambda F_n$, and the components $u_i^k:\mathbb{R}\times\mathbb{R}/n!\mathbb{Z}\to\mathbb{D}^2$ satisfy the Cauchy–Riemann equations. By construction they satisfy the periodicity condition

$$(n!\cdot\mathrm{U}_i)(s,t+1)=\big((n!\cdot\mathrm{U})(s,t)\big)\cdot\sigma, \tag{6.3.1}$$

for all s,t. To the components u_i^k we can apply the a priori estimates in Proposition 4.1. These imply the existence of subsequences $u_{i'}^k\to u^k$, as $i'\to\infty$ in $C^r(I\times\mathbb{R}/n!\mathbb{Z};\mathbb{D}^2)$, for any compact interval $I\subset\mathbb{R}$. The limiting functions $u^k:\mathbb{R}\times\mathbb{R}/n!\mathbb{Z}\to\mathbb{D}^2$ satisfy the Cauchy–Riemann equations and the periodicity condition (6.3.1). Therefore, U rel Y, with $\mathrm{U}=n!\cdot\mathrm{U}|_{[0,1]}$, is a smooth solution to the Cauchy–Riemann equations in the sense of Definition 5.5, with $\mathrm{U}(s,0)$ rel $\mathrm{Y}\in\mathrm{cl}([\mathrm{X}] \text{ rel } \mathrm{Y})$, for all $s\in\mathbb{R}$. The remainder of the proof is to show that $\mathrm{U}(s,\cdot)$ is in the relative braid class $[\mathrm{X}]$ rel Y, by eliminating the possible boundary behaviors.

If $|u^k(s_0,t_0)|=1$, for some (s_0,t_0) and k, then Proposition 5.3 implies that $|u^k|\equiv 1$, hence $|u^k_{i'}|\to 1$ as $i'\to\infty$ uniformly on compact sets in (s,t). This contradicts the fact that $[\mathrm{X}]$ rel Y is proper, and therefore the limit satisfies $|\mathrm{U}(s,t)|<1$, for all s,t.

If $u^k(s_0,t_0)=u^{k'}(s_0,t_0)$ for some (s_0,t_0) and some pair $k\neq k'$, then by Proposition 5.7 either there exists an $\epsilon_0>0$ such that $\mathrm{U}(s_0\pm\epsilon,\cdot)$ rel $\mathrm{Y}\in\Gamma F_n$ rel Y and $\mathrm{Cross}(\mathrm{U}(s_0-\epsilon,\cdot)\text{ rel }\mathrm{Y})>\mathrm{Cross}(\mathrm{U}(s_0+\epsilon,\cdot)\text{ rel }\mathrm{Y})$, for all $0<\epsilon\le\epsilon_0$, or $u^k\equiv u^{k'}$, for at least one pair $k\neq k'$. The former case will be dealt with below. In the latter case the limit has a collapsed singularity, i.e. U rel $\mathrm{Y}\in\Sigma_-[\mathrm{X}]$ rel Y, contradicting the fact that $[\mathrm{X}]$ rel Y is proper.

If $u^k(s_0,t_0)=y^{k'}(t_0)$ for some (s_0,t_0) and k and some $y^{k'}\in\mathrm{Y}$, then by Proposition 5.7 either there exists an $\epsilon_0>0$ such that $\mathrm{U}(s_0\pm\epsilon,\cdot)$ rel $\mathrm{Y}\in\Gamma F_n$ rel Y and $\mathrm{Cross}(\mathrm{U}(s_0-\epsilon,\cdot)\text{ rel }\mathrm{Y})>\mathrm{Cross}(\mathrm{U}(s_0+\epsilon,\cdot)\text{ rel }\mathrm{Y})$, for some $0<\epsilon\le\epsilon_0$, or $u^k\equiv y^{k'}$. Again, the former case will be dealt with below. In the latter case U rel $\mathrm{Y}\in\Sigma_-[\mathrm{X}]$ rel Y, contradicting the fact that X rel Y is proper.

Finally, the remaining two possibilities imply that both $\mathrm{U}(s_0-\epsilon,\cdot)$ rel Y, $\mathrm{U}(s_0+\epsilon,\cdot)$ rel $\mathrm{Y}\in\Gamma F_n$ rel Y, for all $0<\epsilon\le\epsilon_0$, and thus $\mathrm{U}(s_0-\epsilon,\cdot)$ rel Y, $\mathrm{U}(s_0+\epsilon,\cdot)$ rel $\mathrm{Y}\in[\mathrm{X}]$ rel Y (using the fact that $\mathrm{U}(s,\cdot)$ rel $\mathrm{Y}\in\mathrm{cl}([\mathrm{X}]\text{ rel }\mathrm{Y})$ for all s). This implies that

$$\mathrm{Cross}(\mathrm{U}(s_0-\epsilon,\cdot)\text{ rel }\mathrm{Y})=\mathrm{Cross}(\mathrm{U}(s_0+\epsilon,\cdot)\text{ rel }\mathrm{Y}),$$

for all $0<\epsilon\le\epsilon_0$. This contradicts the strict inequalities for the crossing numbers given above and therefore $\mathrm{U}(s,\cdot)$ rel $\mathrm{Y}\in[\mathrm{X}]$ rel Y, for all $s\in\mathbb{R}$, which proves that $\mathscr{M}\big([\mathrm{X}]\text{ rel }\mathrm{Y}\big)$ is compact and $|\mathrm{U}(s,t)|<1$ for all s,t. □

Proposition 6.2 implies an isolating property for the compact set $\mathscr{S}([\mathrm{X}]\text{ rel }\mathrm{Y})$:

$$\mathscr{S}([\mathrm{X}]\text{ rel }\mathrm{Y})\subset\big\{\mathrm{X}'\text{ rel }\mathrm{Y}\in[\mathrm{X}]\text{ rel }\mathrm{Y}\mid|\mathrm{X}'|<1\big\}.$$

Crucial to the construction of Floer homology are the connecting orbit spaces: for $\mathrm{X}_\pm$ rel $\mathrm{Y}\in[\mathrm{X}]$ rel Y we define

$$\begin{aligned}&\mathscr{M}(\mathrm{X}_-\text{ rel }\mathrm{Y},\mathrm{X}_+\text{ rel }\mathrm{Y})=\\&\qquad\Big\{\mathrm{U}\text{ rel }\mathrm{Y}\in\mathscr{M}\big(\mathrm{cl}(\Gamma F_n\text{ rel }\mathrm{Y})\big)\mid\lim_{s\to\pm\infty}\mathrm{U}(s,\cdot)\text{ rel }\mathrm{Y}=\mathrm{X}_\pm\text{ rel }\mathrm{Y}\in[\mathrm{X}]\text{ rel }\mathrm{Y}\Big\}.\end{aligned}\tag{6.3.2}$$

By continuity it holds that $\mathrm{U}(s,\cdot)$ rel $\mathrm{Y}\in[\mathrm{X}]$ rel Y for all $|s|\gg 1$, sufficiently large, and therefore $\mathrm{Cross}(\mathrm{U}(s,\cdot)\text{ rel }\mathrm{Y})=\mathrm{Cross}(\mathrm{X}_\pm\text{ rel }\mathrm{Y})$ for $|s|\gg 1$. If $\mathrm{U}(s_0,\cdot)$ rel $\mathrm{Y}\notin\Gamma F_n$ rel Y for some s_0, then Proposition 5.7 implies that

$$\mathrm{Cross}(\mathrm{X}_-\text{ rel }\mathrm{Y})>\mathrm{Cross}(\mathrm{U}(s_0,\cdot)\text{ rel }\mathrm{Y})>\mathrm{Cross}(\mathrm{X}_+\text{ rel }\mathrm{Y}),$$

which is a contradiction. Consequently, $\mathrm{U}(s,\cdot)$ rel $\mathrm{Y}\in\Gamma F_n$ rel Y, for $s\in\mathbb{R}$ and in the same path component $[\mathrm{X}]$ rel Y. This yields the following corollary.

6.3. Corollary. *For any pair of proper braids* X_- rel Y, X_+ rel $\mathrm{Y}\in[\mathrm{X}]$ rel Y *it holds that* $\mathscr{M}(\mathrm{X}_-\text{ rel }\mathrm{Y},\mathrm{X}_+\text{ rel }\mathrm{Y})\subset\mathscr{M}([\mathrm{X}]\text{ rel }\mathrm{Y})$.

This in turn implies that

$$\bigcup_{\substack{\mathrm{x}_\pm \text{ rel } \mathrm{y} \in \\ \mathrm{Crit}_H([\mathrm{x}] \text{ rel } \mathrm{y})}} \mathscr{M}(\mathrm{x}_- \text{ rel } \mathrm{y}, \mathrm{x}_+ \text{ rel } \mathrm{y}) \subset \mathscr{M}([\mathrm{x}] \text{ rel } \mathrm{y}).$$

It is important to emphasize that, even though $\mathscr{M}([\mathrm{x}] \text{ rel } \mathrm{y})$ is compact, the connecting orbit spaces are not necessarily compact. In the following sections we will discuss the non-compactness of connecting orbits spaces in detail.

7. The Conley–Zehnder index for braids

The action $\mathscr{L}_H$ has the property that stationary braids have a doubly unbounded spectrum, i.e., if we consider the $d^2\mathscr{L}_H(\mathrm{x})$ as a stationary braid x, then $d^2\mathscr{L}_H(\mathrm{x})$ is a self-adjoint operator (on appropriately chosen function spaces) whose (real) spectrum consists of isolated eigenvalues unbounded from above or below. The classical Morse index for stationary braids is therefore not well-defined. The theory of the Maslov index for Lagrangian subspaces is used to replace the classical Morse index [13,30,31], via Fredholm theory. In this section we will recall the standard theory of Maslov indices as developed in [30] and we use these concepts to define an analogue of the Conley–Zehnder index suitable for stationary braids.

7.1. The Maslov index

Let (E, ω) be a (real) symplectic vector space of dimension $\dim E = 2n$, with compatible almost complex structure $J \in \mathrm{Sp}^+(E, \omega)$. An n-dimensional subspace $V \subset E$ is called LAGRANGIAN if $\omega(v, v') = 0$ for all $v, v' \in V$. Denote the space of Lagrangian subspaces of (E, ω) by $\mathcal{L}(E, \omega)$, or $\mathcal{L}$ for short.

It is well-known that a subspace $V \subset E$ is Lagrangian if and only if $V = \mathrm{range}(X)$ for some linear map $X : W \to E$ of rank n and some n-dimensional (real) vector space W, with X satisfying

$$X^T J X = 0, \tag{7.1.1}$$

where the transpose is defined via the inner product $\langle\cdot, \cdot\rangle \stackrel{\text{def}}{=} \omega(\cdot, J\cdot)$.

The map X is called a LAGRANGIAN FRAME for V. If we restrict to the special case $(E, \omega) = (\mathbb{R}^{2n}, \overline{\omega}_0)$, with standard J_0, then for a point x in $\mathbb{R}^{2n}$ one can choose symplectic coordinates $x = (p^1, \cdots, p^n, q^1, \cdots, q^n)$ and the standard symplectic form is given by $\overline{\omega}_0 = dp^1 \wedge dq^1 + \cdots + dp^n \wedge dq^n$. In this case a subspace $V \subset \mathbb{R}^{2n}$ is Lagrangian if $X = \begin{pmatrix} P \\ Q \end{pmatrix}$, with P, Q $n \times n$ matrices satisfying $P^T Q = Q^T P$, and X has rank n. The condition on P and Q follows immediately from Eq. (7.1.1).

For any fixed $V \in \mathcal{L}$, the space $\mathcal{L}$ can be decomposed into strata $\Xi_k(V)$:

$$\mathcal{L} = \bigcup_{k=0}^{n} \Xi_k(V).$$

The strata $\Xi_k(V)$ of Lagrangian subspaces V' which intersect V in a subspace of dimension k are submanifolds of co-dimension $k(k+1)/2$. The MASLOV CYCLE is defined as

$$\Xi(V) = \bigcup_{k=1}^{n} \Xi_k(V).$$

Let $\Lambda(t)$ be a smooth curve of Lagrangian subspaces and $X(t)$ a smooth Lagrangian frame for $\Lambda(t)$. A crossing is a number t_0 such that $\Lambda(t_0) \in \Xi(V)$, i.e., $X(t_0)w = v \in V$, for some $w \in W$, $0 \neq v \in V$. For a curve $\Lambda : [a,b] \to \mathcal{L}$, the set of crossings is compact, and for each crossing $t_0 \in [a,b]$ we can define the crossing form on $\Lambda(t_0) \cap V$:

$$\Gamma(\Lambda, V, t_0)(v) \stackrel{\text{def}}{=} \omega\big(X(t_0)w, X'(t_0)w\big).$$

A crossing t_0 is called REGULAR if Γ is of a nondegenerate form. If $\Lambda : [a,b] \to \mathcal{L}$ is a Lagrangian curve that has only regular crossings then the MASLOV INDEX of the pair (Λ, V) is defined by

$$\mu(\Lambda, V) = \frac{1}{2}\operatorname{sign}\Gamma(\Lambda, V, a) + \sum_{a<t_0<b} \operatorname{sign}\Gamma(\Lambda, V, t_0) + \frac{1}{2}\operatorname{sign}\Gamma(\Lambda, V, b),$$

where $\Gamma(\Lambda, V, a)$ and $\Gamma(\Lambda, V, b)$ are zero when a or b are not crossings. The notation 'sign' is the signature of a quadratic form, i.e. the number of positive minus the number of negative eigenvalues and the sum is over the crossings $t_0 \in (a,b)$. Since the Maslov index is homotopy invariant and every path is homotopic to a regular path the above definition extends to arbitrary continuous Lagrangian paths, using property (iii) below. In the special case of $(\mathbb{R}^{2n}, \overline{\omega}_0)$ we have that

$$\begin{aligned}\Gamma(\Lambda, V, t_0)(v) &= \overline{\omega}_0\big(X(t_0)w, X'(t_0)w\big)\\ &= \langle P(t_0)w, Q'(t_0)w\rangle - \langle P'(t_0)w, Q(t_0)w\rangle.\end{aligned}$$

A list of properties of the Maslov index can be found (and is proved) in [30], of which we mention the most important ones:

(i) for any $\Psi \in \mathrm{Sp}(E)$, $\mu(\Psi\Lambda, \Psi V) = \mu(\Lambda, V)$[3];
(ii) for $\Lambda : [a,b] \to \mathcal{L}$ it holds that $\mu(\Lambda, V) = \mu(\Lambda|_{[a,c]}, V) + \mu(\Lambda|_{[c,b]}, V)$, for any $a < c < b$;
(iii) two paths $\Lambda_0, \Lambda_1 : [a,b] \to \mathcal{L}$ with the same end points are homotopic if and only if $\mu(\Lambda_0, V) = \mu(\Lambda_1, V)$;
(iv) for any path $\Lambda : [a,b] \to \Xi_k(V)$ it holds that $\mu(\Lambda, V) = 0$.

The same can be carried out for pairs of Lagrangian curves $\Lambda, \Lambda^\dagger : [a,b] \to \mathcal{L}$. The crossing form on $\Lambda(t_0) \cap \Lambda^\dagger(t_0)$ is then given by

$$\Gamma(\Lambda, \Lambda^\dagger, t_0) \stackrel{\text{def}}{=} \Gamma(\Lambda, \Lambda^\dagger(t_0), t_0) - \Gamma(\Lambda^\dagger, \Lambda(t_0), t_0).$$

[3] This property shows that we can assume E to be the standard symplectic space without loss of generality.

For pairs $(\Lambda, \Lambda^\dagger)$ with only regular crossings the Maslov index $\mu(\Lambda, \Lambda^\dagger)$ is defined in the same way as above using the crossing form for Lagrangian pairs. By setting $\Lambda^\dagger(t) \equiv V$ we retrieve the previous case, and $\Lambda(t) \equiv V$ yields $\Gamma(V, \Lambda^\dagger, t_0) = -\Gamma(\Lambda^\dagger, V, t_0)$. Consider the symplectic space $(\overline{E}, \overline{\omega}) = (E \times E, (-\omega) \times \omega)$, with almost complex structure $(-J) \times J$. A crossing $\Lambda(t_0) \cap \Lambda^\dagger(t_0) \neq \varnothing$ is equivalent to a crossing $(\Lambda \times \Lambda^\dagger)(t_0) \in \Xi(\Delta)$, where $\Delta \subset \overline{E}$ is the diagonal Lagrangian plane, and $\Lambda \times \Lambda^\dagger$ a Lagragian curve in $\overline{E}$, which follows from Eq. (7.1.1) using the Lagrangian frame $\overline{X}(t) = \begin{pmatrix} X(t) \\ X^\dagger(t) \end{pmatrix}$. Let $\bar{v} = (v, v) = \overline{X}(t_0)w$, then

$$\begin{aligned}
\Gamma(\Lambda \times \Lambda^\dagger, \Delta, t_0)(\bar{v}) &= \overline{\omega}\big(\overline{X}(t_0)w, \overline{X'}(t_0)w\big) \\
&= -\omega\big(X(t_0)w, X'(t_0)w\big) + \omega\big(X^\dagger(t_0)w, {X^\dagger}'(t_0)w\big) \\
&= -\Gamma(\Lambda, \Lambda^\dagger(t_0), t_0)(v) + \Gamma(\Lambda^\dagger, \Lambda(t_0), t_0)(v).
\end{aligned}$$

This justifies the identity

$$\mu(\Lambda, \Lambda^\dagger) = \mu(\Delta, \Lambda \times \Lambda^\dagger). \tag{7.1.2}$$

Eq. (7.1.2) is used to define the Maslov index for continuous pairs of Lagrangian curves, and is a special case of the more general formula below. For $\Psi : [a, b] \to \mathrm{Sp}(E)$ a symplectic curve,

$$\mu(\Psi\Lambda, \Lambda^\dagger) = \mu(\mathrm{gr}(\Psi), \Lambda \times \Lambda^\dagger), \tag{7.1.3}$$

where $\mathrm{gr}(\Psi) = \{(x, \Psi x) \mid x \in E\}$ is the graph of Ψ. The curve $\mathrm{gr}(\Psi)(t)$ is a Lagrangian curve in $(\overline{E}, \overline{\omega})$ and $X_\Psi(t) = \begin{pmatrix} \mathrm{Id} \\ \Psi(t) \end{pmatrix}$ is a Lagrangian frame for $\mathrm{gr}(\Psi)$. Indeed, via (7.1.1) we have

$$\begin{pmatrix} \mathrm{Id} & \Psi^T(t) \end{pmatrix} \begin{pmatrix} -J & 0 \\ 0 & J \end{pmatrix} \begin{pmatrix} \mathrm{Id} \\ \Psi(t) \end{pmatrix} = \Psi^T(t) J \Psi(t) - J = 0,$$

which proves that $\mathrm{gr}(\Psi)(t)$ is a Lagrangian curve in $\overline{E}$. Via $\overline{E} \times \overline{E}$ the crossing form is given by

$$\Gamma\left(\mathrm{gr}(\Psi), \Lambda \times \Lambda^\dagger, t_0\right) = \Gamma\left(\mathrm{gr}(\Psi), (\Lambda \times \Lambda^\dagger)(t_0), t_0\right) - \Gamma\left(\Lambda \times \Lambda^\dagger, \mathrm{gr}(\Psi)(t_0), t_0\right)$$

and upon inspection consists of the three terms making up the crossing form of $(\Psi\Lambda, \Lambda^\dagger)$ in $\overline{E}$. More specifically, let $\xi = X_\Psi(t_0)\xi_0 = \overline{X}(t_0)\eta_0 = \eta$, so that $\Psi X \eta_0 = \Psi \xi_0 = X^\dagger \eta_0$, which yields

$$\begin{aligned}
\Gamma\left(\mathrm{gr}(\Psi), (\Lambda \times \Lambda^\dagger)(t_0), t_0\right)(\xi) &= \omega(\Psi(t_0)\xi_0, \Psi'(t_0)\xi_0) \\
&= \omega(\Psi(t_0)X(t_0)\eta_0, \Psi'(t_0)X(t_0)\eta_0),
\end{aligned}$$

and

$$\begin{aligned}
&\Gamma\left(\Lambda \times \Lambda^\dagger, \mathrm{gr}(\Psi)(t_0), t_0\right)(\eta) \\
&= -\omega\big(X(t_0)\eta_0, X'(t_0)\eta_0\big) + \omega\big(X^\dagger(t_0)\eta_0, {X^\dagger}'(t_0)\eta_0\big) \\
&= -\omega\big(\Psi(t_0)X(t_0)\eta_0, \Psi(t_0)X'(t_0)\eta_0\big) + \omega\big(X^\dagger(t_0)\eta_0, {X^\dagger}'(t_0)\eta_0\big),
\end{aligned}$$

which proves Eq. (7.1.3). The crossing form for a more general Lagrangian pair $(\mathrm{gr}(\Psi), \Lambda)$, where $\overline{\Lambda}(t)$ is a Lagrangian curve in $\overline{E}$, is given by $\Gamma\left(\mathrm{gr}(\Psi), \overline{\Lambda}, t_0\right)$ as described above. In the special case that $\overline{\Lambda}(t) \equiv V \times V$, then

$$\Gamma\left(\mathrm{gr}(\Psi), \overline{\Lambda}, t_0\right)(\bar{v}) = \omega(\Psi(t_0)w, \Psi'(t_0)w),$$

where $\bar{v} = X_\Psi(t_0)w$.

A particular example of the Maslov index for symplectic paths is the CONLEY–ZEHNDER INDEX on $(E, \omega) = (\mathbb{R}^{2n}, \overline{\omega}_0)$, which is defined as $\mu_{CZ}(\Psi) \stackrel{\text{def}}{=} \mu\,(\mathrm{gr}(\Psi), \Delta)$ for paths $\Psi : [a, b] \to \mathrm{Sp}(2n, \mathbb{R})$, with $\Psi(a) = \mathrm{Id}$ and $\mathrm{Id} - \Psi(b)$ invertible. It holds that $\Psi' = \overline{J}_0 K(t)\Psi$, for some smooth path $t \mapsto K(t)$ of symmetric matrices. An intersection of $\mathrm{gr}(\Psi)$ and Δ is equivalent to the condition $\det(\Psi(t_0) - \mathrm{Id}) = 0$, i.e. for $\xi_0 \in \ker\,(\Psi(t_0) - \mathrm{Id})$ it holds that $\Psi(t_0)\xi_0 = \xi_0$. The crossing form is given by

$$\begin{aligned}\Gamma\,(\mathrm{gr}(\Psi), \Delta, t_0)\,(\bar{\xi}_0) &= \overline{\omega}_0(\Psi(t_0)\xi_0, \Psi'(t_0)\xi_0)\\ &= \langle \Psi(t_0)\xi_0, K(t_0)\Psi(t_0)\xi_0\rangle\\ &= \langle \xi_0, K(t_0)\xi_0\rangle.\end{aligned}$$

In the case of a symplectic path $\Psi : [0, \tau] \to \mathrm{Sp}(2n, \mathbb{R})$, with $\Psi(0) = \mathrm{Id}$, the extended Conley–Zehnder index is defined as $\mu_{CZ}(\Psi, \tau) = \mu(\mathrm{gr}(\Psi), \Delta)$.

7.2. *The permuted Conley–Zehnder index*

We now define a variation on the Conley–Zehnder index suitable for the application to braids. Consider the symplectic space

$$E = \mathbb{R}^{2n} \times \mathbb{R}^{2n}, \qquad \omega = (-\overline{\omega}_0) \times \overline{\omega}_0.$$

In E we choose coordinates $(x, \tilde{x})$, with $x = (p^1, \cdots, p^n, q^1, \cdots, q^n)$ and $\tilde{x} = (\tilde{p}^1, \cdots, \tilde{p}^n, \tilde{q}^1, \cdots, \tilde{q}^n)$ both in $\mathbb{R}^{2n}$. Let $\sigma \in S_n$ be a permutation, then the permuted diagonal Δ_σ is defined by:

$$\Delta_\sigma \stackrel{\text{def}}{=} \left\{(x, \tilde{x}) \mid (\tilde{p}^k, \tilde{q}^k) = (p^{\sigma(k)}, q^{\sigma(k)}),\ 1 \le k \le n\right\}. \tag{7.2.1}$$

It holds that $\Delta_\sigma = \mathrm{gr}(\boldsymbol{\sigma})$, where $\boldsymbol{\sigma} = \begin{pmatrix}\sigma & 0\\ 0 & \sigma\end{pmatrix}$ and the permuted diagonal Δ_σ is a Lagrangian subspace of E. Let $\Psi : [0, \tau] \to \mathrm{Sp}(2n, \mathbb{R})$ be a symplectic path with $\Psi(0) = \mathrm{Id}$. A crossing $t = t_0$ is defined by the condition $\ker\,(\Psi(t_0) - \boldsymbol{\sigma}) \neq \{0\}$ and the crossing form is given by

$$\begin{aligned}\Gamma\,(\mathrm{gr}(\Psi), \Delta_\sigma, t_0)\,(\xi_0^\sigma) &= \overline{\omega}_0(\Psi(t_0)\xi_0, \Psi'(t_0)\xi_0)\\ &= \langle \Psi(t_0)\xi_0, K(t_0)\Psi(t_0)\xi_0\rangle\\ &= \langle \boldsymbol{\sigma}\xi_0, K(t_0)\boldsymbol{\sigma}\xi_0\rangle = \langle \xi_0, \boldsymbol{\sigma}^T K(t_0)\boldsymbol{\sigma}\xi_0\rangle,\end{aligned} \tag{7.2.2}$$

where $\xi_0^\sigma = X_\sigma \xi_0$, and X_σ the frame for Δ_σ. The PERMUTED CONLEY–ZEHNDER INDEX is defined as

$$\mu_\sigma(\Psi, \tau) \stackrel{\text{def}}{=} \mu(\mathrm{gr}(\Psi), \Delta_\sigma). \tag{7.2.3}$$

Based on the properties of the Maslov index the following list of basic properties of the index μ_σ can be derived.

7.1. Lemma. *For $\Psi, \Psi^\dagger : [0, \tau] \to \mathrm{Sp}(2n, \mathbb{R})$ symplectic paths with $\Psi(0) = \Psi^\dagger(0) = \mathrm{Id}$,*

(i) $\mu_\sigma(\Psi \times \Psi^\dagger, \tau) = \mu_\sigma(\Psi, \tau) + \mu_\sigma(\Psi^\dagger, \tau)$*;*
(ii) *let $\Phi_k(t) : [0, \tau] \to \mathrm{Sp}(2n, \mathbb{R})$ be a symplectic loop (rotation) given by $\Phi_k(t) = e^{\frac{2\pi k}{\tau} \bar{J}_0 t}$, then* $\mu_\sigma(\Phi_k \Psi, \tau) = \mu_\sigma(\Psi, \tau) + 2kn$.

Proof. Property (i) follows from the fact that the equations for the crossings decouple. As for (ii), consider the symplectic curves (using $\Psi(0) = \mathrm{Id}$)

$$\Psi_0(t) = \begin{cases} \Phi_k(t)\Psi(t) & t \in [0, \tau] \\ \Psi(\tau) & t \in [\tau, 2\tau], \end{cases} \qquad \Psi_1(t) = \begin{cases} \Phi_k(t) & t \in [0, \tau] \\ \Psi(t - \tau) & t \in [\tau, 2\tau]. \end{cases}$$

The curves Ψ_0 and Ψ_1 are homotopic via the homotopy

$$\Psi_\lambda(t) = \begin{cases} \Phi_k(t)\Psi((1 - \lambda)t) & t \in [0, \tau] \\ \Psi\big(\tau + \lambda(t - 2\tau)\big) & t \in [\tau, 2\tau], \end{cases}$$

with $\lambda \in [0, 1]$, and $\mu_\sigma(\Psi_0, 2\tau) = \mu_\sigma(\Psi_1, 2\tau)$. By definition of Ψ_0 it follows that $\mu_\sigma(\Phi_k \Psi, \tau) = \mu_\sigma(\Psi_0, 2\tau)$. Using properties (ii) and (iii) of the Maslov index above, we obtain

$$\begin{aligned} \mu_\sigma(\Phi_k \Psi, \tau) &= \mu_\sigma(\Psi_0, 2\tau) = \mu_\sigma(\Psi_1, 2\tau) = \mu\left(\mathrm{gr}(\Psi_1), \Delta_\sigma\right) \\ &= \mu\left(\mathrm{gr}(\Phi_k)|_{[0,\tau]}, \Delta_\sigma\right) + \mu\left(\mathrm{gr}(\Psi(t - \tau))|_{[\tau,2\tau]}, \Delta_\sigma\right) \\ &= \mu\left(\mathrm{gr}(\Phi_k)|_{[0,\tau]}, \Delta_\sigma\right) + \mu_\sigma(\Psi, \tau). \end{aligned}$$

It remains to evaluate $\mu\left(\mathrm{gr}(\Phi_k)|_{[0,\tau]}, \Delta_\sigma\right)$. Recall from [30, Remark 2.6], that for a Lagrangian loop $\Lambda(t + 1) = \Lambda(t)$ and any Lagrangian subspace V the Maslov index is given by

$$\mu(\Lambda, V) = \frac{\alpha(1) - \alpha(0)}{\pi}, \qquad \det\left(P(t) + i\,Q(t)\right) = e^{i\alpha(t)},$$

where $X = (P, Q)^t$ is a unitary Lagrangian frame for Λ. In particular, the index of the loop is independent of the Lagrangian subspace V. From this we derive that

$$\mu\left(\mathrm{gr}(\Phi_k)|_{[0,\tau]}, \Delta_\sigma\right) = \mu\left(\mathrm{gr}(\Phi_k)|_{[0,\tau]}, \Delta\right),$$

and the latter is computed as follows. Consider the crossings of Φ_k: $\det\left(e^{2\pi k/\tau \bar{J}_0 t_0} - \mathrm{Id}\right) = 0$, which holds for $t_0 = \tau n/k$, $n = 0, \cdots, k$. Since Φ_k satisfies $\Phi_k' = 2\pi k/\tau \bar{J}_0 \Phi_k$, the crossing form is given by $\Gamma\left(\mathrm{gr}(\Phi_k), \Delta, t_0\right)\xi_0 = \langle \xi_0, 2\pi k/\tau \xi_0\rangle = 2\pi k/\tau |\xi_0|^2$, with $\xi_0 \in \ker\left(\Psi(t_0) - \mathrm{Id}\right) \neq \{0\}$, and sign $\Gamma(\mathrm{gr}(\Phi_k), \Delta, t_0) = 2n$ (the dimension of the kernel is $2n$). From this we derive that $\mu\left(\mathrm{gr}(\Phi_k)|_{[0,\tau]}, \Delta\right) = 2kn$ and consequently $\mu\left(\mathrm{gr}(\Phi_k)|_{[0,\tau]}, \Delta_\sigma\right) = 2kn$. $\square$

8. A functional analytic framework

8.1. Function spaces

At the end of Section 6 it is pointed out that connecting orbits in a relative braid class fiber are important for the construction of Floer homology. We will see in the next section that, generically, all bounded solutions are connecting orbits. Consider the Banach spaces ($1 < p < \infty$)

$$W^{1,p}_\sigma([0,1];\mathbb{R}^{2n}) \stackrel{\text{def}}{=} \left\{\eta \in W^{1,p}([0,1];\mathbb{R}^{2n}) \mid \left(\eta(0), \eta(1)\right) \in \Delta_\sigma\right\}$$

$$W^{1,p}_\sigma(\mathbb{R}\times[0,1];\mathbb{R}^{2n}) \stackrel{\text{def}}{=} \left\{\xi \in W^{1,p}(\mathbb{R}\times[0,1];\mathbb{R}^{2n}) \mid \left(\xi(s,0), \xi(s,1)\right) \in \Delta_\sigma\right\}.$$

For functions $\mathrm{U} \in W^{1,p}_\sigma$, with $p > 2$, consider the operator

$$\mathcal{F}_{J,H} : W^{1,p}_\sigma(\mathbb{R}\times[0,1];\mathbb{R}^{2n}) \to L^p(\mathbb{R}\times[0,1];\mathbb{R}^{2n}), \tag{8.1.1}$$

defined as $\left(\mathcal{F}_{J,H}(\mathrm{U})\right)_k = \partial_{J,H}(u^k)$, $k = 1,\dots,n$. Note that stationary braids are s-independent solutions of Eq. (8.1.1). In order to study the spaces $\mathscr{M}(\mathrm{X}_- \text{ rel } \mathrm{Y}, \mathrm{X}_+ \text{ rel } \mathrm{Y})$ we consider Eq. (8.1.1) on the affine spaces

$$\mathscr{U}^{1,p}(\mathrm{X}_- \text{ rel } \mathrm{Y}, \mathrm{X}_+ \text{ rel } \mathrm{Y}) \stackrel{\text{def}}{=} \left\{\mathrm{Z} + \xi \mid \xi \in W^{1,p}_\sigma(\mathbb{R}\times[0,1];\mathbb{R}^{2n})\right\}, \tag{8.1.2}$$

where $\mathrm{Z}(s,t)$ is a fixed, smooth connecting path such that $\lim_{s\to\pm\infty}\mathrm{Z}(s,\cdot) = \mathrm{X}_\pm$ and $|\mathrm{Z}(s,t)| < 1$ for all $(s,t) \in \mathbb{R}\times[0,1]$. When $p > 2$, then all functions $\mathrm{U} \in \mathcal{U}^{1,p}(\mathrm{X}_- \text{ rel } \mathrm{Y}, \mathrm{X}_+ \text{ rel } \mathrm{Y})$ are continuous and satisfy the limits $\lim_{s\to\pm\infty}\mathrm{U}(s,\cdot) = \mathrm{X}_\pm$. The operator $\mathcal{F}_{J,H}$ as mapping from $\mathscr{U}^{1,p}(\mathrm{X}_- \text{ rel } \mathrm{Y}, \mathrm{X}_+ \text{ rel } \mathrm{Y})$ to L^p is a C^1 (nonlinear) operator and $\mathcal{F}_{J,H}(\mathrm{U}) = 0$ then defines the space of connecting orbits $\mathscr{M}(\mathrm{X}_- \text{ rel } \mathrm{Y}, \mathrm{X}_+ \text{ rel } \mathrm{Y})$. Define the set of functions in $\mathscr{U}^{1,p}$ of modulus less than or equal to one by $\mathscr{B}_1 = \left\{\mathrm{U} \in \mathscr{U}^{1,p}(\mathrm{X}_- \text{ rel } \mathrm{Y}, \mathrm{X}_+ \text{ rel } \mathrm{Y}) \mid |\mathrm{U}(s,t)| \le 1\right\}$.

8.1. Lemma. *Let* $\mathrm{X}_\pm$ rel $\mathrm{Y} \in [\mathrm{X}]$ rel Y, *a proper relative braid class, then* $\mathcal{F}^{-1}_{J,H}(0) \cap \mathscr{B}_1 = \mathscr{M}(\mathrm{X}_- \text{ rel } \mathrm{Y}, \mathrm{X}_+ \text{ rel } \mathrm{Y})$.

Proof. Let $\mathrm{U} \in \mathscr{B}_1$ be a zero of $\mathcal{F}_{J,H}(\mathrm{U}) = 0$, then, since $|\mathrm{U}(s,t)| \le 1$, the a priori estimates in Proposition 4.1 imply that U is a smooth mapping satisfying the Cauchy–Riemann equations, limiting to the relative braids $\mathrm{X}_\pm$ rel Y. Moreover, from the monotonicity principle (Proposition 5.7) and since $\lim_{s\to\pm\infty}\mathrm{U}(s,\cdot)$ rel $\mathrm{Y} = \mathrm{X}_\pm$ rel $\mathrm{Y} \in [\mathrm{X}]$ rel Y, it follows that $\mathrm{U}(s,\cdot)$ rel $\mathrm{Y} \in [\mathrm{X}]$ rel Y for all $s \in \mathbb{R}$. $\square$

The linearization $d\mathcal{F}_{J,H}(\mathrm{U}) : W^{1,p}_\sigma \to L^p$ around a zero U is a linear operator of Cauchy–Riemann type:

$$d\mathcal{F}_{J,H}(\mathrm{U}) = \frac{\partial}{\partial s} - \overline{J}\frac{\partial}{\partial t} + \overline{J}\,\overline{J}_0 d^2\overline{H}(t, \mathrm{U}(s,t)) \tag{8.1.3}$$

where $\overline{\omega}_0(\cdot, \overline{J}\cdot) = \omega_0(\cdot, J\cdot) \oplus \cdots \oplus \omega_0(\cdot, J\cdot)$ and $\overline{H}(t, \mathrm{U}(s,t)) = \sum_{k=1}^n H(t, u^k(s,t))$. The standard almost complex structure $\overline{J}_0$ is defined by $\langle \cdot, \cdot \rangle_{\mathbb{R}^{2n}} = \overline{\omega}_0(\cdot, \overline{J}_0 \cdot)$.

Such a linear operator can be transformed into standard type. We carry out the transformation for the more general s-dependent Cauchy–Riemann equations (4.3.1). Consider the change of variables $u = \Phi(s)v$, where $s \mapsto \Phi(s)$ is a path in $\mathrm{GL}(2, \mathbb{R})$, satisfying $J(s)\Phi(s) = \Phi(s)J_0$ and $\Phi^*\omega_0 = \omega_0$, i.e. $\det \Phi(s) = 1$. Such a smooth path in $\mathrm{GL}(2, \mathbb{R})$ can be found by solving the above matrix equation. The almost complex structure $J(s)$ is given by

$$J(s) = \begin{pmatrix} a(s) & b(s) \\ c(s) & -a(s) \end{pmatrix}, \quad \text{with} \quad a^2 + bc = -1.$$

The matrices satisfying $J(s)\Phi(s) = \Phi(s)J_0$ are given by

$$\Phi(s) = \begin{pmatrix} \lambda a(s) + \mu & -\lambda + \mu a(s) \\ \lambda c(s) & \mu c(s) \end{pmatrix}, \quad \lambda, \mu \in \mathbb{R},$$

and $\det(\Phi) = \left(\lambda^2 + \mu^2\right) c(s)$. Since $J(s)$ has limits it holds that $0 < c_- \leq c(s) \leq c_+$ and the functions $b(s)$ and $a(s)$ are bounded as well. Choose λ and μ such that $\lambda^2 + \mu^2 = 1/c(s)$. Then $s \mapsto \Phi(s)$ is a path with $\det \Phi(s) = 1$ and $J(s)\Phi(s) = \Phi(s)J_0$.

The following lemma simplifies the analysis for the case of s-dependent complex structures. We assume throughout that $J(s)$ is constant for sufficiently large $|s|$.

8.2. Lemma. *Under the change of variables described above we have that*

$$\overline{\Phi}^{-1} d\mathcal{F}_{J,H}(\mathrm{U})\overline{\Phi} = \frac{\partial}{\partial s} - \overline{J}_0 \frac{\partial}{\partial t} - \overline{\Phi}^T(s) d^2 \overline{H}(t, \mathrm{U}(s,t))\overline{\Phi}(s) + \overline{\Phi}^{-1}(s)\overline{\Phi}_s(s), \quad (8.1.4)$$

where $\overline{\Phi} = \Phi \oplus \cdots \oplus \Phi$. *The matrix function* $\Phi^T(s) d^2 \overline{H}(t, \mathrm{U}(s,t)\Phi(s)$ *is symmetric and* $\Phi^{-1}(s)\Phi_s(s)$ *has compact support.*

Proof. We need to transform the linearization of Eq. (4.3.1). Using the change of variables given above substitute $\xi = \Phi(s)\eta$ into $\xi_s - J\xi_t + JJ_0 d^2 H(t, u^k(s,t))\xi$. This gives

$$\begin{aligned} \xi_s - J\xi_t + JJ_0 d^2 H(t, u^k(s,t))\xi &= \Phi\eta_s - J\Phi\eta_t + JJ_0 d^2 H(t, u^k(s,t))\Phi\eta + \Phi_s \eta \\ &= \Phi\Big(\eta_s - J_0\eta_t + J_0\Phi^{-1} J_0 d^2 H(t, u^k(s,t))\Phi + \Phi^{-1}\Phi_s \eta\Big), \end{aligned}$$

from which the lemma follows using the identity $J_0 \Phi^{-1} J_0 = -\Phi^T$. $\square$

8.2. Fredholm theory and the Maslov index for closed braids

The main result of this section concerns the relation between the permuted Conley–Zehnder index μ_σ and the Fredholm index of the linearized Cauchy–Riemann operator

$$\partial_{K, \Delta_\sigma} = \frac{\partial}{\partial s} - \overline{J}_0 \frac{\partial}{\partial t} - K(s,t),$$

where $K(s,t)$ is a family of symmetric $2n \times 2n$ matrices parameterized by $\mathbb{R} \times [0,1]$. In Lemma 8.2 we showed that the linearization of the Cauchy–Riemann equations can be transformed into a perturbed Cauchy–Riemann operator of the type $\partial_{K,\Delta_\sigma}$, with $K(s,t) = \overline{\Phi}^T(s) d^2 \overline{H}(t, \mathrm{U}(s,t)) \overline{\Phi}(s) + \overline{\Phi}^{-1}(s) \overline{\Phi}_s(s)$. For $|s| \gg 1$, Φ is independent of s, hence the second term in K vanishes, so that $K(s,t)$ is symmetric for sufficiently large $|s|$. Note that u may be a connecting orbit, hence K is not assumed to be constant for large s. This motivates the study of the perturbed Cauchy–Riemann operators $\partial_{K,\Delta_\sigma}$ below.

The operator $\partial_{K,\Delta_\sigma}$ acts on functions satisfying the non-local boundary conditions $\big(\xi(s,0), \xi(s,1)\big) \in \Delta_\sigma$, or in other words $\xi(s,1) = \xi(s,0) \cdot \sigma$. On K we impose the following hypotheses:

(k1) the matrix-function $K : \mathbb{R} \times [0,1] \to M(2n, \mathbb{R})$ is symmetric for $|s|$ sufficiently large, with uniform limits $\lim_{s \to \pm\infty} K(s,t) = K_\pm(t)$;

(k2) the solutions $\Psi_\pm$ of the initial value problem

$$\frac{d}{dt}\Psi_\pm - \overline{J}_0 K_\pm(t) \Psi_\pm = 0, \qquad \Psi_\pm(0) = \mathrm{Id},$$

have the property that $\mathrm{gr}\big(\Psi_\pm(1)\big)$ is transverse to Δ_σ.

Hypothesis (k2) can be rephrased as $\det\big(\Psi_\pm(1) - \boldsymbol{\sigma}\big) \neq 0$. It follows from the proof below that this is equivalent to saying that the mappings $L_\pm = \overline{J}_0 \frac{d}{dt} + K_\pm(t)$ are invertible.

In [31] (Theorem 7.42) the following result was proved.

8.3. Proposition. *Suppose that hypotheses (k1) and (k2) are satisfied. Then the operator $\partial_{K,\Delta_\sigma} : W^{1,2}_\sigma \to L^2$ is Fredholm and the Fredholm index is given by*

$$\mathrm{ind}\, \partial_{K,\Delta_\sigma} = \mu_\sigma(\Psi_-, 1) - \mu_\sigma(\Psi_+, 1).$$

As a matter of fact $\partial_{K,\Delta_\sigma}$ is a Fredholm operator from $W^{1,p}_\sigma$ to L^p, $1 < p < \infty$, with the same Fredholm index.

Proof. We will sketch the proof adjusted to the special situation here. Regard the linearized Cauchy–Riemann operator as an unbounded operator

$$D_L = \frac{d}{ds} - L(s),$$

on $L^2\big(\mathbb{R}; L^2([0,1]; \mathbb{R}^{2n})\big)$, where $L(s) = \overline{J}_0 \frac{d}{dt} + K(s,t)$. Assume that $K(s,t)$ is symmetric, so that $L(s)$ is a family of unbounded, self-adjoint operators on $L^2([0,1]; \mathbb{R}^{2n})$, with (dense) domain $W^{1,2}_\sigma([0,1]; \mathbb{R}^{2n})$. In general, $L(s)$ is only self-ajdoint for $|s| \gg 1$. However, $L(s)$ is a compact perturbation of a self-adjoint operator, which does not change the Fredholm property or index. It therefore suffices to consider the self-adjoint case. In this special case the result follows from the spectral flow of $L(s)$: for the path $s \mapsto L(s)$ a number $s_0 \in \mathbb{R}$ is a crossing if $\ker L(s) \neq \{0\}$. On $\ker L(s)$ we have the crossing form

$$\Gamma(L, s_0)\xi \overset{\text{def}}{=} (\xi, L'(s)\xi)_{L^2} = \int_0^1 \Big\langle \xi(t), \frac{\partial K(s,t)}{\partial s} \xi(t) \Big\rangle dt,$$

with $\xi \in \ker L(s)$. If the path $s \mapsto L(s)$ has only regular crossings — crossings for which Γ is non-degenerate — then the main result in [31] states that D_L is Fredholm with

$$\operatorname{ind} D_L = -\sum_{s_0} \operatorname{sign} \Gamma(L, s_0) \stackrel{\text{def}}{=} -\mu_{\text{spec}}(L).$$

Let $\Psi(s,t)$ be the solution of the s-parametrized family of ODEs

$$\begin{cases} L(s)\Psi(s,t) = 0, \\ \Psi(s,0) = \mathrm{Id}. \end{cases}$$

Note that $\xi \in \ker\ L(s)$ if and only if $\xi(t) = \Psi(s,t)\xi_0$ and $\Psi(s,1)\xi_0 = \boldsymbol{\sigma}\xi_0$, i.e., $\xi_0 \in \ker(\Psi(s,1) - \boldsymbol{\sigma})$. The crossing form for L can be related to the crossing form for $(\mathrm{gr}(\Psi), \Delta_\sigma)$. We have that $L(s)\Psi(s,\cdot) = 0$ and thus by differentiating

$$\frac{\partial K(s,t)}{\partial s}\Psi(s,t) + K(s,t)\frac{\partial \Psi(s,t)}{\partial s} = -\overline{J}_0 \frac{\partial^2 \Psi(s,t)}{\partial s \partial t}.$$

From this we derive

$$\begin{aligned}
&-\Big\langle \Psi(s,t)\xi_0, \frac{\partial K(s,t)}{\partial s}\Psi(s,t)\xi_0\Big\rangle \\
&= \Big\langle \Psi(s,t)\xi_0, K(s,t)\frac{\partial \Psi(s,t)}{\partial s}\xi_0\Big\rangle + \Big\langle \Psi(s,t)\xi_0, \overline{J}_0\frac{\partial^2 \Psi(s,t)}{\partial s\partial t}\xi_0\Big\rangle \\
&= \Big\langle K(s,t)\Psi(s,t)\xi_0, \frac{\partial \Psi(s,t)}{\partial s}\xi_0\Big\rangle + \Big\langle \Psi(s,t)\xi_0, \overline{J}_0\frac{\partial^2 \Psi(s,t)}{\partial s\partial t}\xi_0\Big\rangle \\
&= -\Big\langle \overline{J}_0\frac{\partial \Psi(s,t)}{\partial t}\xi_0, \frac{\partial \Psi(s,t)}{\partial s}\xi_0\Big\rangle + \Big\langle \Psi(s,t)\xi_0, \overline{J}_0\frac{\partial^2 \Psi(s,t)}{\partial s\partial t}\xi_0\Big\rangle,
\end{aligned}$$

which yields that

$$-\Big\langle \Psi(s,t)\xi_0, \frac{\partial K(s,t)}{\partial s}\Psi(s,t)\xi_0\Big\rangle = \frac{\partial}{\partial t}\Big\langle \Psi(s,t)\xi_0, \overline{J}_0\frac{\partial \Psi(s,t)}{\partial s}\xi_0\Big\rangle.$$

We substitute this identity in the integral crossing form for $L(s)$ at a crossing $s = s_0$:

$$\begin{aligned}
\Gamma(L, s_0)(\xi) &= \int_0^1 \Big\langle \xi(t), \frac{\partial K(s,t)}{\partial s}\xi(t)\Big\rangle dt \\
&= \int_0^1 \Big\langle \Psi(s,t)\xi_0, \frac{\partial K(s,t)}{\partial s}\Psi(s,t)\xi_0\Big\rangle dt \\
&= -\Big\langle \Psi(s,t)\xi_0, \overline{J}_0\frac{\partial \Psi(s,t)}{\partial s}\xi_0\Big\rangle\Big|_0^1 = -\Big\langle \Psi(s,1)\xi_0, \overline{J}_0\frac{\partial \Psi(s,1)}{\partial s}\xi_0\Big\rangle \\
&= \overline{\omega}_0\Big(\Psi(s,1)\xi_0, \frac{\partial \Psi(s,1)}{\partial s}\xi_0\Big) = \Gamma\big(\mathrm{gr}(\Psi(s,1), \Delta_\sigma, s_0\big)(\xi_0^\sigma).
\end{aligned}$$

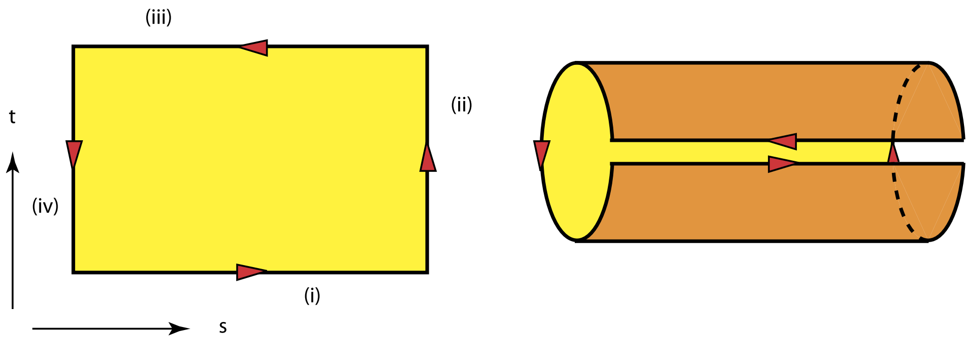

Fig. 8.2.1. The symplectic contour in $\mathbb{R}^2$ and as cylinder $[-T, T] \times \mathbb{R}/\mathbb{Z}$.

The boundary term at $t = 0$ is zero since $\Psi(s, 0) = \mathrm{Id}$ for all s. The relation between the crossing forms proves that the curves $s \mapsto L(s)$ and $s \mapsto \Psi(s, 1)$ have the same crossings and $\mu(\mathrm{gr}(\Psi(s, 1)), \Delta_\sigma) = \mu_{\mathrm{spec}}(L)$. We assume that $\Psi(\pm T, t) = \Psi_\pm(t)$, and that the crossings $s = s_0$ are regular, as the general case follows from homotopy invariance. Consider $\Psi(s, t)$ as a symplectic operator parametrized by s and t. The symplectic path along the boundary of the cylinder $[-T, T] \times \mathbb{R}/\mathbb{Z} \subset \mathbb{R} \times \mathbb{R}/\mathbb{Z}$ yields

$$-\mu(\Delta, \Delta_\sigma) - \mu_\sigma(\Psi_+, 1) + \mu_{\mathrm{spec}}(L) + \mu_\sigma(\Psi_-, 1) = 0.$$

Indeed, since the loop is contractible the sum of the terms is zero. The individual terms along the boundary components are found as follows, see Fig. 8.2.1: (i) for $-T \le s \le T$, it holds that $\Psi(s, 0) = \mathrm{Id}$, and thus $\mathrm{gr}(\Psi(s, 0)) = \Delta$ and $\mu(\mathrm{gr}(\Psi(s, 0)), \Delta_\sigma) = \mu(\Delta, \Delta_\sigma)$; (ii) for $0 \le t \le 1$, we have $\Psi(T, t) = \Psi_+(t)$, and therefore $\mu(\mathrm{gr}(\Psi_+), \Delta_\sigma) = \mu_\sigma(\Psi_+, 1)$; (iii) for $-T \le s \le T$ (opposite direction) the previous calculations with the crossing form for $L(s)$ show that $\mu(\mathrm{gr}(\Psi(s, 1)), \Delta_\sigma) = \mu_{\mathrm{spec}}(L)$; (iv) for $0 \le t \le 1$ (opposite direction), it holds that $\Psi(-T, t) = \Psi_-(t)$, and therefore $\mu(\mathrm{gr}(\Psi_-), \Delta_\sigma) = \mu_\sigma(\Psi_-, 1)$. Since $\mathrm{ind}\, D_L = -\mu_{\mathrm{spec}}(L)$ we obtain

$$\mathrm{ind}\, D_L = \mathrm{ind}\, \partial_{K,\Delta_\sigma} = \mu_\sigma(\Psi_-, 1) - \mu_\sigma(\Psi_+, 1) + \mu(\Delta, \Delta_\sigma).$$

Since Δ_σ and Δ are both constant Lagrangian curves, it follows that $\mu(\Delta, \Delta_\sigma) = 0$, which concludes the proof of the theorem. □

We recall from Section 4 that the Hamiltonian for multi-strand braids is defined as $\overline{H}(t, \mathrm{x}(t)) = \sum_{k=1}^{n} H(t, x^k(t))$. The linearization around a braid x is given by

$$L_{\mathrm{x}} \stackrel{\mathrm{def}}{=} -d^2\mathscr{L}_H(\mathrm{x}) = \overline{J}_0 \frac{d}{dt} + d^2\overline{H}(t, \mathrm{x}). \tag{8.2.1}$$

Define the symplectic path $\Psi : [0, 1] \to \mathrm{Sp}(2n, \mathbb{R})$ by

$$\frac{d\Psi}{dt} - \overline{J}_0 d^2\overline{H}(t, \mathrm{x}(t))\Psi = 0, \qquad \Psi(0) = \mathrm{Id}. \tag{8.2.2}$$

For convenience we write $K(t) = d^2\overline{H}(t, \mathrm{x}(t))$, so that the linearized equation becomes $\frac{d}{dt}\Psi - \overline{J}_0 K(t)\Psi = 0$.

8.4. Lemma. *If* $\det\big(\Psi(1) - \boldsymbol{\sigma}\big) \neq 0$*, then* $\mu_\sigma(\Psi, 1)$ *is an integer.*

Proof. Since crossings between $\mathrm{gr}(\Psi)$ and Δ_σ occur when $\det\big(\Psi(t) - \boldsymbol{\sigma}\big) = 0$, the only endpoint that may lead to a non-integer contribution is the starting point. There the crossing form is, as in (7.2.2), given by

$$\Gamma(\mathrm{gr}(\Psi), \Delta_\sigma, 0)(\xi^\sigma) = \langle \xi, \boldsymbol{\sigma}^T K(0)\boldsymbol{\sigma}\xi \rangle,$$

for all $\xi \in \ker(\Psi(0) - \boldsymbol{\sigma})$. The kernel of $\Psi(0) - \boldsymbol{\sigma} = \mathrm{Id} - \boldsymbol{\sigma}$ is even dimensional, since in coordinates (7.2.1) it is of the form $\ker(\mathrm{Id}_{2n} - \boldsymbol{\sigma}) = \ker(\mathrm{Id}_n - \sigma) \times \ker(\mathrm{Id}_n - \sigma)$. Therefore, $\mathrm{sign}\, \Gamma(\mathrm{gr}(\Psi), \Delta_\sigma, 0)$ is always even, and $\mu_\sigma(\Psi, 1)$ is an integer. $\square$

The non-degeneracy condition leads to an integer valued Conley–Zehnder index for braids.

8.5. Definition. A stationary braid x is said to be NON-DEGENERATE if $\det\big(\Psi(1) - \boldsymbol{\sigma}\big) \neq 0$. The Conley–Zehnder index of a non-degenerate stationary braid x is defined by $\mu(\mathrm{x}) \stackrel{\mathrm{def}}{=} \mu_\sigma(\Psi, 1)$, where $\sigma \in S_n$ is the associated permutation of x.

The proof of the following remark can be found in Thm. 3.3 in [35].

8.6. Remark. If $\mathrm{x} = \{x^k(t)\}$ is a stationary non-degenerate braid, then $\mu(\mathrm{x})$ can be related to the Morse indices $\mu_H(x^k)$ provided that the matrix norm of $K = d^2H(x^k)$ is not too large, e.g. if $\|K\| < 2\pi$. For $\mu_H(\mathrm{x}) \stackrel{\mathrm{def}}{=} \sum_k \mu_H(x^k)$,

$$\mu(\mathrm{x}) = \mu_\sigma(\Psi, 1) = \sum_k \Big(1 - \mu_H(x^k)\Big) = n - \mu_H(\mathrm{x}). \tag{8.2.3}$$

This relation can be useful in some instances for computing Floer homology, see Section 13.2. Indeed, in dimension two Ψ satisfies: $\Psi(t) = \exp(J_0 K t)$, where $K = d^2H(x^k(t))$ is a constant matrix. Then $\mu(x^k) = \mu_{\mathrm{CZ}}(\Psi) = 1 - \mu^-(K)$, where $\mu^-(K)$ is the number of negative eigenvalues of eigenvalues.

9. Transversality and connecting orbit spaces

Central to the analysis of the Cauchy–Riemann equations are various generic non-degeneracy and transversality properties. The first important statement in this direction involves the generic non-degeneracy of critical points.

9.1. Generic properties of critical points

The Hamiltonians in this section are chosen from the class $\mathscr{H} = \mathscr{H}(\mathbb{D}^2)$, which is defined at the beginning of Section 3.2. On $C^\infty(\mathbb{R}/\mathbb{Z} \times \mathbb{D}^2; \mathbb{R})$ we define the norm

$$\|h\|_{C^\infty} \stackrel{\mathrm{def}}{=} \sum_{k=0}^{\infty} \epsilon_k \|h\|_{C^k},$$

for a sufficiently fast decaying sequence $\epsilon_k > 0$, such that (part of) C^∞ equipped with this norm is a separable Banach space, dense in L^2. The set $\mathscr{H}$ is a closed linear subspace.

The genericity result below is special in the way that perturbations are with respect to Hamiltonians on $\mathbb{R}/\mathbb{Z} \times \mathbb{D}^2$, *not* on $\mathbb{R}/\mathbb{Z} \times \mathbb{D}^2 \times \cdots \times \mathbb{D}^2$. For completeness we carry out the subtle constructions.

9.1. Proposition. *Let* $[\mathrm{X} \text{ rel } \mathrm{Y}]$ *be a proper relative braid class and* $[\mathrm{X}] \text{ rel } \mathrm{Y}$ *a fiber. Then, for any Hamiltonian* $H \in \mathscr{H}$*, with* $\mathrm{Y} \in \mathrm{Crit}_H(\Gamma F_m)$*, there exists a* $\delta_* > 0$ *such that for any* $\delta < \delta_*$ *there exists a nearby Hamiltonian* $H' \in \mathscr{H}$ *satisfying*

(i) $\|H - H'\|_{C^\infty} < \delta$;
(ii) $\mathrm{Y} \in \mathrm{Crit}_{H'}(\Gamma F_m)$,

such that $\mathrm{Crit}_{H'}([\mathrm{X}] \text{ rel } \mathrm{Y})$ *consists of only finitely many non-degenerate critical points for the action* $\mathscr{L}_{H'}$.

The property that $\mathrm{Crit}_H([\mathrm{X}] \text{ rel } \mathrm{Y})$ consists of only non-degenerate critical points is called a generic property, and is satisfied by generic Hamiltonians in the above sense, denoted by $\mathscr{H}^{\mathrm{reg}}$ and which are dense in $\mathscr{H}$.

Proof of Proposition 9.1. Given $H \in \mathscr{H}$ we start off with defining a class of perturbations. For a braid $\mathrm{Y} \in \Gamma F_m$, define the tubular neighborhood $N_\epsilon(\mathrm{Y})$ of Y in $[0,1] \times \mathbb{D}^2$ by :

$$N_\epsilon(\mathrm{Y}) = \bigcup_{\substack{k=1,\cdots,m \\ t\in[0,1]}} B_\epsilon(y^k(t)).$$

If $\epsilon > 0$ is sufficiently small, then a neighborhood $N_\epsilon(\mathrm{Y})$ consists of m disjoint cylinders. Let $D_\epsilon = \{x \in \mathbb{D}^2 \mid 1 - \epsilon < |x| \le 1\}$ be a small neighborhood of the boundary, and define

$$A_\epsilon = N_\epsilon(\mathrm{Y}) \cup \big([0,1] \times D_\epsilon\big), \quad A_\epsilon^c = \big([0,1] \times \mathbb{D}^2\big)\backslash A_\epsilon.$$

Let $\mathscr{T}([\mathrm{X}] \text{ rel } \mathrm{Y}; J, H)$ represent the paths in the cylinder traced out by the elements of $\mathscr{S}([\mathrm{X}] \text{ rel } \mathrm{Y}; J, H)$:

$$\mathscr{T}([\mathrm{X}] \text{ rel } \mathrm{Y}; J, H) \stackrel{\text{def}}{=} \big\{(t, x^k(t)) \,\big|\, 1 \le k \le n,\ t \in [0,1],\ \mathrm{X} \in \mathscr{S}([\mathrm{X}] \text{ rel } \mathrm{Y}; J, H)\big\}.$$

Since $[\mathrm{X}] \text{ rel } \mathrm{Y}$ is proper, by Proposition 6.2 there exists an $\epsilon_* > 0$, such that for all $\epsilon \le \epsilon_*$ it holds that $\mathscr{T}([\mathrm{X}] \text{ rel } \mathrm{Y}; J, H) \subset \mathrm{int}(A_{2\epsilon}^c)$. Now fix $\epsilon \in (0, \epsilon_*]$. Let

$$\begin{aligned} \mathscr{V}_\epsilon &\stackrel{\text{def}}{=} \{h \in C^\infty(\mathbb{R}/\mathbb{Z} \times \mathbb{D}^2; \mathbb{R}) \mid \operatorname{supp} h \subset A_\epsilon^c\}, \\ \mathscr{V}_{\delta,\epsilon} &\stackrel{\text{def}}{=} \{h \in \mathscr{V}_\epsilon \mid \|h\|_{C^\infty} < \delta\}, \end{aligned}$$

and consider Hamiltonians of the form $H' = H + h_\delta \in \mathscr{H}$, with $h_\delta \in \mathscr{V}_{\delta,\epsilon}$. Then, by construction, $\mathrm{Y} \in \mathrm{Crit}_{H'}(\Gamma F_m)$, and by Proposition 6.2 the set $\mathscr{S}([\mathrm{X}] \text{ rel } \mathrm{Y}; J, H')$ is compact and isolated in the proper braid class $[\mathrm{X}]$ rel Y for all perturbation $h_\delta \in \mathscr{V}_{\delta,\epsilon}$. A straightforward compactness argument using the compactness result of Proposition 6.2 shows that $\mathscr{T}([\mathrm{X}] \text{ rel } \mathrm{Y}; J, H + h_\delta)$ converges to $\mathscr{T}([\mathrm{X}] \text{ rel } \mathrm{Y}; J, H)$ in the Hausdorff metric as $\delta \to 0$. Therefore, there exists a $\delta_* > 0$, such that $\mathscr{T}([\mathrm{X}] \text{ rel } \mathrm{Y}; J, H + h_\delta) \subset \mathrm{int}(A^c_{2\epsilon})$, for all $0 \le \delta \le \delta_*$. In particular $\mathrm{Crit}_{H+h_{\delta,\epsilon}} \subset \mathrm{int}(A^c_{2\epsilon})$, for all $0 \le \delta \le \delta_*$. Now fix $\delta \in (0, \delta_*]$.

The Hamilton equations for H' are $x^k_t - J_0 \nabla H(t, x^k) - J_0 \nabla h(t, x) = 0$, with periodic boundary conditions in t. Define $\mathscr{U}_\epsilon \subset W^{1,2}_\sigma([0,1]; \mathbb{R}^{2n})$ to be the open subset of functions $\mathrm{X} = \{x^k\}$ such that $x^k(t) \in \mathrm{int}(A^c_{2\epsilon})$ and define the nonlinear mapping

$$\mathcal{G} : \mathscr{U}_\epsilon \times \mathscr{V}_{\delta,\epsilon} \to L^2([0,1]; \mathbb{R}^{2n}),$$

which represents the above system of equations and boundary conditions. Explicitly,

$$\mathcal{G}(\mathrm{X}, h) = \overline{J}_0 \mathrm{X}_t + \nabla \overline{H}(t, \mathrm{X}) + \nabla \bar{h}(t, \mathrm{X}),$$

where $\overline{H}(t, \mathrm{X}) = \sum_k H(t, x^k)$, and likewise for $\bar{h}$. The mapping $\mathcal{G}$ is linear in h. Since $\mathcal{G}$ is defined on $\mathscr{U}_\epsilon$ and both H and h are of class C^∞, the mapping $\mathcal{G}$ is of class C^1. The derivative with respect to variations $(\xi, \delta h) \in W^{1,2}_\sigma([0,1]; \mathbb{R}^{2n}) \times \mathscr{V}_\epsilon$ is given by

$$\begin{aligned} d\mathcal{G}(\mathrm{X}, h)(\xi, \delta h) &= \overline{J}_0 \xi_t + d^2 \overline{H}(t, \mathrm{X}) \xi + d^2 \bar{h}(t, \mathrm{X}) \xi + \nabla \overline{\delta h}(t, \mathrm{X}) \\ &= L_\mathrm{X} \xi + \nabla \overline{\delta h}(t, \mathrm{X}), \end{aligned}$$

where $L_\mathrm{X} = \overline{J}_0 \frac{d}{dt} + d^2 \overline{H}(t, \mathrm{X}) + d^2 \bar{h}(t, \mathrm{X})$, by analogy with Eq. (8.2.1). We see that there is a one-to-one correspondence between elements Ψ in the kernel of L_X and symplectic paths described by Eq. (8.2.2) with $\det\big(\Psi(1) - \boldsymbol{\sigma}\big) = 0$. In other words, the stationary braid X is non-degenerate if and only if L_X has trivial kernel.

The operator L_X is self-adjoint on $L^2([0,1]; \mathbb{R}^{2n})$ with domain $W^{1,2}_\sigma([0,1]; \mathbb{R}^{2n})$ and is Fredholm with $\mathrm{ind}(L_\mathrm{X}) = 0$. Therefore $\mathcal{G}_h \stackrel{\text{def}}{=} \mathcal{G}(\cdot, h)$ is a (proper) nonlinear Fredholm operator with

$$\mathrm{ind}(\mathcal{G}_h) = \mathrm{ind}(L_\mathrm{X}) = 0.$$

Define the set

$$\mathscr{Z} = \big\{(\mathrm{X}, h) \in \mathscr{U}_\epsilon \times \mathscr{V}_{\delta,\epsilon} \mid \mathcal{G}(\mathrm{X}, h) = 0\big\} = \mathcal{G}^{-1}(0).$$

We show that $\mathscr{Z}$ is a Banach manifold by demonstrating that $d\mathcal{G}(\mathrm{X}, h)$ is surjective for all $(\mathrm{X}, h) \in \mathscr{Z}$. Since $d\mathcal{G}(\mathrm{X}, h)(\xi, \delta h) = L_\mathrm{X} \xi + \nabla \overline{\delta h}(t, \mathrm{X})$, and the (closed) range of L_X has finite codimension, we need to show there is a (finite dimensional) complement of $R(L_\mathrm{X})$ in the image of $\nabla \overline{\delta h}(t, \mathrm{X})$. It suffices to show that $\big\{\nabla \overline{\delta h}(t, \mathrm{X})\big\}_{\delta h \in \mathscr{V}_\epsilon}$ is dense in $L^2([0,1]; \mathbb{R}^{2n})$.

Recall that for any pair $(\mathrm{X}, h) \in \mathscr{Z}$, it holds that $\mathrm{X} \in \mathrm{Crit}_{H'} \subset \mathrm{int}(A^c_{2\epsilon})$. As before consider a neighborhood $N_\epsilon(\mathrm{X})$, so that $N_\epsilon(\mathrm{X}) \subset \mathrm{int}(A^c_\epsilon)$ and consists of n disjoint cylinders $N_\epsilon(x^k)$. Let $\phi^k_\epsilon(t, x) \in C^\infty_0(N_\epsilon(x^k))$, such that $\phi^k_\epsilon \equiv 1$ on $N_{\epsilon/2}(x^k)$. Define, for arbitrary $f^k \in C^\infty(\mathbb{R}/\mathbb{Z}; \mathbb{R}^2)$,

$$(\delta h)(t, x) = \sum_{k=1}^n \phi^k_\epsilon(t, x) \langle f^k(t), x \rangle_{L^2}.$$

Since $\phi_\epsilon^k(t, x^k(t)) \equiv 1$ it holds that $\overline{\delta h}(t, \mathrm{X}) = \sum_{k=1}^n \langle f^k(t), x^k \rangle_{L^2}$ for $\mathrm{X} \in \mathrm{Crit}_{H'}$, and therefore the gradient satisfies $\nabla \overline{\delta h}(t, \mathrm{X}) = (f^k) \in C^\infty(\mathbb{R}/\mathbb{Z}; \mathbb{R}^{2n})$. Moreover, $\delta h \in \mathscr{V}_\epsilon$ by construction, and because $C^\infty(\mathbb{R}/\mathbb{Z}; \mathbb{R}^{2n})$ is dense in $L^2([0,1]; \mathbb{R}^{2n})$ it follows that $d\mathcal{G}(\mathrm{X}, h)$ is surjective.

Consider the projection $\pi : \mathscr{Z} \to \mathscr{V}_{\delta,\epsilon}$, defined by $\pi(\mathrm{X}, h) = h$. The projection π is a Fredholm operator. Indeed, $d\pi : T_{(\mathrm{X},h)}\mathscr{Z} \to \mathscr{V}_\epsilon$, with $d\pi(\mathrm{X}, h)(\xi, \delta h) = \delta h$, and

$$T_{(\mathrm{X},h)}\mathscr{Z} = \left\{ (\xi, \delta h) \in W^{1,2}_\sigma \times \mathscr{V}_\epsilon \mid L_\mathrm{X}\xi - \nabla\overline{\delta h} = 0 \right\}.$$

From this it follows that $\mathrm{ind}(d\pi) = \mathrm{ind}(L_\mathrm{X}) = 0$. The Sard–Smale Theorem implies that the set of perturbations $h \in \mathscr{V}^{\mathrm{reg}}_{\delta,\epsilon} \subset \mathscr{V}_{\delta,\epsilon}$ for which h is a regular value of π being a countable intersection of open and dense subsets, and thus a residual set and dense by the Baire category theorem. It remains to show that $h \in \mathscr{V}^{\mathrm{reg}}_{\delta,\epsilon}$ yields that L_X is surjective. Let $h \in \mathscr{V}^{\mathrm{reg}}_{\delta,\epsilon}$, and $(\mathrm{X}, h) \in Z$, then $d\mathcal{G}(\mathrm{X}, h)$ is surjective, i.e., for any $\zeta \in L^2([0,1]; \mathbb{R}^{2n})$ there are $(\xi, \delta h)$ such that $d\mathcal{G}(\mathrm{X}, h)(\xi, \delta h) = \zeta$. On the other hand, since h is a regular value for π, there exists a $\widehat{\xi}$ such that $d\pi(\mathrm{X}, h)(\widehat{\xi}, \delta h) = \delta h$, $(\widehat{\xi}, \delta h) \in T_{(\mathrm{X},h)}\mathscr{Z}$, i.e. $L_\mathrm{X}\widehat{\xi} - \nabla\overline{\delta h} = 0$. Now

$$\begin{aligned} L_\mathrm{X}(\xi - \widehat{\xi}) &= d\mathcal{G}(\mathrm{X}, h)(\xi - \widehat{\xi}, 0) \\ &= d\mathcal{G}(\mathrm{X}, h)\big((\xi, \delta h) - (\widehat{\xi}, \eta)\big) = \zeta - 0 = \zeta, \end{aligned}$$

which proves that for all $h \in \mathscr{V}^{\mathrm{reg}}_{\delta,\epsilon}$ the operator L_X is surjective, and hence also injective, implying that X is non-degenerate. □

In Corollary 6.3 we showed that for any pair of proper braids X_- rel Y, X_+ rel $\mathrm{Y} \in [\mathrm{X}]$ rel Y it holds that $\mathscr{M}(\mathrm{X}_- \text{ rel } \mathrm{Y}, \mathrm{X}_+ \text{ rel } \mathrm{Y}) \subset \mathscr{M}([\mathrm{X}] \text{ rel } \mathrm{Y})$ and thus

$$\bigcup_{\substack{\mathrm{X}_\pm \text{ rel } \mathrm{Y} \in \\ \mathrm{Crit}_H([\mathrm{X}] \text{ rel } \mathrm{Y})}} \mathscr{M}(\mathrm{X}_- \text{ rel } \mathrm{Y}, \mathrm{X}_+ \text{ rel } \mathrm{Y}) \subset \mathscr{M}([\mathrm{X}] \text{ rel } \mathrm{Y}). \tag{9.1.1}$$

For generic Hamiltonians $H \in \mathscr{H}^{\mathrm{reg}}$ we also have the reversed inclusion.

9.2. Lemma. *Let* $[\mathrm{X} \text{ rel } \mathrm{Y}]$ *be a proper braid class,* $[\mathrm{X}]$ rel Y *a fiber and let* $H \in \mathscr{H}^{\mathrm{reg}}$ *be a generic Hamiltonian with* $\mathrm{Y} \in \mathrm{Crit}_H(\Gamma F_m)$. *Then,*

$$\mathscr{M}([\mathrm{X}] \text{ rel } \mathrm{Y}) \subset \bigcup_{\substack{\mathrm{X}_\pm \text{ rel } \mathrm{Y} \in \\ \mathrm{Crit}_H([\mathrm{X}] \text{ rel } \mathrm{Y})}} \mathscr{M}(\mathrm{X}_- \text{ rel } \mathrm{Y}, \mathrm{X}_+ \text{ rel } \mathrm{Y}),$$

which yields equality in view of (9.1.1).

Proof. See [33, Proposition 4.2], for a detailed proof. □

As pointed out before the sets $\mathscr{M}_{\mathrm{X}_-,\mathrm{X}_+}([\mathrm{X}] \text{ rel } \mathrm{Y})$ are not necessarily compact. The following corollary gives a more precise statement about the compactness of the spaces $\mathscr{M}_{\mathrm{X}_-,\mathrm{X}_+}([\mathrm{X}] \text{ rel } \mathrm{Y})$, which will be referred to as geometric convergence.

9.3. Corollary. *Let* $[\mathrm{X} \text{ rel } \mathrm{Y}]$ *be a proper relative braid class,* $[\mathrm{X}] \text{ rel } \mathrm{Y}$ *a fiber and let* $H \in \mathscr{H}^{\mathrm{reg}}$ *be a generic Hamiltonian with* $\mathrm{Y} \in \mathrm{Crit}_H(\Gamma F_m)$*. Then for any sequence* $\{\mathrm{U}_i \text{ rel } \mathrm{Y}\} \subset \mathscr{M}(\mathrm{X}_- \text{ rel } \mathrm{Y}, \mathrm{X}_+ \text{ rel } \mathrm{Y})$ *(along a subsequence) there exist stationary braids* $\mathrm{X}^j \in \mathrm{Crit}_H([\mathrm{X}] \text{ rel } \mathrm{Y})$*,* $j = 0, \ldots, d$*, orbits* $\mathrm{U}^j \text{ rel } \mathrm{Y} \in \mathscr{M}(\mathrm{X}^j \text{ rel } \mathrm{Y}, \mathrm{X}^{j-1} \text{ rel } \mathrm{Y})$ *and times* s_i^j*,* $j = 1, \ldots, d$*, such that*

$$(\mathrm{U}_i \text{ rel } \mathrm{Y})(\cdot + s_i^j, \cdot) \longrightarrow \mathrm{U}^j \text{ rel } \mathrm{Y}, \quad i \to \infty,$$

in $C^r_{\mathrm{loc}}(\mathbb{R} \times \mathbb{R}/\mathbb{Z})$*, for any* $r \geq 1$*. Moreover,* $\mathrm{X}^0 \text{ rel } \mathrm{Y} = \mathrm{X}_+ \text{ rel } \mathrm{Y}$ *and* $\mathrm{X}^d \text{ rel } \mathrm{Y} = \mathrm{X}_- \text{ rel } \mathrm{Y}$ *and* $\mathscr{L}_H(\mathrm{X}^j) > \mathscr{L}_H(\mathrm{X}^{j-1})$ *for* $j = 1, \ldots, d$*. The sequence* $\mathrm{U}_i \text{ rel } \mathrm{Y}$ *is said to geometrically converge to the broken trajectory* $(\mathrm{U}^1 \text{ rel } \mathrm{Y}, \ldots, \mathrm{U}^d \text{ rel } \mathrm{Y})$.

Proof. See [33, Proposition 4.2], for a detailed proof. □

9.2. Generic properties for connecting orbits

As for critical points, non-degeneracy can also be defined for connecting orbits. This closely follows the ideas in the previous subsection. Set $W^{1,p}_\sigma = W^{1,p}_\sigma(\mathbb{R} \times [0,1]; \mathbb{R}^{2n})$ and $L^p = L^p(\mathbb{R} \times [0,1]; \mathbb{R}^{2n})$.

Let $\mathrm{X}_-, \mathrm{X}_+ \in \mathrm{Crit}_H(\Gamma F_n)$ be non-degenerate stationary braids. A connecting orbit $\mathrm{U} \in \mathscr{M}^{J,H}_{\mathrm{X}_-,\mathrm{X}_+}$ is said to be NON-DEGENERATE, or TRANSVERSE, if the linearized Cauchy–Riemann operator

$$\frac{\partial}{\partial s} - \overline{J}\frac{\partial}{\partial t} + \overline{J}\,\overline{J}_0 d^2 \overline{H}(t, \mathrm{U}(s,t)) : \; W^{1,p}_\sigma \to L^p,$$

is a surjective operator (for all $1 < p < \infty$).

As before we equip $C^\infty(\mathbb{R}/\mathbb{Z} \times \mathbb{D}^2; \mathbb{R})$ with a Banach structure.

9.4. Proposition. *Let* $[\mathrm{X}] \text{ rel } \mathrm{Y}$ *be a proper relative braid class, and* $H \in \mathscr{H}$ *be a generic Hamiltonian such that* $\mathrm{Y} \in \mathrm{Crit}_H(\Gamma F_m)$*. Then, there exists a* $\delta_* > 0$ *such that for any* $\delta \leq \delta_*$ *there exists a nearby Hamiltonian* $H' \in \mathscr{H}$ *with* $\|H - H'\|_{C^\infty} < \delta$ *and* $\mathrm{Y} \in \mathrm{Crit}_{H'}(\Gamma F_m)$ *such that*

(i) $\mathrm{Crit}_{H'}([\mathrm{X}] \text{ rel } \mathrm{Y}) = \mathrm{Crit}_H([\mathrm{X}] \text{ rel } \mathrm{Y})$ *and consists of only non-degenerate stationary points for the action* $\mathscr{L}_{H'}$*;*

and for any pair $\mathrm{X}_-, \mathrm{X}_+ \in \mathrm{Crit}_{H'}([\mathrm{X}] \text{ rel } \mathrm{Y})$

(ii) $\mathscr{S}(\mathrm{X}_- \text{ rel } \mathrm{Y}, \mathrm{X}_+ \text{ rel } \mathrm{Y}; J, H')$ *is isolated in* $[\mathrm{X}] \text{ rel } \mathrm{Y}$*;*
(iii) $\mathscr{M}(\mathrm{X}_- \text{ rel } \mathrm{Y}, \mathrm{X}_+ \text{ rel } \mathrm{Y}; J, H')$ *consists of non-degenerate connecting orbits;*
(iv) $\mathscr{M}(\mathrm{X}_- \text{ rel } \mathrm{Y}, \mathrm{X}_+ \text{ rel } \mathrm{Y}; J, H')$ *are smooth manifolds without boundary and*

$$\dim \mathscr{M}(\mathrm{X}_- \text{ rel } \mathrm{Y}, \mathrm{X}_+ \text{ rel } \mathrm{Y}; J, H') = \mu(\mathrm{X}_-) - \mu(\mathrm{X}_+),$$

where μ *is the Conley–Zehnder index defined in Definition 8.5.*

Proof. Since $[\mathrm{X}] \text{ rel } \mathrm{Y}$ is a proper braid class it follows from Proposition 6.2 that $\mathscr{S}_{\mathrm{X}_-,\mathrm{X}_+}$ is isolated in $[\mathrm{X}] \text{ rel } \mathrm{Y}$ for any $H' \in \mathscr{H}$ provided $\mathrm{Y} \in \mathrm{Crit}_{H'}$.

As for the transversality properties we follow Salamon and Zehnder [35], where perturbations in $\mathbb{R}^{2n}$ are considered. We adapt the proof for Hamiltonians in $\mathbb{R}^2$. The proof is similar in spirit to the genericity of critical points. We restrict attention to the transversality properties of connecting orbits for a single pair $(\mathrm{x}_-, \mathrm{x}_+)$, and refer to Remark 9.5 for the general statement.

As in the proof of Proposition 9.1 we denote by $\widetilde{\mathscr{V}}_\epsilon$ the set of perturbations $h \in C^\infty(\mathbb{R}/\mathbb{Z} \times \mathbb{D}^2; \mathbb{R})$ whose support is bounded away from $(t, \mathrm{y}(t))$, $(t, \mathrm{x}_\pm(t))$ and $\partial\mathbb{D}^2$ (this yields a corresponding set A_ϵ as in the proof of Proposition 9.1). It follows that there exists a δ_* such that if we choose an $h \in \widetilde{\mathscr{V}}_{\delta,\epsilon} = \{h \in \widetilde{\mathscr{V}}_\epsilon \mid \|h\|_{C^\infty} < \delta\}$ with $\delta \le \delta_*$, then $\mathrm{Crit}_{H'}([\mathrm{x}] \text{ rel } \mathrm{y}) = \mathrm{Crit}_H([\mathrm{x}] \text{ rel } \mathrm{y})$ and it consists of only non-degenerate stationary points for the action $\mathscr{L}_{H'}$. For details of this construction we refer to the proof of Proposition 9.1.

Define the Cauchy–Riemann operator

$$\mathcal{G}(\mathrm{u}, h) = \mathrm{u}_s - \overline{J}\mathrm{u}_t + \overline{J}\,\overline{J}_0\nabla\overline{H}(t, \mathrm{u}) + \overline{J}\,\overline{J}_0\nabla\overline{h}(t, \mathrm{u}).$$

Then the mapping (see (8.1.2))

$$\mathcal{G} : \mathscr{U}^{1,p}(\mathrm{x}_- \text{ rel } \mathrm{y}, \mathrm{x}_+ \text{ rel } \mathrm{y}) \times \widetilde{\mathscr{V}}_{\delta,\epsilon} \to L^p(\mathbb{R} \times [0, 1]; \mathbb{R}^{2n}),$$

is smooth (as before, we shall only consider $\mathrm{u} \in \mathscr{U}^{1,p}$ with $|\mathrm{u}(s, t)| \le 1$ for all s, t). Define

$$\mathscr{Z}_{\mathrm{x}_-,\mathrm{x}_+} \stackrel{\text{def}}{=} \left\{(\mathrm{u}, h) \in \mathscr{U}^{1,p}(\mathrm{x}_- \text{ rel } \mathrm{y}, \mathrm{x}_+ \text{ rel } \mathrm{y}) \times \widetilde{\mathscr{V}}_{\delta,\epsilon} \mid \mathcal{G}(\mathrm{u}, h) = 0\right\} = \mathcal{G}^{-1}(0),$$

which is Banach manifold provided that $d\mathcal{G}(\mathrm{u}, h)$ is onto on for all $(\mathrm{u}, h) \in \mathscr{Z}_{\mathrm{x}_-,\mathrm{x}_+}$, where

$$d\mathcal{G}(\mathrm{u}, h)(\xi, \delta h) = d_1\mathcal{G}(\mathrm{u}, \delta h)\xi + \overline{J}\,\overline{J}_0\nabla\overline{\delta h}.$$

Assume that $d\mathcal{G}(\mathrm{u}, h)$ is not onto. Then there exists a non-zero function $\eta \in L^q$ which annihilates the range of $d\mathcal{G}(\mathrm{u}, h)$ and thus also the range of $d_1\mathcal{G}(\mathrm{u}, h)$, which is a Fredholm operator of index $\mu(\mathrm{x}_-) - \mu(\mathrm{x}_+)$; see Proposition 8.3. If we identify the dual pairing $\langle\cdot, \cdot\rangle$ with the g-inner product we obtain

$$\begin{aligned}
\langle d\mathcal{G}(\mathrm{u}, h)(\xi, \delta h), \eta\rangle &= \int_{-\infty}^{\infty}\int_0^1 \langle d\mathcal{G}(\mathrm{u}, h)(\xi, \delta h), \eta\rangle_g dt ds \\
&= \int_{-\infty}^{\infty}\int_0^1 \langle d\mathcal{G}(\mathrm{u}, h)(\xi, \delta h), -\overline{J}_0\overline{J}\eta\rangle_{\mathbb{R}^{2n}} dt ds = 0.
\end{aligned}$$

The relation $\langle d_1\mathcal{G}(\mathrm{u}, h)\xi, \eta\rangle = 0$ for all ξ then implies that

$$d_1\mathcal{G}(\mathrm{u}, h)^*\eta = -\eta_s - \overline{J}\eta_t + \overline{J}\,\overline{J}_0 d^2\overline{H}(t, \mathrm{u})\eta + \overline{J}\,\overline{J}_0 d^2\overline{h}(t, \mathrm{u})\eta = 0.$$

Since $\langle d\mathcal{G}(\mathrm{u}, h)(\xi, \delta h), \eta\rangle = 0$ it follows that

$$\int_{-\infty}^{\infty}\int_0^1 \langle \eta(s,t), \nabla_g \overline{\delta h}\rangle_g dtds = -\int_{-\infty}^{\infty}\int_0^1 \langle \eta(s,t), \overline{J}\overline{J}_0 \nabla \overline{\delta h}\rangle_g dtds$$
$$= \int_{-\infty}^{\infty}\int_0^1 \langle \eta(s,t), \nabla \overline{\delta h}\rangle_{\mathbb{R}^{2n}} dtds = 0, \tag{9.2.1}$$

for all δh. Due to the assumptions on h and H, the regularity theory for the linear Cauchy–Riemann operator implies that η is smooth. It remains to show that no such non-zero function η exists.

Step 1: The function η satisfies the following perturbed Laplace's equation: $\Delta\eta = \partial_{\overline{J}}\bar{\partial}_{\overline{J}}\eta = \partial_{\overline{J}}\overline{J}\overline{J}_0 d^2\overline{H}(t, \mathrm{U})\eta + \partial_{\overline{J}}\overline{J}\overline{J}_0 d^2\bar{h}(t, \mathrm{U})\eta$. If at some (s_0, t_0) all derivatives of η vanish, it follows from Aronszajn's unique continuation [6] that $\eta \equiv 0$ is a neighborhood of (s_0, t_0). Therefore $\eta(s, t) \neq 0$ for almost all $(s, t) \in \mathbb{R} \times [0, 1]$.

Step 2: The vectors $\eta(s, t) = (\eta^k(s, t))$ and $\mathrm{U}_s(s, t) = (u_s^k(s, t))$ are linearly dependent for all s and t. Suppose not, then these vectors are linearly independent at some point (s_0, t_0). By Theorem 8.2 in [35] we may assume without loss of generality that $\mathrm{U}_s(s_0, t_0) \neq 0$ and $\mathrm{U}(s_0, t_0) \neq \mathrm{X}_\pm(t_0)$ — a regular point. We carry out the argument for one component of U and we therefore assume without loss of generality that U consists of one component. As before let $\underline{\mathrm{U}}$ denote the concatenation of the strands:

$$\underline{\mathrm{U}}(s, \tau + k - 1) = u^k(s, \tau), \quad \underline{\eta}(s, \tau + k - 1) = \eta^k(s, \tau), \quad \tau \in [0, 1],\ k = 1, \cdots, m.$$

Then both $\underline{\mathrm{U}} : \mathbb{R} \times \mathbb{R}/m\mathbb{Z} \to \mathbb{D}^2$ and $\underline{\eta} : \mathbb{R} \times \mathbb{R}/m\mathbb{Z} \to \mathbb{R}^2$ satisfy the equations

$$\underline{\mathrm{U}}_s - J\underline{\mathrm{U}}_t + JJ_0\nabla H(t, \underline{\mathrm{U}}) + JJ_0\nabla h(t, \underline{\mathrm{U}}) = 0, \quad \text{and}$$
$$-\underline{\eta}_s - J\underline{\eta}_t + JJ_0 d^2H(t, \underline{\mathrm{U}})\underline{\eta} + JJ_0 d^2h(t, \underline{\mathrm{U}})\underline{\eta} = 0,$$

on $\mathbb{D}^2$ and $\underline{\mathrm{U}}_s$ and $\underline{\eta}$ are linearly independent at (s_0, t_0). We can now apply the arguments as in the proof of Theorem 8.4 in [35] with $M = \mathbb{D}^2$. Since (s_0, t_0) is a regular point there exists small neighborhood $U_0 \subset \mathbb{R}/m\mathbb{Z} \times \mathbb{D}^2$, such that $V_0 = \{(s, t) \mid (t, \underline{\mathrm{U}}(s, t)) \in U_0\}$ is a open neighborhood of (s_0, t_0) and $U_0 \cap A_\epsilon = \varnothing$. The set U_0 is a neighborhood of $u^k(s_0, t_0 - k + 1)$ — note that $t_0 - k + 1 \in [0, 1]$ —, for some $1 \le k \le m$ and has the important property that

$$u^{k'}(s, t - k' + 1) \notin U_0, \quad \forall \quad k \neq k', \text{ and } \quad \forall \quad t \in [k' - 1, k']. \tag{9.2.2}$$

The proof in [35] shows that the map $(s, t) \mapsto (t, \underline{\mathrm{U}}(s, t))$ from V_0 to U_0 is a diffeomorphism onto its image. By choosing V_0 small enough, $\underline{\eta}(s, t)$ and $\underline{\mathrm{U}}_s(s, t)$ are linearly independent of V_0. As in [35] this yields the existence of coordinates $\phi_t : \mathbb{D}^2 \to \mathbb{R}^2$ in a neighborhood of $(t_0, \underline{\mathrm{U}}(s_0, t_0)) \in U_0$ such that

$$\phi_t(\underline{u}(s, t)) = (s - s_0, 0), \qquad d\phi_t(\underline{\mathrm{U}}(s, t))\underline{\eta}(s, t) = (0, 1).$$

Define $g : \mathbb{R}/\mathbb{Z} \times \mathbb{R}^2 \to \mathbb{R}$ as follows:

$$g(t, y_1, y_2) = \beta(t - t_0)\beta(y_1)\beta(y_2)y_2, \quad \text{for} \quad t \in \mathbb{R},$$

where $\beta \geq 0$ is a C^∞ cutoff function such that $\beta = 1$ on a ball $B_{\delta_1}(0)$ centered at zero for sufficiently small positive δ_1, and $\beta = 0$ outside of $B_{2\delta_1}(0)$. We define a Hamiltonian $\delta h : \mathbb{R}/\mathbb{Z} \times \mathbb{D}^2 \to \mathbb{R}$ via

$$\delta h(t, x) = g(t, \phi_t(x)), \quad (t, x) \in U_0.$$

By construction, δh can be extended smoothly by setting it identically zero on the set $[t_0 - 1/2, t_0 + 1/2] \times \mathbb{D}^2 \setminus U_0$. We extend δh periodically to all of $\mathbb{R} \times \mathbb{D}^2$. Due to (9.2.2) it holds that $d\delta h(t, \underline{\mathrm{U}}(s, t)) \equiv 0$ for all $(t, \underline{\mathrm{U}}(s, t)) \in \mathbb{R}/m\mathbb{Z} \times \mathbb{D}^2 \setminus U_0$. This is the very reason why the 1-periodic extension above does not cause additional terms and allows for the construction of a Hamiltonian which is 1-periodic in t. On the set U_0 the above calculation yields

$$d\delta h(t, \underline{\mathrm{U}}(s, t))\underline{\eta}(s, t) = \beta(s - s_0)\beta(t - t_0),$$

for all $(s, t) \in V_0$. This implies

$$\int_{-\infty}^{\infty}\int_0^1 d\overline{\delta h}(t, \mathrm{U}(s, t))\eta(s, t) = \int_{-\infty}^{\infty}\int_0^1 d\delta h(t, \underline{\mathrm{U}}(s, t))\underline{\eta}(s, t) > 0,$$

and since $d\overline{\delta h}(t, \mathrm{U}(s, t))\eta(s, t) = \langle \eta(s, t), \nabla\overline{\delta h}\rangle_{\mathbb{R}^{2n}}$ we obtain

$$\int_{-\infty}^{\infty}\int_0^1 \langle \eta(s, t), \nabla\overline{\delta h}\rangle_{\mathbb{R}^{2n}} dt ds > 0,$$

which contradicts Eq. (9.2.1).

The remaining steps are identical to those in the proof of Theorem 8.4 in [35]: we outline these for completeness.

Step 3: The previous step implies the existence of a function $\lambda : \mathbb{R} \times [0, 1] \to \mathbb{R}$ such that $\eta(s, t) = \lambda(s, t)\frac{\partial \mathrm{U}}{\partial s}(s, t)$, for all s, t for which $\eta(s, t) \neq 0$. Using a contradiction argument with respect to Eq. (9.2.1) yields $\frac{\partial\lambda}{\partial s}(s, t) = 0$, for almost all (s, t). In particular we obtain that λ is s-independent and we can assume that $\lambda(t) \geq \delta > 0$ for all $t \in [0, 1]$ (invoking again unique continuation).

Step 4: This final step provides a contradiction to the assumption that $d\mathcal{G}$ is not onto. It holds that

$$\int_0^1 \left\langle \frac{\partial \mathrm{U}}{\partial s}(s, t), \eta(s, t)\right\rangle_g dt = \int_0^1 \lambda(t)\left|\frac{\partial \mathrm{U}}{\partial s}(s, t)\right|_g^2 dt \geq \delta\int_0^1 \left|\frac{\partial \mathrm{U}}{\partial s}(s, t)\right|_g^2 dt > 0.$$

The functions U_s and η satisfy the equations $d_1\mathcal{G}(\mathrm{U}, h)\mathrm{U}_s = 0$ and $d_1\mathcal{G}(\mathrm{U}, h)^*\eta = 0$, respectively. From these equations we can derive expressions for U_{ss} and η_s, from which it follows that

$$\frac{d}{ds}\int_0^1 \left\langle \frac{\partial \mathrm{U}}{\partial s}(s, t), \eta(s, t)\right\rangle_g dt = 0.$$

Combining this with the previous estimate yields that $\int_{-\infty}^{\infty}\int_0^1 |\mathrm{U}_s(s,t)|^2 dt = \infty$, which, combined with the compactness properties, contradicts the fact that $\mathrm{U} \in \mathscr{M}_{\mathrm{X}_-,\mathrm{X}_+}$ (Corollary 4.3); thus $d\mathcal{G}(\mathrm{U},h)$ is onto for all $(\mathrm{U},h) \in \mathscr{Z}_{\mathrm{X}_-,\mathrm{X}_+}$.

We can now apply the Sard–Smale theorem as in the proof of Proposition 9.1. The only difference here is that application of the Sard–Smale requires $(\mu(\mathrm{X}_-) - \mu(\mathrm{X}_+) + 1)$-smoothness of $\mathcal{G}$ which is guaranteed by the smoothness of Y, H and h. $\square$

9.5. Remark. We can label a Hamiltonian to be generic now if both $\mathrm{Crit}_H([\mathrm{X}] \text{ rel } \mathrm{Y})$ and $\mathscr{M}(\mathrm{X}_- \text{ rel } \mathrm{Y}, \mathrm{X}_+ \text{ rel } \mathrm{Y})$ for all $\mathrm{X}_\pm \in \mathrm{Crit}_H([\mathrm{X}] \text{ rel } \mathrm{Y})$, are non-degenerate. The terminology 'generic' is justified since the proofs of Propositions 9.1 and 9.4 provide a residual set of Hamiltonians $\mathscr{H}_{\mathrm{reg}}$. Namely, we may take the (finitely many) intersections for the different pairs $\mathrm{X}_\pm$, which yields a dense set of Hamiltonians for which properties (i)–(iv) in Proposition 9.4 hold for all pairs $(\mathrm{X}_-, \mathrm{X}_+)$ and thus for all of $\mathscr{M}([\mathrm{X}] \text{ rel } \mathrm{Y}; J, H)$.

For generic Hamiltonians $H \in \mathscr{H}_{\mathrm{reg}}$ the convergence of Corollary 9.3 can be extended with estimates on the Conley–Zehnder indices of the stationary braids.

9.6. Corollary. *Let* $[\mathrm{X}]$ rel Y *be a proper relative braid class and* $H \in \mathscr{H}_{\mathrm{reg}}$ *be a generic Hamiltonian with* $\mathrm{Y} \in \mathrm{Crit}_H(\Gamma F_m)$. *If* U_n *geometrically converges to the broken trajectory* $(\mathrm{U}^1, \ldots, \mathrm{U}^m)$, *with* $\mathrm{U}^i \in \mathscr{M}(\mathrm{X}^i \text{ rel } \mathrm{Y}, \mathrm{X}^{i-1} \text{ rel } \mathrm{Y}; J, H)$, $i = 1, \ldots, m$ *and* $\mathrm{X}^i \in \mathrm{Crit}_H([\mathrm{X}] \text{ rel } \mathrm{Y})$, $i = 0, \ldots, m$, *then*

$$\mu(\mathrm{X}^i) > \mu(\mathrm{X}^{i-1}),$$

for $i = 1, \ldots, m$.

Proof. See [33] for a detailed proof of this statement. $\square$

The above proof also carries over to the Cauchy–Riemann equations with s-dependent Hamiltonians $H(s,\cdot,\cdot)$. Exploiting the Fredholm index property for the s-dependent case we obtain the following corollary. Let $s \mapsto H(s,\cdot,\cdot)$ be a smooth path in $\mathscr{H}$ with the property $H_s = 0$ for $|s| \geq R$. We have the following non-autonomous version of Proposition 9.4, see [35].

9.7. Corollary. *Let* $[\mathrm{X} \text{ rel } \mathrm{Y}]$ *be a proper relative braid class with fibers* $[\mathrm{X}]$ rel Y, $[\mathrm{X}']$ rel Y' *in* $[\mathrm{X} \text{ rel } \mathrm{Y}]$. *Let* $s \mapsto H(s,\cdot,\cdot)$ *be a smooth path in* $\mathscr{H}$ *as described above with* $H_\pm = H(\pm\infty,\cdot,\cdot) \in \mathscr{H}_{\mathrm{reg}}$ *and* $\mathrm{Y} \in \mathrm{Crit}_{H_-}(\Gamma F_m)$, $\mathrm{Y}' \in \mathrm{Crit}_{H_+}(\Gamma F_m)$. *Then there exists a* $\delta_* > 0$ *such that for any* $\delta \leq \delta_*$ *there exists a path of Hamiltonians* $s \mapsto H'(s,\cdot,\cdot)$ *in* $\mathscr{H}$, *with* $H'_s = 0$ *for* $|s| \geq R$, $H(\pm\infty,\cdot,\cdot) = H_\pm$ *and* $\|H - H'\|_{C^\infty} < \delta$ *such that*

(i) $\mathscr{S}([\mathrm{X} \text{ rel } \mathrm{Y}]; J, H')$ *is isolated in* $[\mathrm{X} \text{ rel } \mathrm{Y}]$;
(ii) $\mathscr{M}(\mathrm{X}_- \text{ rel } \mathrm{Y}, \mathrm{X}'_+ \text{ rel } \mathrm{Y}'; J, H')$ *consist of non-degenerate connecting orbits with respect to the* s*-dependent Cauchy–Riemann equations;*
(iii) $\mathscr{M}(\mathrm{X}_- \text{ rel } \mathrm{Y}, \mathrm{X}'_+ \text{ rel } \mathrm{Y}'; J, H')$ *are smooth manifolds without boundary with*

$$\dim \mathscr{M}(\mathrm{X}_- \text{ rel } \mathrm{Y}, \mathrm{X}'_+ \text{ rel } \mathrm{Y}'; J, H') = \mu(\mathrm{X}_-) - \mu(\mathrm{X}'_+),$$

where μ *is the Conley–Zehnder indices with respect to the Hamiltonians* $H_\pm$.

10. Floer homology for proper braid classes

10.1. Definition

Let $\mathrm{Y} \in \Gamma F_m$ be a smooth braid and $[\mathrm{X}]$ rel Y a proper relative braid class. Let $H \in \mathscr{H}_{\mathrm{reg}}$ be a generic Hamiltonian with respect to the proper braid class $[\mathrm{X}]$ rel Y (as per Propositions 9.1 and 9.4). Then the set of bounded solutions $\mathscr{M}([\mathrm{X}] \text{ rel } \mathrm{Y})$ is compact and non-degenerate, $\mathrm{Crit}_H([\mathrm{X}] \text{ rel } \mathrm{Y})$ is non-degenerate, and $\mathscr{S}([\mathrm{X}] \text{ rel } \mathrm{Y})$ is isolated in $[\mathrm{X}]$ rel Y. Since $\mathrm{Crit}_H([\mathrm{X}] \text{ rel } \mathrm{Y})$ is a finite set we can define the chain groups

$$C_k\big([\mathrm{X}] \text{ rel } \mathrm{Y}, H; \mathbb{Z}_2\big) \stackrel{\text{def}}{=} \bigoplus_{\substack{\mathrm{X}' \in \mathrm{Crit}_H([\mathrm{X}] \text{ rel } \mathrm{Y}) \\ \mu(\mathrm{X}')=k}} \mathbb{Z}_2 \cdot \mathrm{X}', \tag{10.1.1}$$

as products of $\mathbb{Z}_2$. We define the boundary operator $\partial_k : C_k \to C_{k-1}$ in the standard manner as follows. By Proposition 9.4, the orbits $\mathrm{U} \in \mathscr{M}(\mathrm{X}_- \text{ rel } \mathrm{Y}, \mathrm{X}_+ \text{ rel } \mathrm{Y})$ are non-degenerate for all pairs $\mathrm{X}_-, \mathrm{X}_+ \in \mathrm{Crit}_H([\mathrm{X}] \text{ rel } \mathrm{Y})$. Let $\widehat{\mathscr{M}}(\mathrm{X}_- \text{ rel } \mathrm{Y}, \mathrm{X}_+ \text{ rel } \mathrm{Y}) = \mathscr{M}(\mathrm{X}_- \text{ rel } \mathrm{Y}, \mathrm{X}_+ \text{ rel } \mathrm{Y})/\mathbb{R}$ be the equivalence classes of orbits identified by translation in the s-variable. Consequently, the $\widehat{\mathscr{M}}(\mathrm{X}_- \text{ rel } \mathrm{Y}, \mathrm{X}_+ \text{ rel } \mathrm{Y})$ are smooth manifolds of dimension $\dim \widehat{\mathscr{M}}(\mathrm{X}_- \text{ rel } \mathrm{Y}, \mathrm{X}_+ \text{ rel } \mathrm{Y}) = \mu(\mathrm{X}_-) - \mu(\mathrm{X}_+) - 1$.

10.1. Lemma. *If $\mu(\mathrm{X}_-) - \mu(\mathrm{X}_+) = 1$, then $\widehat{\mathscr{M}}(\mathrm{X}_- \text{ rel } \mathrm{Y}, \mathrm{X}_+ \text{ rel } \mathrm{Y})$ consists of finitely many equivalence classes.*

Proof. From the compactness Theorem 4.1 and the geometric convergence in Corollaries 9.3 and 9.6, we derive that any sequence $\{\mathrm{U}_n\} \subset \mathscr{M}(\mathrm{X}_- \text{ rel } \mathrm{Y}, \mathrm{X}_+ \text{ rel } \mathrm{Y})$ has a subsequence which geometrically converges to a broken trajectory $(\mathrm{U}^1, \dots, \mathrm{U}^m)$, with $\mathrm{U}^i \in \mathscr{M}(\mathrm{X}^i \text{ rel } \mathrm{Y}, \mathrm{X}^{i-1} \text{ rel } \mathrm{Y})$, $i = 1, \dots, m$ and $\mathrm{X}^i \in \mathrm{Crit}_H([\mathrm{X}] \text{ rel } \mathrm{Y})$, $i = 0, \dots, m$, such that $\mu(\mathrm{X}^i) > \mu(\mathrm{X}^{i-1})$, for $i = 1, \dots, m$. Since by assumption $\mu(\mathrm{X}_-) = \mu(\mathrm{X}_+) + 1$, it follows that $m = 1$ and U_n converges to a single orbit $\mathrm{U}^1 \in \mathscr{M}(\mathrm{X}_- \text{ rel } \mathrm{Y}, \mathrm{X}_+ \text{ rel } \mathrm{Y})$. Therefore, the set $\widehat{\mathscr{M}}(\mathrm{X}_- \text{ rel } \mathrm{Y}, \mathrm{X}_+ \text{ rel } \mathrm{Y})$ is compact. From Proposition 9.4 it follows that the orbits in $\widehat{\mathscr{M}}(\mathrm{X}_- \text{ rel } \mathrm{Y}, \mathrm{X}_+ \text{ rel } \mathrm{Y})$ occur as isolated points and therefore $\widehat{\mathscr{M}}(\mathrm{X}_- \text{ rel } \mathrm{Y}, \mathrm{X}_+ \text{ rel } \mathrm{Y})$ is a finite set. □

Define the boundary operator by

$$\partial_k(J, H)\mathrm{X} \stackrel{\text{def}}{=} \sum_{\substack{\mathrm{X}' \in \mathrm{Crit}_H([\mathrm{X}] \text{ rel } \mathrm{Y}) \\ \mu(x')=k-1}} n(\mathrm{X}, \mathrm{X}'; J, H)\mathrm{X}', \tag{10.1.2}$$

where $n(\mathrm{X}, \mathrm{X}'; J, H) = \Big[\# \widehat{\mathscr{M}}_{\mathrm{X},\mathrm{X}'}\Big] \bmod 2 \in \mathbb{Z}_2$. The final property that the boundary operator has to satisfy is $\partial_{k-1} \circ \partial_k = 0$. The composition counts the number of 'broken connections' from X to X'' modulo 2.

10.2. Lemma. *If $\mu(\mathrm{X}_-) - \mu(\mathrm{X}_+) = 2$, then $\widehat{\mathscr{M}}(\mathrm{X}_- \text{ rel } \mathrm{Y}, \mathrm{X}_+ \text{ rel } \mathrm{Y})$ is a smooth 1-dimensional manifold with finitely many connected components. The non-compact components can be* *identified with* $(0, 1)$ *and the closure with* $[0, 1]$. *The limits* $\{0, 1\}$ *correspond to unique pairs of distinct broken trajectories (this follows from "gluing", see [13,33])*

$$(\mathrm{U}^1, \mathrm{U}^2) \in \mathscr{M}(\mathrm{X}_- \text{ rel } \mathrm{Y}, \mathrm{X}' \text{ rel } \mathrm{Y}) \times \mathscr{M}(\mathrm{X}' \text{ rel } \mathrm{Y}, \mathrm{X}_+ \text{ rel } \mathrm{Y}),$$

and

$$(\tilde{\mathrm{U}}^1, \tilde{\mathrm{U}}^2) \in \mathscr{M}(\mathrm{X}_- \text{ rel } \mathrm{Y}, \mathrm{X}'' \text{ rel } \mathrm{Y}) \times \mathscr{M}(\mathrm{X}'' \text{ rel } \mathrm{Y}, \mathrm{X}_+ \text{ rel } \mathrm{Y}),$$

with $\mu(\mathrm{X}'') = \mu(\mathrm{X}') = \mu(\mathrm{X}_-) - 1$.

We point out that properness of $[\mathrm{X} \text{ rel } \mathrm{Y}]$ and thus the isolation of $\mathscr{S}^{J,H}$ is crucial for the validity of Lemma 10.2. From Lemma 10.2 it follows that the total number of broken connections from X to X'' is even; hence $\partial_{k-1} \circ \partial_k = 0$, and consequently,

$$\Big(C_*\big([\mathrm{X}] \text{ rel } \mathrm{Y}, H; \mathbb{Z}_2\big), \partial_*(J, H)\Big)$$

is a (finite) chain complex. The Floer homology of $\big([\mathrm{X}] \text{ rel } \mathrm{Y}, J, H\big)$ is the homology of the chain complex (C_*, ∂_*):

$$\mathrm{HF}_k\big([\mathrm{X}] \text{ rel } \mathrm{Y}, J, H; \mathbb{Z}_2\big) \stackrel{\text{def}}{=} \frac{\ker \partial_k}{\mathrm{im}\partial_{k+1}}. \tag{10.1.3}$$

This Floer homology is finite. It is not yet established that HF_* is independent of J, H and whether HF_* is an invariant for proper relative braid class $[\mathrm{X} \text{ rel } \mathrm{Y}]$.

10.2. Continuation

Floer homology has a powerful invariance property with respect to 'large' variations in its parameters [13]. Let $[\mathrm{X}]$ rel Y be a proper relative braid class and consider almost complex structures $J, \widetilde{J} \in \mathscr{J}$, and generic Hamiltonians $H, \widetilde{H} \in \mathscr{H}_{\mathrm{reg}}$ such that $\mathrm{Y} \in \mathrm{Crit}_H \cap \mathrm{Crit}_{\widetilde{H}}$. Then the Floer homologies $\mathrm{HF}_*\big([\mathrm{X}] \text{ rel } \mathrm{Y}, J, H; \mathbb{Z}_2\big)$ and $\mathrm{HF}_*\big([\mathrm{X}] \text{ rel } \mathrm{Y}, \widetilde{J}, \widetilde{H}; \mathbb{Z}_2\big)$ are well-defined.

10.3. Proposition. *Given a proper relative braid class* $[\mathrm{X}]$ rel Y,

$$\mathrm{HF}_*\big([\mathrm{X}] \text{ rel } \mathrm{Y}, J, H; \mathbb{Z}_2\big) \cong \mathrm{HF}_*\big([\mathrm{X}] \text{ rel } \mathrm{Y}, \widetilde{J}, \widetilde{H}; \mathbb{Z}_2\big),$$

under the hypotheses on (J, H) *and* $(\widetilde{J}, \widetilde{H})$ *as stated above.*

In order to prove the isomorphism we follow the standard procedure in Floer homology. The main steps can be summarized as follows. Consider the chain complexes

$$\Big(C_*\big([\mathrm{X}] \text{ rel } \mathrm{Y}, H; \mathbb{Z}_2\big), \partial_*(J, H)\Big) \text{ and } \Big(C_*\big([\mathrm{X}] \text{ rel } \mathrm{Y}, \widetilde{H}; \mathbb{Z}_2\big), \partial_*(\widetilde{J}, \widetilde{H})\Big),$$

and construct homomorphisms h_k satisfying the commutative diagram

$$\begin{tikzcd}
\cdots \arrow[r] & C_k(H) \arrow[r, "\partial_k(J,H)"] \arrow[d, "h_k"'] & C_{k-1}(H) \arrow[r, "\partial_{k-1}(J,H)"] \arrow[d, "h_{k-1}"'] & C_{k-2}(H) \arrow[r] \arrow[d, "h_{k-2}"'] & \cdots \\
\cdots \arrow[r] & C_k(\widetilde{H}) \arrow[r, "\partial_k(\widetilde{J},\widetilde{H})"] & C_{k-1}(\widetilde{H}) \arrow[r, "\partial_{k-1}(\widetilde{J},\widetilde{H})"] & C_{k-2}(\widetilde{H}) \arrow[r] & \cdots
\end{tikzcd}$$

To define h_k consider the homotopies $\lambda \mapsto (J_\lambda, H_\lambda)$ in $\mathscr{J} \times \mathscr{H}$ with $\lambda \in [0,1]$. In particular choose $H_\lambda = (1-\lambda)H + \lambda\widetilde{H}$ such that $\mathrm{Y} \in \mathrm{Crit}_{H_\lambda}$ for all $\lambda \in [0,1]$. Note that at the end points $\lambda = 0, 1$ the systems are generic, i.e., $H_0 = H \in \mathscr{H}_{\mathrm{reg}}$ and $H_1 = \widetilde{H} \in \mathscr{H}_{\mathrm{reg}}$; this is not necessarily true for all $\lambda \in (0,1)$. Define the smooth function $\lambda(s)$ such that $\lambda(s) = 0$ for $s \le -R$ and $\lambda(s) = 1$ for $s \ge R$, for some $R > 0$ and $0 \le \lambda(s) \le 1$ on $\mathbb{R}$. The non-autonomous Cauchy–Riemann equations become

$$u_s - J_{\lambda(s)} u_t - \nabla_{g_s} H_{\lambda(s)}(t,u) = 0. \tag{10.2.1}$$

By setting $J(s,\cdot,\cdot) = J_{\lambda(s)}(\cdot,\cdot)$ and $H(s,\cdot,\cdot) = H_{\lambda(s)}$, Eq. (10.2.1) fits in the framework of Eq. (4.3.1). By Corollary 9.7 the path $s \mapsto H(s,\cdot,\cdot)$ can be chosen to be generic with the same limits.

Denote the space of bounded solutions by $\mathscr{M}([x] \text{ rel } \mathrm{Y}; J_\lambda, H_\lambda)$. The required basic compactness result is as follows:

10.4. Proposition. *The space $\mathscr{M}([x] \text{ rel } \mathrm{Y}; J_\lambda, H_\lambda)$ is compact in the topology of uniform convergence on compact sets in $(s,t) \in \mathbb{R}^2$, with derivatives up to order arbitrary order. Moreover, $\mathscr{L}_{H_\lambda}$ is uniformly bounded along trajectories $\mathrm{U} \in \mathscr{M}([x] \text{ rel } \mathrm{Y}; J_\lambda, H_\lambda)$, and*

$$\lim_{s\to\pm\infty} |\mathscr{L}_{H_\lambda}(\mathrm{U}(s,\cdot))| = |c_\pm(\mathrm{U})| \le C(J, \tilde{J}, H, \tilde{H}),$$

$$\int_{\mathbb{R}}\int_0^1 |\mathrm{U}_s|^2 dt ds = \sum_{k=1}^n \int_{\mathbb{R}}\int_0^1 |u_s^k|^2 dt ds \le C'(J' \tilde{J}, H, \tilde{H}).$$

Moreover, $\lim_{s\to-\infty} \mathrm{U}(s,\cdot) \in \mathrm{Crit}_H$ *and* $\lim_{s\to+\infty} \mathrm{U}(s,\cdot) \in \mathrm{Crit}_{\widetilde{H}}$.

Proof. Compactness follows from the estimates in Section 4.3 and the compactness in Proposition 4.1. Due to genericity, bounded solutions have limits in $\mathrm{Crit}_H \cup \mathrm{Crit}_{\widetilde{H}}$, see Corollary 9.2. □

We define a homomorphism $h_k = h_k(J_\lambda, H_\lambda)$ as follows:

$$h_k \mathrm{X} = \sum_{\substack{\mathrm{X}' \in \mathrm{Crit}_{\widetilde{H}} \\ \mu(\mathrm{X}')=k}} n(\mathrm{X}, \mathrm{X}'; J_\lambda, H_\lambda)\mathrm{X}',$$

where $n(\mathrm{x}, \mathrm{x}'; J_\lambda, H_\lambda) = \Big[\# \mathscr{M}(\mathrm{x} \text{ rel } \mathrm{y}, \mathrm{x}' \text{ rel } \mathrm{y}; J_\lambda, H_\lambda)\Big] \bmod 2 \in \mathbb{Z}_2$. Using similar gluing constructions and the isolation of the sets $\mathscr{S}^{J,H}$ and $\mathscr{S}^{J',H'}$, it is straightforward to show that the mappings h_k are chain homomorphisms and induce a homomorphisms h_k^* on Floer homology:

$$h_k^*(J_\lambda, H_\lambda) : \mathrm{HF}_*(\mathrm{x} \text{ rel } \mathrm{y}; J, H) \to \mathrm{HF}_*(\mathrm{x} \text{ rel } \mathrm{y}; \widetilde{J}, \widetilde{H}).$$

Further analysis of the non-autonomous Cauchy–Riemann equations and standard procedures in Floer theory show that any two homotopies (J_λ, H_λ) and $(\hat{J}_\lambda, \hat{H}_\lambda)$ between (J, H) and $(\widetilde{J}, \widetilde{H})$ descend to the same homomorphism in Floer homology:

10.5. Proposition. *For any two homotopies (J_λ, H_λ) and $(\hat{J}_\lambda, \hat{H}_\lambda)$ between (J, H) and $(\widetilde{J}, \widetilde{H})$,*

$$h_k^*(J_\lambda, H_\lambda) = h_k^*(\hat{J}_\lambda, \hat{H}_\lambda).$$

Moreover, for a homotopy (J_λ, H_λ) between (J, H) and $(\widetilde{J}, \widetilde{H})$, and a homotopy $(\hat{J}_\lambda, \hat{H}_\lambda)$ between $(\widetilde{J}, \widetilde{H})$ and $(\check{J}, \check{H})$, the induced homomorphism between the Floer homologies is given by

$$h_k^* : \mathrm{HF}_*([\mathrm{x}] \text{ rel } \mathrm{y}, J, H) \to \mathrm{HF}_*([\mathrm{x}] \text{ rel } \mathrm{y}, \check{J}, \check{H}),$$

where $h_k^* = h_k^*(\hat{J}_\lambda, \hat{H}_\lambda) \circ h_k^*(J_\lambda, H_\lambda)$ and h_k^* is thus an isomorphism.

Proof of Proposition 10.3. Consider the homotopties

$$h_k^* : \mathrm{HF}_*(\mathrm{x} \text{ rel } \mathrm{y}; J, H) \to \mathrm{HF}_*(\mathrm{x} \text{ rel } \mathrm{y}; \widetilde{J}, \widetilde{H}),$$

and

$$\bar{h}_k^* : \mathrm{HF}_*(\mathrm{x} \text{ rel } \mathrm{y}; \widetilde{J}, \widetilde{H}) \to \mathrm{HF}_*(\mathrm{x} \text{ rel } \mathrm{y}; J, H),$$

then

$$\bar{h}_k^* \circ h_k^* : \mathrm{HF}_*(\mathrm{x} \text{ rel } \mathrm{y}; J, H) \to \mathrm{HF}_*(\mathrm{x} \text{ rel } \mathrm{y}; J, H).$$

Since a homotopy from (J, H) to itself induces the identity homomorphism on homology, it holds that $\bar{h}_k^* \circ h_k^* = \mathrm{Id}$. By the same token it follows that $h_k \circ \bar{h}_k^* = \mathrm{Id}$, which proves $\bar{h}_k^* = (h_k^*)^{-1}$ and thus the proposition. □

10.3. Admissible pairs and independence of the skeleton

By Proposition 10.3 the Floer homology of $[\mathrm{x}]$ rel y is independent of a generic pair (J, H), which justifies the notation $\mathrm{HF}_*([\mathrm{x}] \text{ rel } \mathrm{y}; \mathbb{Z}_2) \stackrel{\text{def}}{=} \varprojlim \mathrm{HF}_*([\mathrm{x}] \text{ rel } \mathrm{y}; J, H)$. It remains to show that, firstly, for any braid class $[\mathrm{x}]$ rel y an appropriate Hamiltonian (having stationary y) exists, and thus the Floer homology is defined, and secondly that the Floer homology depends only on the braid class $[\mathrm{x} \text{ rel } \mathrm{y}]$.

10.6. Lemma. *Let* $\mathrm{Y} \in \Gamma F_m \cap C^\infty$*, then there exists a Hamiltonian* $H \in \mathscr{H}$ *such that* $\mathrm{Y} \in \mathrm{Crit}_H(\Gamma F_m)$*, i.e.* Y *is a braided solution of the associated Hamilton equations.*

Proof. Let $\mathrm{Y} = \{y^k\}_{k=1}^m$ and define $H^k(t,x) = \langle y_t^k, J_0 x\rangle$, which is a C^∞-function on $\mathbb{R}\times\mathbb{R}^2$. Note that $H^k(t+1,x) = H^{\sigma(k)}(t,x)$, and H^k is smooth, since Y is smooth. The strand y^k is a solution of the Hamilton equations for H^k. Define tubular neighborhoods $A_\epsilon^k = \bigcup_{t\in\mathbb{R}} B_\epsilon(y^k(t)) \subset \mathbb{R}\times\mathbb{D}^2$, and $D_\epsilon = \{x \in \mathbb{D}^2 \mid 1-\epsilon < |x| \le 1\}$. Choose $\epsilon > 0$ so small that the sets $\{A_\epsilon^k\}_{k=1}^m$ and D_ϵ are all disjoint. Define a cut-off function $\lambda_\epsilon \in C^\infty([0,\infty),\mathbb{R})$ such that $\lambda(r) = 1$ for $0 \le r \le \epsilon/4$ and $\lambda(r) = 0$ for $r \ge \epsilon/2$. Let $\lambda_\epsilon^k(t,x) = \lambda_\epsilon(|x - y^k(t)|)$. Then λ_ϵ^k is a smooth function with support in A_ϵ^k, and $\lambda_\epsilon^k(t+1,x) = \lambda_\epsilon^{\sigma(k)}(t,x)$. Now define

$$H(t,x) \stackrel{\text{def}}{=} \sum_{k=1}^m \lambda_\epsilon^k(t,x) H^k(t,x).$$

We claim that $H \in \mathscr{H}$. Indeed,

$$H(t+1,x) = \sum_{k=1}^m \lambda_\epsilon^k(t+1,x) H^k(t+1,x) = \sum_{k=1}^m \lambda_\epsilon^{\sigma(k)}(t,x) H^{\sigma(k)}(t,x) = H(t,x).$$

By the construction of H, it holds that $y_t^k = X_{H^k}(t,y^k) = X_H(t,y^k)$, since H restricts to H^k in a neighborhood of y^k. □

This establishes that $\mathrm{HF}_*([\mathrm{X}] \text{ rel } \mathrm{Y})$ is well-defined for any proper relative braid class $[\mathrm{X}] \text{ rel } \mathrm{Y} \in \Gamma F_n \text{ rel } \mathrm{Y}$, with $\mathrm{Y} \in \Gamma F_m \cap C^\infty$. We still need to establish independence of the braid class in $[\mathrm{X} \text{ rel } \mathrm{Y}]$, i.e., that the Floer homology is the same for any two relative braid classes $[\mathrm{X}] \text{ rel } \mathrm{Y}$, $[\mathrm{X}'] \text{ rel } \mathrm{Y}'$ such that $[\mathrm{X} \text{ rel } \mathrm{Y}] = [\mathrm{X}' \text{ rel } \mathrm{Y}']$. This leads to the first main result of this paper.

10.7. Theorem. *Let* $[\mathrm{X} \text{ rel } \mathrm{Y}]$ *be a proper relative braid class. Then,*

$$\mathrm{HF}_*([\mathrm{X}] \text{ rel } \mathrm{Y}) \cong \mathrm{HF}_*([\mathrm{X}'] \text{ rel } \mathrm{Y}'),$$

for any two fibers $[\mathrm{X}] \text{ rel } \mathrm{Y}$ *and* $[\mathrm{X}'] \text{ rel } \mathrm{Y}'$ *in* $[\mathrm{X} \text{ rel } \mathrm{Y}]$*. In particular,*

$$\mathrm{HB}_*([\mathrm{X} \text{ rel } \mathrm{Y}]) \stackrel{\text{def}}{=} \varprojlim \mathrm{HF}_*([\mathrm{X}] \text{ rel } \mathrm{Y})$$

is an invariant of $[\mathrm{X} \text{ rel } \mathrm{Y}]$*, which will be referred to as the* BRAID FLOER HOMOLOGY *of* $[\mathrm{X} \text{ rel } \mathrm{Y}]$.

Proof. Let $\mathrm{Y}, \mathrm{Y}' \in \Gamma F_m \cap C^\infty$ and let $(\mathrm{X}(\lambda), \mathrm{Y}(\lambda))$, $\lambda \in [0,1]$ be a smooth path $[\mathrm{X} \text{ rel } \mathrm{Y}]$ which connects the pairs $\mathrm{X} \text{ rel } \mathrm{Y}$ and $\mathrm{X}' \text{ rel } \mathrm{Y}'$. Since $\mathrm{X}(\lambda) \text{ rel } \mathrm{Y}(\lambda) \in [\mathrm{X} \text{ rel } \mathrm{Y}]$, for all $\lambda \in [0,1]$, the sets $\mathscr{N}_\lambda = [\mathrm{X}(\lambda)] \text{ rel } \mathrm{Y}(\lambda)$ are isolating neighborhoods for all λ. The proof of Lemma 10.6 shows that we can choose smooth Hamiltonians H_λ such that $\mathrm{Y}(\lambda) \in \mathrm{Crit}_{H_\lambda}$.

There are two philosophies one can follow to prove this theorem. On the one hand, using the genericity theory in Section 9 (Corollary 9.7), we can choose a generic family (J_λ, H_λ) for any smooth homotopy of almost complex structures J_λ. Then by repeating the proof (of Proposition 10.3) and incorporating continuously changing skeletons for this homotopy, we conclude that $\mathrm{HF}_*([\mathrm{X}] \text{ rel } \mathrm{Y}) \cong \mathrm{HF}_*([\mathrm{X}'] \text{ rel } \mathrm{Y}')$. On the other hand, without having to redo the homotopy theory we note that $\mathscr{S}([\mathrm{X}(\lambda)] \text{ rel } \mathrm{Y}(\lambda); J_\lambda, H_\lambda)$ is compact and isolated in $\mathscr{N}_\lambda$; thus, there exists an ϵ_λ for each $\lambda \in [0,1]$ such that $\mathscr{N}_\lambda$ isolates $\mathscr{S}([\mathrm{X}(\lambda')] \text{ rel } \mathrm{Y}(\lambda'))$ for all λ' in $[\lambda - \epsilon_\lambda, \lambda + \epsilon_\lambda]$. Fix $\lambda_0 \in (0,1)$; then, by arguments similar to those of Proposition 10.3, we have

$$\mathrm{HF}_*(\mathscr{N}_{\lambda_0}; J_{\lambda_0}, H_{\lambda_0}) \cong \mathrm{HF}_*(\mathscr{N}_{\lambda_0}; J_{\lambda'}, H_{\lambda'}),$$

for all $\lambda' \in [\lambda_0 - \epsilon_{\lambda_0}, \lambda_0 + \epsilon_{\lambda_0}]$ (i.e. using a fixed skeleton). A compactness argument shows that, for ϵ'_{λ_0} sufficiently small, the sets of bounded solutions $\mathscr{M}(\mathscr{N}_{\lambda'}; J_{\lambda'}, H_{\lambda'})$ and $\mathscr{M}(\mathscr{N}_{\lambda_0}; J_{\lambda'}, H_{\lambda'})$ are identical, for all $\lambda' \in [\lambda_0 - \epsilon'_{\lambda_0}, \lambda_0 + \epsilon'_{\lambda_0}]$. Together these imply that

$$\mathrm{HB}_*\big([\mathrm{X}(\lambda') \text{ rel } \mathrm{Y}(\lambda')]\big) \cong \mathrm{HB}_*\big([\mathrm{X}(\lambda_0) \text{ rel } \mathrm{Y}(\lambda_0)]\big)$$

for $|\lambda' - \lambda_0| \le \min\{\epsilon_{\lambda_0}, \epsilon'_{\lambda_0}\}$. Since $[0,1]$ is compact, any covering has a finite subcovering, which proves that $\mathrm{HF}_*([\mathrm{X}] \text{ rel } \mathrm{Y}) \cong \mathrm{HF}_*([\mathrm{X}'] \text{ rel } \mathrm{Y}')$.

Finally, since any skeleton Y in π ([X rel Y]) can be approximated by a smooth skeleton Y', the isolating neighborhood $\mathscr{N} = \pi^{-1}(\mathrm{Y}) \cap [\mathrm{X} \text{ rel } \mathrm{Y}]$ is also isolating for Y', i.e., we can define $\mathrm{HF}_*(\mathscr{N}) \stackrel{\text{def}}{=} \mathrm{HF}_*(\mathscr{N}')$. This defines $\mathrm{HF}_*([\mathrm{X}] \text{ rel } \mathrm{Y}) = \mathrm{HF}_*(\mathscr{N})$ for any $\mathrm{Y} \in \pi([\mathrm{X} \text{ rel } \mathrm{Y}])$. $\square$

11. Properties and interpretation of the braid class invariant

The braid Floer homology $\mathrm{HB}_*([\mathrm{X} \text{ rel } \mathrm{Y}])$ entwines braiding and dynamical features of solutions of the Hamilton equations (2.1.1) on the 2-disc $\mathbb{D}^2$. One such property — the non-triviality of the invariant yields braided solutions — will form the basis of a forcing theory.

11.1. Theorem. *Let* $H \in \mathscr{H}$ *and let* $\mathrm{Y} \in \mathrm{Crit}_H(\Gamma F_m)$. *Let* $[\mathrm{X} \text{ rel } \mathrm{Y}]$ *be a proper relative braid class. If* $\mathrm{HB}_*([\mathrm{X} \text{ rel } \mathrm{Y}]) \neq 0$, *then* $\mathrm{Crit}_H([\mathrm{X}] \text{ rel } \mathrm{Y}) \neq \varnothing$.

Proof. Let $H_n \in \mathscr{H}$ be a sequence of Hamiltonians such that $H_n \to H$, i.e. $H_n - H \to 0$ in C^∞, see Section 9. If $\mathrm{HB}_*([\mathrm{X} \text{ rel } \mathrm{Y}]) \neq 0$, then $C_*([\mathrm{X}] \text{ rel } \mathrm{Y}, H_n; \mathbb{Z}_2) \neq 0$ for any n, since

$$H_*\big(C_*([\mathrm{X}] \text{ rel } \mathrm{Y}, H_n; \mathbb{Z}_2), \partial_*\big) \cong \mathrm{FH}_*([\mathrm{X} \text{ rel } \mathrm{Y}]) \neq 0,$$

where $\partial_* = \partial_*(J, H_n)$ (see Section 10). Consequently, $\mathrm{Crit}_{H_n}([\mathrm{X}] \text{ rel } \mathrm{Y}) \neq \varnothing$. The strands x_n^k satisfy the equation $(x_n^k)' = X_{H_n}(t, x_n^k)$ and therefore $\|x_n^k\|_{C^1([0,1])} \le C$. By the compactness of $C^1([0,1]) \hookrightarrow C^0([0,1])$ it follows that (along a subsequence) $x_n^k \to x^k \in C^0([0,1])$. The right hand side of the Hamilton equations now converges $X_{H_n}(t, x_n^k(t)) \to X_H(t, x(t))$ pointwise in $t \in [0,1]$; thus $x_n^k \to x^k$ in $C^1([0,1])$. This holds for any strand x^k and therefore produces a limit $\mathrm{X} \in \mathrm{Crit}_H([\mathrm{X}] \text{ rel } \mathrm{Y})$. $\square$

Let $\beta_k = \dim \mathrm{HB}_k([\mathrm{X} \text{ rel } \mathrm{Y}]; \mathbb{Z}_2)$ be the $\mathbb{Z}_2$-Betti numbers of the braid class invariant. Its Poincaré series is defined as

$$P_t([\mathrm{X} \text{ rel } \mathrm{Y}]) = \sum_{k \in \mathbb{Z}} \beta_k([\mathrm{X} \text{ rel } \mathrm{Y}]) t^k.$$

11.2. Theorem. *The braid Floer homology of any proper relative braid class is finite.*

Proof. Assume without loss of generality that Y is a smooth skeleton and choose a smooth generic Hamiltonian H such that $\mathrm{Y} \in \mathrm{Crit}_H$. Since the Floer homology is the same for all braid classes $[\mathrm{X}]$ rel $\mathrm{Y} \in [\mathrm{X} \text{ rel } \mathrm{Y}]$ and all Hamiltonians H satisfying the above, we have $\mathrm{HF}_*([\mathrm{X}] \text{ rel } \mathrm{Y}, J, H) \cong \mathrm{HB}_*([\mathrm{X} \text{ rel } \mathrm{Y}])$. Let $c_k = \dim C_k$; then,

$$c_k([\mathrm{X}] \text{ rel } \mathrm{Y}, H) \geq \dim\ker C_k \geq \beta_k([\mathrm{X}] \text{ rel } \mathrm{Y}, J, H) = \beta_k([\mathrm{X} \text{ rel } \mathrm{Y}]).$$

Since H is generic it follows from compactness that $\sum_k c_k < \infty$. Therefore $c_k < \infty$ and $c_k \neq 0$ for finitely many k. By the above bound $\beta_k \leq c_k < \infty$. $\square$

In the case that H is a generic Hamiltonian a more detailed result follows. Both $\bigoplus_k \mathrm{HF}_k([\mathrm{X} \text{ rel } \mathrm{Y}]; \mathbb{Z}_2)$ and $\bigoplus_k C_k([\mathrm{X}] \text{ rel } \mathrm{Y}, H; \mathbb{Z}_2)$ are graded $\mathbb{Z}_2$-modules, and their Poincaré series are well-defined:

$$P_t\big(\mathrm{Crit}_H([\mathrm{X}] \text{ rel } \mathrm{Y})\big) = \sum_{k\in\mathbb{Z}} c_k([\mathrm{X}] \text{ rel } \mathrm{Y}, H)t^k,$$

where $c_k = \dim C_k([\mathrm{X}] \text{ rel } \mathrm{Y}, H; \mathbb{Z}_2)$.

11.3. Theorem. *Let* $[\mathrm{X} \text{ rel } \mathrm{Y}]$ *be a proper relative braid class and H a generic Hamiltonian such that* $\mathrm{Y} \in \mathrm{Crit}_H$ *for a given skeleton* Y. *Then*

$$P_t\big(\mathrm{Crit}_H([\mathrm{X}] \text{ rel } \mathrm{Y})\big) = P_t([\mathrm{X} \text{ rel } \mathrm{Y}]) + (1+t)Q_t, \tag{11.0.1}$$

where $Q_t \geq 0$. In addition, $\#\,\mathrm{Crit}_H([\mathrm{X}] \text{ rel } \mathrm{Y}) \geq P_1([\mathrm{X} \text{ rel } \mathrm{Y}])$.

Proof. Define $Z_k = \ker \partial_k$, $B_k = \mathrm{im}\ \partial_{k+1}$ and $Z_k \subset B_k \subset C_k([\mathrm{X}] \text{ rel } \mathrm{Y}, H)$ by the fact that ∂_* is a boundary map. This yields the following short exact sequence

$$0 \xrightarrow{\mathrm{Id}} B_k \xrightarrow{i_k} Z_k \xrightarrow{j_k} \mathrm{HF}_k = \frac{Z_k}{B_k} \xrightarrow{0} 0.$$

The maps i_k and j_k are defined as follows: $i_k(\mathrm{X}) = \mathrm{X}$ and $j_k(\mathrm{X}) = \{\mathrm{X}\}$, the equivalence class in HF_k. Exactness is satisfied since $\ker i_k = 0 = \mathrm{im}\ \mathrm{Id}$, $\ker j_k = B_k = \mathrm{im}\ i_k$ and $\ker 0 = \mathrm{HF}_k = \mathrm{im}\ j_k$. Upon inspection of the short exact sequence we obtain

$$\dim Z_k = \dim B_k + \dim \mathrm{HF}_k.$$

Indeed, by exactness, $Z_k \supset \ker j_k = B_k$ and $\mathrm{im}\ j_k = \mathrm{HF}_k$ (onto) and therefore $\dim Z_k = \dim\ker j_k + \dim\mathrm{im}\ j_k = \dim B_k + \dim \mathrm{HF}_k$. Since $C_k \cong Z_k \oplus B_{k-1}$ it holds that

$$\dim C_k = \dim Z_k + \dim B_{k-1}.$$

Combining these equalities gives $\dim C_k = \dim \mathrm{HF}_k + \dim B_{k-1} + B_k$. On the level of Poincaré series this gives

$$P_t(\oplus_k C_k) = P_t(\oplus_k \mathrm{HF}_k) + (1+t)P_t(\oplus_k B_k),$$

which proves (11.0.1). $\square$

An important question is whether $\mathrm{HB}_*([\mathrm{X} \text{ rel } \mathrm{Y}])$ also contains information about $\mathrm{Crit}_H([\mathrm{X}] \text{ rel } \mathrm{Y})$ in the non-generic case besides the result in Theorem 11.1. In [18] such a result was indeed obtained in a related setting, and detailed study of the spectral properties of stationary braids will most likely reveal a similar property. We conjecture that $\#\,\mathrm{Crit}_H([\mathrm{X}] \text{ rel } \mathrm{Y}) \geq \mathrm{length}\big(\mathrm{HB}_*([\mathrm{X} \text{ rel } \mathrm{Y}])\big)$, where $\mathrm{length}(\mathrm{HB}_*)$ equals the number of monomial terms in $P_t([\mathrm{X} \text{ rel } \mathrm{Y}])\big)$.

12. Homology shifts and Garside's normal form

In this section we show that composing a braid class with full twists yields a shift in braid Floer homology. Consider the symplectic twist $S : [0, 1] \to \mathrm{Sp}(2, \mathbb{R})$ defined by $S(t) = e^{2\pi J_0 t}$, which rotates the variables counter clock wise over 2π as t goes from 0 to 1. On the product $\mathbb{R}^2 \times \cdots \times \mathbb{R}^2 \cong \mathbb{R}^{2n}$ this yields the product rotation $\bar{S}(t) = e^{2\pi \bar{J}_0 t}$ in $\mathrm{Sp}(2n, \mathbb{R})$.

Lifting to the Hamiltonian gives $\bar{S}\mathrm{X} \in \mathrm{Crit}_{\bar{S}H}$, where the rotated Hamiltonian $\bar{S}H \in \mathscr{H}$ is given by $\bar{S}H(t, \bar{S}\mathrm{X}) = H\big(t, \mathrm{X}\big) + \pi\big(|\bar{S}x|^2 - 1\big)$. Substitution yields the transformed Hamilton equations (by using that $S^T = S^{-1}$):

$$(\bar{S}\mathrm{X})_t - \bar{S}J_0\nabla H(t, \mathrm{X}) - 2\pi J_0\bar{S}\mathrm{X} = 0, \tag{12.0.2}$$

which are the Hamilton equations for the Hamiltonian $\bar{S}H$. This twisting induces a shift between the Conley–Zehnder indices $\mu(\mathrm{X})$ and $\mu(\bar{S}x)$:

12.1. Lemma. *For $\mathrm{X} \in \mathrm{Crit}_H$, $\mu(\bar{S}\mathrm{X}) = \mu(\mathrm{X}) + 2n$, where n equals the number of strands in X.*

Proof. In Definition 8.5 the Conley–Zehnder index of a stationary braid $\mathrm{X} \in \mathrm{Crit}_H$ was given as the permuted Conley–Zehnder index of the symplectic path $\Psi : [0, 1] \to \mathrm{Sp}(2n, \mathbb{R})$ defined by

$$\frac{d\Psi}{dt} - \bar{J}_0 d^2\bar{H}(t, \mathrm{X}(t))\Psi = 0, \qquad \Psi(0) = \mathrm{Id}. \tag{12.0.3}$$

In order to compute the Conley–Zehnder index of $\bar{S}\mathrm{X}$ we linearize Eq. (12.0.2) in $\bar{S}\mathrm{X}$, which yields

$$\frac{d(\bar{S}(t)\Psi)}{dt} - \bar{S}(t)J_0 d^2\bar{H}(t, \mathrm{X}(t))\Psi - 2\pi\bar{J}_0\bar{S}(t)\Psi = 0, \qquad \bar{S}(0)\Psi(0) = \mathrm{Id}.$$

From Lemma 7.1(ii) it follows that

$$\begin{aligned}\mu(\bar{S}\mathrm{X}) &= \mu_\sigma(\bar{S}\Psi, 1)\\ &= \mu_\sigma(\Psi, 1) + \mu(\bar{S}) = \mu_\sigma(\Psi, 1) + n\mu(e^{2\pi J_0 t})\\ &= \mu(\mathrm{X}) + 2n,\end{aligned}$$

which proves the lemma. □

We relate the Floer homologies of $[\mathrm{X} \text{ rel } \mathrm{Y}]$ with Hamiltonian H and $[\bar{S}\mathrm{X} \text{ rel } \bar{S}\mathrm{Y}]$ with Hamiltonian $\bar{S}H$ via the index shift in Lemma 12.1. Since the Floer homologies do not depend on the choice of Hamiltonian we obtain the following:

12.2. Theorem *(Shift Theorem). Let* $[\mathrm{X} \text{ rel } \mathrm{Y}]$ *denote a braid class with* X *having n strands. Then*

$$\mathrm{HB}_*\big([(\mathrm{X} \text{ rel } \mathrm{Y})\cdot\Delta^2]\big) \cong \mathrm{HB}_{*-2n}([\mathrm{X} \text{ rel } \mathrm{Y}]).$$

Proof. It is clear that the application of $\bar{S}$ acts on braids by concatenating with the full positive twist Δ^2. As Δ^2 generates the center of the braid group, we do not need to worry about whether the twist occurs before, during, or after the braid. It therefore suffices to show that $\mathrm{HB}_*([\bar{S}\mathrm{X} \text{ rel } \bar{S}\mathrm{Y}]) \cong \mathrm{HB}_{*-2n}([\mathrm{X} \text{ rel } \mathrm{Y}])$.

The Floer homology for $[\mathrm{X} \text{ rel } \mathrm{Y}]$ is defined by choosing a generic Hamiltonian H. From Lemma 12.1 we have that $\mu(\bar{S}\mathrm{X}) = \mu(\mathrm{X}) + 2n$ and therefore

$$C_k([\bar{S}\mathrm{X}] \text{ rel } \bar{S}\mathrm{Y}, \bar{S}H; \mathbb{Z}_2) = C_{k-2n}([\mathrm{X}] \text{ rel } \mathrm{Y}, H; \mathbb{Z}_2).$$

Since the solutions in $\mathscr{M}^{J,\bar{S}H}$ are obtained via $\bar{S}$ through (12.0.2), it also holds that

$$\partial_k(J, \bar{S}H) = \partial_{k-2n}(J, H),$$

and thus $\mathrm{HF}_k([\bar{S}\mathrm{X}] \text{ rel } \bar{S}\mathrm{Y}) \cong \mathrm{HF}_{k-2n}([\mathrm{X}] \text{ rel } \mathrm{Y})$. □

Recall that a positive braid is one that has all its crossings of the same ('left-over-right') sign; equivalently, in the standard (Artin) presentation of the braid group $\mathcal{B}_n$, only positive powers of generators are utilized. Positive braids possess a number of remarkable and usually restrictive properties. Such is not the case for braid Floer homology.

12.3. Corollary. *Positive braids realize, up to shifts, all possible braid Floer homologies.*

Proof. It follows from GARSIDE'S THEOREM [17,7] that every braid $\beta \in \mathcal{B}_n$ has a unique presentation as the product of a positive braid along with a (minimal) number of *negative* full twists Δ^{-2g} for some $g \geq 0$. From Theorem 12.2, the braid Floer homology of any given relative braid class is equal to that of its (positive!) Garside normal form, shifted to the left by degree $2gn$, where n is the number of free strands. □

This reduces the problem of computing braid Floer homology to the subclass of positive braid pairs. We believe this to be a considerable simplification.

13. Cyclic braid classes and their Floer homology

In this section we compute examples of braid Floer homology for cyclic type braid classes. The cases we consider can be computed by continuing the skeleton and the Hamiltonians to a Hamiltonian system for which the space of bounded solutions can be determined explicitly: the integrable case.

13.1. Single-strand rotations and symplectic polar coordinates

Choose complex coordinates $x = p + iq$ and consider Hamiltonians of the form

$$H(x) = F(|x|) + \phi_\delta(|x|)G\big(\arg(x)\big), \tag{13.1.1}$$

where $\arg(x)=\theta$ is the argument and $G(\theta+2\pi)=G(\theta)$. The cut-off function ϕ_δ is chosen such that $\phi_\delta(|x|)=0$ for $|x|\le\delta$ and $|x|\ge 1-\delta$, and $\phi_\delta(|x|)=1$ for $2\delta\le|x|\le 1-2\delta$. In the special case that $G(\theta)\equiv 0$, then the Hamilton equations are given by

$$x_t = i\nabla H(x) = i\frac{F'(|x|)}{|x|}x.$$

Solutions of the Hamilton equations are given by $x(t)=r\exp\left(i\frac{F'(r)}{r}t\right)$, where $r=|x|$. This gives the period $T=\frac{2\pi r}{F'(r)}$. Since H is autonomous all solutions of the Hamilton equations occur as circles of solutions. The Cauchy–Riemann equations are given by

$$u_s - iu_t - \nabla H(u) = 0. \tag{13.1.2}$$

Consider the natural change to symplectic polar coordinates (I,θ) via the relation $p=\sqrt{2I}\cos(\theta)$, $q=\sqrt{2I}\sin(\theta)$, and define $\hat{H}(I,\theta)=H(p,q)$. In particular, $\hat{H}(I,\theta)=F(\sqrt{2I})+\phi_\delta(\sqrt{2I})G(\theta)$. The Cauchy–Riemann equations become

$$\begin{aligned} I_s + 2I\theta_t - 2I\hat{H}_I(I,\theta) &= 0,\\ \theta_s - \frac{1}{2I}I_t - \frac{1}{2I}\hat{H}_\theta(I,\theta) &= 0. \end{aligned}$$

If we restrict x to the annulus $\mathbb{A}_{2\delta}=\{x\in\mathbb{D}^2 : 2\delta\le|x|\le 1-2\delta\}$, the particular choice of H described above yields

$$\begin{aligned} I_s + 2I\theta_t - \sqrt{2I}F'(\sqrt{2I}) &= 0,\\ \theta_s - \frac{1}{2I}I_t - \frac{1}{2I}G'(\theta) &= 0. \end{aligned}$$

Before giving a general result for braid classes for which x is a single-strand rotation we employ the above model to get insight into the Floer homology of the annulus.

13.2. Floer homology of the annulus

Consider an annulus $\mathbb{A}=\mathbb{A}_\delta$ with Hamiltonians H satisfying the hypotheses:

(a1) $H\in C^\infty(\mathbb{R}\times\mathbb{R}^2;\mathbb{R})$;
(a2) $H(t+1,x)=H(t,x)$ for all $t\in\mathbb{R}$ and all $x\in\mathbb{R}^2$;
(a3) $H(t,x)=0$ for all $x\in\partial\mathbb{A}$ and all $t\in\mathbb{R}$.

This class of Hamiltonians is denoted by $\mathscr{H}(\mathbb{A})$. We consider Floer homology of the annulus in the case that H has prescribed behavior on $\partial\mathbb{A}$. The boundary orientation is the canonical Stokes orientation and the orientation form on $\partial\mathbb{A}$ is given by $\lambda=i_{\mathbf{n}}\omega$, with $\mathbf{n}$ the outward pointing normal. We will consider the Floer homology of the annulus in the case that H has prescribed behavior on $\partial\mathbb{A}$:

(a4$^+$) $i_{X_H}\lambda>0$ on $\partial\mathbb{A}$;
(a4$^-$) $i_{X_H}\lambda<0$ on $\partial\mathbb{A}$.

The class of Hamiltonians that satisfy (a1)–(a3), ($a4^+$) is denoted by $\mathscr{H}^+$ and those satisfying (a1)–(a3), ($a4^-$) are denoted by $\mathscr{H}^-$. For Hamiltonians in $\mathscr{H}^+$ the boundary orientation induced by X_H is coherent with the canonical orientation of $\partial\mathbb{A}$, while for Hamiltonians in $\mathscr{H}^-$ the boundary orientation induced by X_H is opposite to the canonical orientation of $\partial\mathbb{A}$.

For pairs $(J,H) \in \mathscr{J} \times \mathscr{H}^+(\mathbb{A})$ let $\mathrm{HF}_*^+(\mathbb{A}; J, H)$ denote the Floer homology for contractible loops in $\mathbb{A}$, which can be constructed analogously to $\mathrm{HF}_*([\mathrm{x}] \text{ rel } \mathrm{Y}, J, H)$. Similarly, for $H \in \mathscr{H}^-$ the Floer homology is denoted by $\mathrm{HF}_*^-(\mathbb{A}; J, H)$. For Hamiltonians of the form (13.1.1) it can also be interpreted as the Floer homology of the space of single strand braids x that wind zero times around the annulus, i.e. any constant strand is a representative. One difference compared to the construction of $\mathrm{HF}_*([\mathrm{x}] \text{ rel } \mathrm{Y}, J, H)$ is that isolation of the invariant sets involved in $\mathrm{HF}_*^\pm(\mathbb{A}; J, H)$ follows from a variant of properness in which the hypothesis ($a4^\pm$) forces the invariant set to be away from the boundary of the annulus.

13.1. Theorem. *The Floer homology* $\mathrm{HF}_*^+(\mathbb{A}; J, H)$ *is independent of the pair* $(J,H) \in \mathscr{J} \times \mathscr{H}^+(\mathbb{A})$ *and is denoted by* $\mathrm{HF}_*^+(\mathbb{A})$. *There is a natural isomorphism*

$$\mathrm{HF}_k^+(\mathbb{A}) \cong H_{k+1}(\mathbb{A}, \partial\mathbb{A}) = \begin{cases} \mathbb{Z}_2 & \text{for } k = 0, 1 \\ 0 & \text{otherwise.} \end{cases}$$

Similarly, the Floer homology $\mathrm{HF}_*^-(\mathbb{A}; J, H)$ *is independent of the pairs* $(J,H) \in \mathscr{J} \times \mathscr{H}^-(\mathbb{A})$ *and is denoted by* $\mathrm{HF}_*^-(\mathbb{A})$ *and there is a natural isomorphism*

$$\mathrm{HF}_k^-(\mathbb{A}) \cong H_{k+1}(\mathbb{A}) = \begin{cases} \mathbb{Z}_2 & \text{for } k = -1, 0 \\ 0 & \text{otherwise,} \end{cases}$$

where H_* *denotes the singular homology with coefficients in* $\mathbb{Z}_2$.

Proof. Let us start with Hamiltonians in the class $\mathscr{H}^+$. Consider $\mathbb{A} = \mathbb{A}_\delta$ and choose $H = F + \phi_\delta G$, with $F(r) = \frac{1}{2}\left(r - \frac{1}{2}\right)^2 - \frac{1}{2}\left(\delta - \frac{1}{2}\right)^2$ and $G(\theta) = \epsilon\cos(\theta)$. Using symplectic polar coordinates we obtain that

$$\nabla_g \hat{H}(I,\theta) = \begin{pmatrix} 2I - \frac{1}{2}\sqrt{2I} + \epsilon\sqrt{2I}\phi_\delta'(\sqrt{2I})\cos(\theta) \\ -\frac{\epsilon}{2I}\phi_\delta(\sqrt{2I})\sin(\theta) \end{pmatrix}.$$

For $\frac{1}{2}\delta^2 \le I \le 2\delta^2$ and for $\frac{1}{2}(1-2\delta)^2 \le I \le \frac{1}{2}(1-\delta)^2$ it holds that $|\sqrt{2I} - \frac{1}{2}| \ge \frac{1}{2} - 2\delta$ and thus if we choose sufficiently small $\epsilon < \frac{1}{4\delta} - 1$ all zeroes of $\nabla_g \hat{H}$ lie in the annulus set $\mathbb{A}_{2\delta} \subset \mathbb{A}_\delta$. The zeroes of $\nabla_g \hat{H}$ are found at $I = \frac{1}{8}$ and $\theta = 0, \pi$, which are both non-degenerate critical points. Linearization yields

$$d\nabla_g \hat{H}(1/8, 0) = \begin{pmatrix} 1 & 0 \\ 0 & -4\epsilon \end{pmatrix}, \quad d\nabla_g \hat{H}\,(1/8, \pi) = \begin{pmatrix} 1 & 0 \\ 0 & 4\epsilon \end{pmatrix},$$

i.e. a saddle point (index 1) and a minimum (index 0) of H respectively. For $\tau \le 1$ it follows from Remark 8.6 that the Conley–Zehnder indices of the associated symplectic paths defined by $\Psi_t = J_0 d\nabla_g \hat{H}\Psi$ are given $\mu_\sigma(\Psi,\tau) = 1 - \mu_H = 0, 1$ for (I,θ) equal to $(\frac{1}{8}, 0)$ and $(\frac{1}{8}, \pi)$, respectively.

Next consider Hamiltonians of the form τH and the associated Cauchy–Riemann equations $u_s - J_0 u_t - \tau \nabla H(u) = 0$. Rescale $\tau s \to s$, $\tau t \to t$ and $u(s/\tau, t/\tau) \to u(s,t)$; then u satisfies (13.1.2) again with periodicity $u(s, t+\tau) = u(s,t)$. The 1-periodic solutions of the Cauchy–Riemann equations with τH are transformed to τ-periodic solutions of (13.1.2). Note that if τ is sufficiently small then all τ-periodic solutions of the stationary Cauchy–Riemann equations are independent of t and thus critical points of H.

If we linearize around t-independent solutions of (13.1.2), then $\frac{d}{ds} - d\nabla H(u(s))$ is Fredholm and thus also

$$\partial_{K,\Delta} = \frac{\partial}{\partial s} - J\frac{\partial}{\partial t} - K,$$

with $K = d\nabla H(u(s))$, is Fredholm, see [35]. We claim that if τ is sufficiently small then all contractible τ-periodic bounded solutions $u(s, t+\tau) = u(s,t)$ of (13.1.2) are t-independent, i.e. solutions of the equation $u_s = \nabla H(u)$, see [35].

Note that $\mathbb{A}$ is an isolating neighborhood for the gradient flow generated by $u_s = \tau\nabla H(u) = f(u)$, and for $H \in \mathscr{H}^+$ the exit set is $\partial\mathbb{A}$. Using the Morse relations for the Conley index we obtain for any generic $H \in \mathscr{H}^+$ that

$$\sum_{x\in \mathrm{Fix}(f)} t^{\mathrm{ind}(x)} = P_t(\mathbb{A}, \partial\mathbb{A}) + (1+t)Q_t,$$

where $\mathrm{ind}(x) = \dim W^u(x)$. Using [33] the Poincaré polynomial follows, provided we choose the appropriate grading. By the previous considerations on the Conley–Zehnder index in Remark 8.6 we have that $\mu_\sigma(\Psi, \tau) = 1 - \mu_H(x) = 1 - (2 - \mathrm{ind}(x)) = \mathrm{ind}(x) - 1$. This yields $\mathrm{ind}(x) = k+1$, where k is the grading of Floer homology and therefore $H_k(C, \partial; J_0, \tau H) \cong H_{k+1}(\mathbb{A}, \partial\mathbb{A})$, which proves the first statement.

As for Hamiltonians in $\mathcal{H}^-$ we choose $F(r) = -\frac{1}{2}\left(r - \frac{1}{2}\right)^2 + \frac{1}{2}\left(\delta - \frac{1}{2}\right)^2$. The proof is identical to the previous case except for the indices of the stationary points. The Morse indices of $(\frac{1}{8}, 0)$ and $(\frac{1}{8}, \pi)$ are 1 and 2 and $\mu_\sigma(\Psi, \tau) = 1 - \mu_H = 0, -1$ for (I, θ) equal to $(\frac{1}{8}, 0)$ and $(\frac{1}{8}, \pi)$ respectively. As before $\mathbb{A}$ is an isolating neighborhood for the gradient flow of τH and for $H \in \mathscr{H}^-$ the exit set is $\varnothing$. This yields the slightly different Morse relations

$$\sum_{x\in \mathrm{Fix}(f)} t^{\mathrm{ind}(x)} = P_t(\mathbb{A}) + (1+t)Q_t.$$

By the same grading as before we obtain that $H_k(C, \partial; J_0, \tau H) \cong H_{k+1}(\mathbb{A})$, which proves the second statement. □

13.3. Floer homology for single-strand cyclic braid classes

We apply the results in the previous subsection to compute the Floer homology of families of cyclic braid classes $[\mathrm{X}\ \mathrm{rel}\ \mathrm{Y}]$. The skeletons Y consist of two braid components Y^1 and Y^2, which are given by (in complex notation)

$$\mathrm{Y}^1 = \left\{ r_1 e^{\frac{2\pi n}{m} it}, \cdots, r_1 e^{\frac{2\pi n}{m} i(t-m+1)} \right\}, \quad \mathrm{Y}^2 = \left\{ r_2 e^{\frac{2\pi n'}{m'} it}, \cdots, r_2 e^{\frac{2\pi n'}{m'} i(t-m'+1)} \right\}, \quad (13.3.1)$$

where $0 < r_1 < r_2 \le 1$, $m, m' \in \mathbb{N}$, $n, n' \in \mathbb{Z}$ and $n \ne 0$, $m \ge 2$. Without loss of generality we take both pairs (n, m) and (n', m') relatively prime. In the braid group $\mathcal{B}_m$ the braid Y^1 is represented by the word $\beta^1 = (\sigma_1 \cdots \sigma_{m-1})^n$, $m \ge 2$, and $n \in \mathbb{Z}$, and similarly for Y^2. In order to describe the relative braid class $[\mathrm{X} \text{ rel } \mathrm{Y}]$ with the skeleton defined above we consider a single strand braid $\mathrm{X} = \{x^1(t)\}$ with

$$x^1(t) = re^{2\pi \ell i t}$$

where $r_1 < r < r_2$ and $\ell \in \mathbb{Z}$. We now consider two cases for which X rel Y is a representative.

13.3.1. The case $\frac{n}{m} < \ell < \frac{n'}{m'}$

The relative braid class $[\mathrm{X} \text{ rel } \mathrm{Y}]$ is a proper braid class since the inequalities are strict.

13.2. Lemma. *The Floer homology is given by*

$$\mathrm{HB}_k([\mathrm{X} \text{ rel } \mathrm{Y}], \mathbb{Z}_2) = \begin{cases} \mathbb{Z}_2 & \text{for } k = 2\ell, 2\ell+1 \\ 0 & \text{otherwise.} \end{cases}$$

The Poincaré polynomial is given by $P_t([\mathrm{X} \text{ rel } \mathrm{Y}]) = t^{2\ell} + t^{2\ell+1}$.

Proof. Since $\mathrm{HB}_*([\mathrm{X} \text{ rel } \mathrm{Y}], \mathbb{Z}_2)$ is independent of the representative we consider the class $[x]$ rel Y with X and Y as defined above. Apply $-\ell$ full twists to X rel Y: $(\tilde{\mathrm{X}}, \tilde{\mathrm{Y}}) = \overline{S}^{-\ell}(\mathrm{X}, \mathrm{Y})$. Then by Theorem 12.2

$$\mathrm{HB}_k([\tilde{\mathrm{X}} \text{ rel } \tilde{\mathrm{Y}}]) \cong \mathrm{HB}_{k+2\ell}([\mathrm{X} \text{ rel } \mathrm{Y}]). \tag{13.3.2}$$

We now compute the homology $\mathrm{HF}_k([\tilde{\mathrm{X}} \text{ rel } \tilde{\mathrm{Y}}])$ using Theorem 13.1. The free strand $\tilde{\mathrm{X}}$, given by $\tilde{x}^1(t) = r$, in $\tilde{\mathrm{X}}$ rel $\tilde{\mathrm{Y}}$ is unlinked with the $\tilde{y}^1$. Consider an explicit Hamiltonian $H(x) = F(|x|) + \omega(|x|)G(\arg(x))$ and choose F such that $F(r_1) = F(r_2) = 0$ and

$$\frac{F'(r_1)}{2\pi r_1} = \frac{n}{m} - \ell < 0, \quad \text{and} \quad \frac{F'(r_2)}{2\pi r_2} = \frac{n'}{m'} - \ell > 0. \tag{13.3.3}$$

Clearly $\mathrm{Y} \in \mathrm{Crit}_H$ and the circles $|x| = r_1$ and $|x| = r_2$ are invariant for the Hamiltonian vector field X_H. For $\tau > 0$ sufficiently small it holds that $\mathscr{M}^{J_0,\tau H}([\tilde{\mathrm{x}}] \text{ rel } \mathrm{Y}) = \mathscr{M}^{J_0,\tau H}(\mathbb{A})$. This follows from the arguments developed in [35] (all solutions are t-independent). From the boundary conditions in Eq. (13.3.3) it follows that $H \in \mathscr{H}^+$. From Theorem 13.1 we deduce that $\mathrm{HF}_0([\tilde{\mathrm{x}}] \text{ rel } \tilde{\mathrm{Y}}) \cong \mathrm{HF}_0^+(\mathbb{A}) = \mathbb{Z}_2$ and $\mathrm{HF}_1([\tilde{\mathrm{x}}] \text{ rel } \tilde{\mathrm{Y}}) \cong \mathrm{HF}_1^+(\mathbb{A}) = \mathbb{Z}_2$. This proves, using Eq. (13.3.2), that $\mathrm{HF}_{2\ell}([\mathrm{x}] \text{ rel } \mathrm{Y}) = \mathbb{Z}_2$ and $\mathrm{HF}_{2\ell+1}([\mathrm{x}] \text{ rel } \mathrm{Y}) = \mathbb{Z}_2$, which completes the proof. □

13.3.2. The case $\frac{n}{m} > \ell > \frac{n'}{m'}$

The relative braid class $[\mathrm{X} \text{ rel } \mathrm{Y}]$ with the reversed inequalities is also a proper braid class. We have

13.3. Lemma. *The Floer homology is given by*

$$\mathrm{HB}_k([\mathrm{X} \text{ rel } \mathrm{Y}], \mathbb{Z}_2) = \begin{cases} \mathbb{Z}_2 & \text{for } k = 2\ell-1, 2\ell \\ 0 & \text{otherwise.} \end{cases}$$

The Poincaré polynomial is given by $P_t([\mathrm{X} \text{ rel } \mathrm{Y}]) = t^{2\ell-1} + t^{2\ell}$.

Proof. The proof is identical to the proof of Lemma 13.2. Because the inequalities are reversed we construct a Hamiltonian such that

$$\frac{F'(r_1)}{2\pi r_1} = \frac{n}{m} - \ell > 0, \quad \text{and} \quad \frac{F'(r_2)}{2\pi r_2} = \frac{n'}{m'} - \ell < 0.$$

This yields a Hamiltonian in $\mathscr{H}^-$ for which we repeat the above argument using the homology $\mathrm{HF}_*^-(\mathbb{A})$. □

13.4. Applications to disc maps

We demonstrate how these simple computed examples of HB_* yield forcing results at the level of dynamics. The following results are not so much novel (cf. Franks' work on rotation sets) as illustrative of how one uses a Floer-type forcing theory.

13.4. Theorem. *Let $f : \mathbb{D}^2 \to \mathbb{D}^2$ be an area-preserving diffeomorphism with invariant set $A \subset \mathbb{D}^2$ having as braid class representative* Y*, where* $[\mathrm{Y}]$ *is as described in Eq.* (13.3.1)*, with $\frac{n}{m} \neq \frac{n'}{m'}$ relatively prime. Then, for each $l \in \mathbb{Z}$ and $k \in \mathbb{N}$, satisfying*

$$\frac{n}{m} < \frac{l}{k} < \frac{n'}{m'}, \quad \text{or} \quad \frac{n}{m} > \frac{l}{k} > \frac{n'}{m'},$$

there exists a distinct period k orbit of f. In particular, f has infinitely many distinct periodic orbits.

Proof. By Proposition 2.1 there exists a Hamiltonian $H \in \mathscr{H}(\mathbb{D}^2)$ such that $f = \psi_{1,H}$, where $\psi_{t,H}$ is the Hamlitonian flow generated by the Hamiltonian system $x_t = X_H(t,x)$ on $(\mathbb{D}^2, \omega_0)$. Up to full twists Δ^2, the invariant set A generates a braid $\psi_{t,H}(A)$ of braid class $[\psi_{t,H}(A)] = [\mathrm{Y}] \bmod \Delta^2$ with

$$\psi_{t,H}(A) = \tilde{\mathrm{Y}} = \{\tilde{\mathrm{Y}}^1, \tilde{\mathrm{Y}}^2\}.$$

We begin with the case $k = 1$. There exists an integer N (depending on the choice of H) such that the numbers of turns of the strands $\tilde{y}^1$ and $\tilde{y}^2$ are $\frac{\tilde{n}}{m} = \frac{n}{m} + N$ and $\frac{\tilde{n}'}{m'} = \frac{n'}{m'} + N$ respectively. Consider a free strand $\tilde{\mathrm{X}}$ such that $\tilde{\mathrm{X}} \text{ rel } \tilde{\mathrm{Y}} \sim (\mathrm{X} \text{ rel } \mathrm{Y}) \cdot \Delta^{2N}$, with $[\mathrm{X} \text{ rel } \mathrm{Y}]$ as in Lemmas 13.2 and 13.3, and with l satisfying the inequalities above. By Lemmas 13.2 and 13.3 the Floer homology of $[\mathrm{X} \text{ rel } \mathrm{Y}]$ is non-trivial, and Theorem 12.2 implies that $\mathrm{HB}_k\big([\tilde{\mathrm{X}} \text{ rel } \tilde{\mathrm{Y}}]\big) \cong \mathrm{HB}_{k-2nN}([\mathrm{X} \text{ rel } \mathrm{Y}])$. Therefore the Floer homology of $[\tilde{\mathrm{X}} \text{ rel } \tilde{\mathrm{Y}}]$ is non-trivial. From Theorem 11.1 the existence of a stationary relative braid $\tilde{\mathrm{X}}$ follows, which yields a fixed point for f.

For the case $k > 1$, consider the Hamiltonain kH; the time-1 map associated with Hamiltonian system $x_t = X_{kH}$ is equal to f^k. The fixed point implied by the proof above descends to a k-periodic point of f. □

13.5. Remark. As pointed out in Section 11, we conjecture that the Floer homology $\mathrm{HB}_*([\mathrm{X} \text{ rel } \mathrm{Y}]) = \mathbb{Z}_2 \oplus \mathbb{Z}_2$ implies the existence of at least two fixed points of different indices. This agrees with the generic setting where centers and saddles occur in pairs.

14. Remarks and future steps

In this section we outline a number of remarks and future directions. The results in this paper are a first step to a more in depth theory.

14.1. Floer homology, Morse homology and the Conley index

The perennial problem with Floer homologies is their general lack of computability.[4] We outline a strategy for algorithmic computation of braid Floer homology.

(1) Use Garside's Theorem and Theorem 12.2 to reduce computation to the case of positive braids.
(2) Prove that braid Floer homology is isomorphic to the Conley braid index of [18] in the case of positive braids.
(3) Invoke the computational results in [18].

Steps 1 and 3 above are in place; Step 2 is conjectural.

To be more precise, let $[\text{X rel Y}]$ be a proper relative braid class. It follows from the Garside theorem that for $g \geq 0$ sufficiently large, the braid class $[\text{P rel Q}] = [\text{X rel Y}] \cdot \Delta^{2g}$ is positive — there exist representatives of the braid class which are positive braids. Given such a positive braid, its LEGENDRIAN REPRESENTATIVE (roughly speaking, an image of the braid under a certain planar projection, lifted back via a 1-jet extension) captures the braid class. From [18], one starts with a Legendrian braid representative and performs a spatial discretization, reducing the braid to a finite set of points which can be reconstructed into a piecewise-linear braid in $[\text{P rel Q}]$.

There is an analogous HOMOLOGICAL INDEX for Legendrian braids, introduced in [18]. This index, $\mathrm{HC}_*([\text{P rel Q}])$, is defined as the (homological) Conley index of the discretized braid class under an appropriate class of parabolic dynamics. This index has finite-dimensionality built in and computation of several classes of examples has been implemented using current computational homology code.

This index shares some features with the braid Floer homology. Besides finite dimensionality of the index, there is a precise analogue of the Shift Theorem for products with full twists. Even the underlying dynamical constructs are consonant. For Legendrian braid classes the same construction as in this paper can be carried out using a nonlinear heat equation instead of the nonlinear Cauchy–Riemann equations. Consider the scalar parabolic equation

$$u_s - u_{tt} - g(t, u) = 0, \tag{14.1.1}$$

where $u(s, t)$ takes values in the interval $[-1, 1]$. Such equations can be obtained as a limiting case of the nonlinear Cauchy–Riemann equations, see [32]. For the function g we assume the following hypotheses:

(g1) $g \in C^\infty(\mathbb{R} \times \mathbb{R}; \mathbb{R})$;

[4] Much excitement in Heegaard Floer homology surrounds recent breakthroughs in combinatorial formulae for (still challenging!) computation of examples.

(g2) $g(t+1,q) = g(t,q)$ for all $(t,q) \in \mathbb{R} \times \mathbb{R}$;
(g3) $g(t,-1) = g(t,1) = 0$ for $t \in \mathbb{R}$.

This equation generates a local semi-flow ψ^s on periodic functions in $C^\infty(\mathbb{R}/\mathbb{Z};\mathbb{R})$. For a braid diagram P we define the intersection number $I(\mathrm{P})$ as the total number of intersections, and since all intersections in a Legendrian braid of this type correspond to positive crossings, the total intersection number is equal to the crossing number defined above. The classical LAP-NUMBER PROPERTY [4] of nonlinear scalar heat equations states that the number of intersections between two graphs can only decrease as time $s \to \infty$. As before let $[\mathrm{P}]_L$ rel Q be a relative braid class fiber with skeleton Q; we can choose a nonlinearity g such that the skeletal strands in Q are solutions of the equation $q_{tt} + g(t,q) = 0$. Let $\mathbf{v}(s)$ rel Q denote a local solution (in s) of Eq. (14.1.1), then $I(\mathbf{v} \text{ rel } \mathrm{Q})|_{s_0-\epsilon} > I(\mathbf{v} \text{ rel } \mathrm{Q})|_{s_0+\epsilon}$, whenever $u(s_0,t_0) = u^k(s_0,t_0)$ for some k, and as before we define the sets of all bounded solutions in $[\mathrm{P}]_L$ rel Q, which we denote by $\mathscr{M}([\mathrm{P}]_L \text{ rel } \mathrm{Q})$. Similarly, $\mathscr{S}([\mathrm{P}]_L \text{ rel } \mathrm{Q}) \subset C^\infty(\mathbb{R}/\mathbb{Z};\mathbb{R})$, the image under the map $u \mapsto u(0,\cdot)$, is compact in the appropriate sense. We can now build a chain complex in the usual way which yields the Morse homology $\mathrm{HM}_*([\mathrm{P} \text{ rel } \mathrm{Q}])$.

14.1. Conjecture. *Let* $[\mathrm{X} \text{ rel } \mathrm{Y}]$ *be a proper relative braid class with* X *having* n *strands. Let* $[\mathrm{P} \text{ rel } \mathrm{Q}] = [\mathrm{X} \text{ rel } \mathrm{Y}] \cdot \Delta^{2g}$ *be sufficiently twisted so as to be positive. Then,*

$$\mathrm{HB}_{*-2ng}([\mathrm{X} \text{ rel } \mathrm{Y}]) \cong \mathrm{HM}_*([\mathrm{P} \text{ rel } \mathrm{Q}]) \cong \mathrm{HC}_*([\mathrm{P} \text{ rel } \mathrm{Q}]). \tag{14.1.2}$$

A result of this type would be crucial for computing the Floer and Morse homology via the discrete invariants.

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