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Multitransition Homoclinic and Heteroclinic Solutions of the Extended Fisher–Kolmogorov Equation

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1. INTRODUCTION

We investigate the structure of the homoclinic and heteroclinic solutions of the scalar fourth-order equation

$$\gamma u^{(4)} - \beta u'' + F'(u) = 0, \tag{1}$$

with $\gamma, \beta > 0$. For the purposes of this paper we will concentrate on two types of potentials, double-well and periodic. The two primary examples we have in mind are

$$F_1(u) = \frac{1}{4}(u^2-1)^2 \qquad \text{and} \qquad F_2(u) = \frac{2}{\pi^2}[1+\cos \pi u].$$

Equation (1) with the double-well potential F_1 is the extended Fisher–Kolmogorov equation which has been proposed as a model for phase transitions and other bistable phenomena, cf. Zimmerman [28], Coullet *et al.* [8], and Dee and van SaarLoos [9]. Our results will hold for a general class of those potentials which have even symmetry about the midpoints between the wells and whose wells are superquadratic and have equal depth. By an affine transformation, the wells can be moved to the odd integers (± 1 for the double-well) so that F is even and nonnegative (with period 2 in the periodic case), and thus without loss of generality we will assume this for the remainder of this paper. The specific hypotheses for the behavior of F at its wells are

$$\begin{aligned} &F(1) = F'(1) = 0,\ F''(1) > 0, \qquad \text{and} \\ &F(u) \geqslant \delta(u-1)^2 \text{ for } u \in (0,2) \text{ for some } \delta > 0. \end{aligned} \tag{2}$$

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We are interested in the orbits in the four-dimensional flow generated by (1) which connect those equilibrium points corresponding to the wells of F. The function

$$H(u, u', u'', u''') = \gamma u''' u' - \frac{\gamma}{2}(u'')^2 - \frac{\beta}{2}(u')^2 + F(u)$$

is a Hamiltonian for this flow, and due to the hypotheses on F these minimum equilibrium points are saddle-foci when $\gamma > \beta^2/4F''(1)$ and saddle-nodes otherwise. The nature of the dynamics of a four-dimensional Hamiltonian system with an orbit homoclinic to a saddle-focus or a heteroclinic loop between two saddle-foci has been studied by several authors. Devaney [10] has shown that if the homoclinic (heteroclinic) orbit is a transversal intersection of the stable and unstable manifolds, then there is horseshoe-type dynamics in a neighborhood of this orbit, see also Sil'nikov [25]. We will restrict our attention to the saddle-focus case; the other case is more delicate, cf. Turaev and Sil'nikov [26] and Wiggins [27] for a general discussion.

The results of Devaney show that generically four-dimensional autonomous Hamiltonian systems with connections between saddle-foci have complicated dynamics, but verification of transversality for specific systems such as (1) is difficult. Buffoni and Séré [5] have presented a weaker geometric condition which combined with topological information about the homoclinic (heteroclinic) orbit in an appropriate function space can be used to find multibump solutions. We will use this approach to study the dynamics of (1) and show, for example, that the system has positive entropy. However we expect the dynamics to be much richer than established below, see Section 5.

Since Eq. (1) has a variational structure, the homoclinic and heteroclinic connections are all critical points of the functional

$$J[u] = \int_{\mathbb{R}} j(u)\, dt = \int_{\mathbb{R}} \left[\frac{\gamma}{2}(u'')^2 + \frac{\beta}{2}(u')^2 + F(u) \right] dt \tag{3}$$

in an appropriate function space. In the next section we show that this functional with either a periodic or double-well potential has a minimizer which is a heteroclinic orbit of (1). To show that this solution is isolated in a suitable function space, we will also require $F(u)$ to be analytic. This topological information is subsequently used to apply the gluing method of Buffoni and Séré [5] to establish the existence of a countable family of multitransition heteroclinic and homoclinic orbits. For various classes of Hamiltonian systems, the existence of homoclinic multibump solutions was proven by Rabinowitz [22, 23] under certain nondegeneracy conditions.

See also Séré [24], Ambrosetti and Coti-Zelati [2], and the references therein.

THEOREM 1.1. *For the extended Fisher–Kolmogorov equation, i.e., Eq.* (1) *with* $F(u)=(u^2-1)^2/4$, *there exist primary heteroclinic connections* u_+ *from* -1 *to* 1 *and* u_- *from* 1 *to* -1 *which occur as minimizers of J. Moreover let* $\mu=\tan^{-1}(\sqrt{8\gamma\beta^{-2}-1})/2$ *so that the period of rotation of the linearized saddle-focus is* $2\pi/\mu$. *Then for any* $\varepsilon \ll 1$ *there is a constant* $C>0$ *such that the following hold.*

(i) *For every* $k>0$, *integers* $n_1, ..., n_{2k-1}>0$, *and points* $s_1<\cdots<s_{2k}\in\mathbb{R}$ *satisfying* $|s_{i+1}-s_i-2n_i\pi/\mu-C|<\varepsilon$, *the ball of radius* ε *in* $W^{1,\infty}$ *centered at*

$$\pm 1+\sum_{i=1}^{k}\{u_\pm(t-s_{2i})+u_\mp(t-s_{2i-1})\} \qquad t\in\mathbb{R}$$

contains at least one solution to (1); *i.e., there exists a countable family of solutions homoclinic to* ± 1.

(ii) *For every* $k>0$, *integers* $n_1, ..., n_{2k}>0$, *and points* $s_1<\cdots<s_{2k+1}\in\mathbb{R}$ *satisfying* $|s_{i+1}-s_i-2n_i\pi/\mu-C|<\varepsilon$, *the ball of radius* ε *in* $W^{1,\infty}$ *centered at*

$$\sum_{i=1}^{k}\{u_\pm(t-s_{2i})+u_\mp(t-s_{2i-1})\}+u_\mp(t-s_{2k+1}) \qquad t\in\mathbb{R}$$

contains at least one solution to (1); *i.e., there exists a countable family of heteroclinic solutions between* ± 1 *in both directions.*

(iii) *Statements* (i) *and* (ii) *hold for any consecutive pair of minimum equilibria of the system* (1) *with a periodic potential. In this case there are also countable families of heteroclinic connections between any pair of these equilibria.*

Peletier and Troy [17] have constructed a family of heteroclinic orbits of the extended Fisher–Kolmogorov equation distinct from those described above. However their shooting method does not provide the information about the geometry of the functional near these solutions necessary to extend the gluing construction used below. We conclude with a description of the dynamics which can be established using these variational methods and a comparison with the more conventional dynamical systems methods as used by Devaney.

2. HETEROCLINIC ORBITS AS GLOBAL MINIMA

In the procedure described below, we will minimize the functional (3) with a periodic potential F. Analogous arguments also apply to the double-well case, and the results will be stated without proof, see also Peletier *et al.* [1994].

We find a heteroclinic orbit from -1 to 1 by minimizing the functional $J[u]$ on the metric space $M=\{u : u=\chi+\xi \text{ for } \xi\in H^2(\mathbb{R}) \text{ and } u(0)=0\}$ where $\chi\in C^\infty(\mathbb{R})$ is any fixed odd function with $\chi(x)=1$ for $x\geqslant 1$. First we fix some notation. Let $\sigma=\inf_{u\in M} J[u]$. A subscript $+$ or $-$ will denote the restriction of a function or functional to $\mathbb{R}^+$ or $\mathbb{R}^-$. For example,

$$J_+[u]=\int_{\mathbb{R}^+} j(u)\,dt.$$

LEMMA 2.1. *There exist minimizing sequences for J which are odd.*

Proof. Let $\sigma_\pm=\inf_{u\in M_\pm} J_\pm[u]$ where $M_\pm=\{u : u=\chi_\pm+\xi$ for $\xi\in H^2\cap H_0^1(\mathbb{R}^\pm)\}$. Since the potential $F(u)$ is even, the functionals $J_\pm$ have the symmetry $J_+[u(t)]=J_-[-u(-t)]$. Therefore $\sigma_+=\sigma_-$, and odd minimizing sequences for J can be constructed from minimizing sequences for J_+ using this symmetry. ∎

From Lemma 2.1 it suffices to minimize J_+ on M_+.

LEMMA 2.2. *There exist minimizing sequences $\{u_n\}\subset M_+$ of J_+ with the property that $u_n(t)\in(0,2)$ for $t>0$.*

Proof. Since $u_n\in M_+$, $\lim_{t\to\infty} u_n(t)=1$, and hence there exists a T_n such that $u_n(t)\in(0,2)$ for $t>T_n$ and $u_n(T_n)=0$ or 2. We define a new sequence $\hat{u}_n\in M_+$ by

$$\hat{u}_n(t)=\begin{cases} u_n(t+T_n) & \text{if } \ u_n(T_n)=0 \\ 2-u_n(t+T_n) & \text{if } \ u_n(T_n)=2 \end{cases}$$

for which $u_n(t)\in(0,2)$ for $t>0$. Estimating the functional along this sequence yields

$$\begin{aligned} J_+[u_n] &= \int_0^{T_n} j(u_n)\,dt+\int_{T_n}^\infty j(u_n)\,dt \\ &> \int_{T_n}^\infty j(u_n)\,dt=\int_{\mathbb{R}^+} j(\hat{u}_n)\,dt=J_+[\hat{u}_n], \end{aligned}$$

since F is even with period 2. Hence $\hat{u}_n$ is also a minimizing sequence. ∎

LEMMA 2.3. *Minimizing sequences of* J_+ *for which* $u_n(t) \in (0, 2)$ *have weakly convergent subsequences.*

Proof. Since $J_+[u_n]$ is bounded, it follows that

$$\int_{\mathbb{R}^+} |u_n''|^2 \, dt \leqslant C \quad \text{and} \qquad \int_{\mathbb{R}^+} F(u_n) \, dt \leqslant C.$$

Since $u_n(t) \in (0, 2)$, by assumption there exists a $\delta > 0$ such that

$$\delta(u_n - 1)^2 \leqslant F(u_n),$$

and hence $\|u_n - 1\|_{H^2} \leqslant C$. Therefore u_n has a weakly convergent subsequence. ∎

LEMMA 2.4. *The functional* J_+ *is sequentially weakly lower semicontinuous* (*swlsc*).

Proof. Let $u_n \subset M_+$ be a weakly convergent sequence. Then passing to a subsequence if necessary,

$$u_n \rightharpoonup u \text{ in } M_+ \qquad \text{and} \qquad u_n \to u \text{ in } C^1_{\text{loc}},$$

and thus $F(u_n) \to F(u)$ in C^1_{loc}. Consequently,

$$\liminf_{n \to \infty} \int_{\mathbb{R}^+} [|u_n''|^2 + F(u_n)] \, dt \geqslant \int_{\mathbb{R}^+} [|u''|^2 + F(u)] \, dt \geqslant 0$$

by Fatou's lemma which proves that J_+ is swlsc. ∎

LEMMA 2.5. *The functional* J_+ *attains a global minimum in* M_+ *and every minimizer* u *satisfies* $u(t) \in (0, 2)$ *for* $t > 0$.

Proof. By Lemmas 2.2 and 2.3, we can choose a minimizing sequence $\{u_n\} \subset M_+$ such that $u_n \rightharpoonup u \in M_+$ since M_+ is weakly closed. Thus by Lemma 2.4,

$$\sigma_+ = \liminf_{n \to \infty} J_+[u_n] \geqslant J_+[u] \geqslant \sigma_+,$$

which proves the existence of a minimizer. Suppose u is a minimizer with $u(t) \notin (0, 2)$ for some $t > 0$. The construction in Lemma 2.2 gives a $\hat{u} \in M_+$ with $J_+[\hat{u}] < J_+[u] = \sigma_+$ which is a contradiction. ∎

Using the previous lemmas, we can now characterize the minimizers of J.

THEOREM 2.6. *The functional J attains a global minimum on M, and all minimizers u are odd with $u(t) \in (0, 2)$ for $t > 0$.*

Proof. The existence of a minimizer follows from Lemmas 2.1 and 2.5. Also from Lemma 2.1, $\sigma = \sigma_+ + \sigma_-$ since odd extensions of minimizing sequences for J_+ are minimizing sequences for J. Let $u \in M$ be a minimizer of J. Then $J[u] = \sigma = \sigma_+ + \sigma_-$, and $u_+ = u|_{\mathbb{R}^+}$ and $u_- = u|_{\mathbb{R}^-}$ are minimizers of J_+ and J_- respectively. Therefore $u(0) = 0$, and by symmetry $J_+[-u_-(-t)] = \sigma_+$. Also $u''(0) = 0$ is a natural boundary condition satisfied by all critical points of $J_\pm$ with variations in $H^2 \cap H_0^1(\mathbb{R}^\pm)$. Define $\hat{u}_+(t) = -u_-(-t)$. Then, $\hat{u}'_+(0) = u'_+(0)$ and $\hat{u}'''_+(0) = u'''_+(0)$ which proves that $u_+ = \hat{u}_+ = -u_-$. ∎

This theorem establishes the existence of a heteroclinic orbit u of (1) connecting -1 and 1. Using the symmetries $u \to -u$ and $u \to u + 2$ under which J is invariant, we can generate heteroclinic connections in both directions between any consecutive pair of wells given by

$$u_k^\pm = \pm u + 2k, \qquad k \in \mathbb{Z}. \tag{4}$$

All of these solutions will be minima of J in a suitable class of functions. When $\gamma > \beta^2/4F''(1)$, the equilibrium solutions $u \equiv 2k + 1$, $k \in \mathbb{Z}$ are saddle-foci (four complex eigenvalues, two with positive real part and two with negative real part). The existence of the family (4) implies that the global stable (unstable) manifold of each of these equilibria does not coincide with either of the unstable (stable) manifolds of its neighbors. A minor modification of a theorem by Buffoni and Séré [5] implies that the family $u_k^\pm$ are isolated critical points (in the appropriate function spaces) of J up to translation invariance.

We now turn to the case of a double-well potential where using similar techniques a result analogous to Theorem 2.6 can be proven, cf. Peletier *et al.* [21].

THEOREM 2.7. *The functional J attains a global minimum on M, and all minimizers u are odd with $u(t) > 0$ for $t > 0$.*

Therefore by symmetry, a loop exists between the equilibria ± 1 in which the heteroclinic solutions are minimizers of J. However the following construction of multitransition solutions requires these primary solutions to be isolated in M which is equivalent to the geometric condition that the stable (unstable) manifold of 1 does not coincide with the unstable (stable) manifold of -1, (Theorem 7 of Buffoni and Séré [5]). We verify this specifically for the extended Fisher–Kolmogorov equation

$$\gamma u^{(4)} - \beta u'' - u + u^3 = 0. \tag{5}$$

For the more general case see Section 5. It should be noted that even for (5) the transversality of the intersection of the stable and unstable manifolds is difficult to check.

THEOREM 2.8. *For the extended Fisher–Kolmogorov equation* (5) *with* $\gamma > \beta^2/8$, *the manifolds* $W^u(-1)$ *and* $W^s(1)$ *do not coincide. Therefore all heteroclinic orbits connecting* -1 *and* 1 *are isolated as functions in* M.

Proof. Using shooting methods Peletier and Troy [17] have established the existence of a family $\{u_n\}_{n>0}$ of odd heteroclinics with the property that for every $n>0$ there exists $L_n>0$ such that $|u_n|<1$ on $(0, L_n)$, $0<|u_n| \leqslant \sqrt{2}$ on (L_n, ∞), and u_n has exactly n zeroes in the interval $(0, L_n)$.

CLAIM. $L_n \to \infty$ as $n \to \infty$.

Suppose $L_n < L < \infty$. Then for each $n \geqslant 4$, we can choose points $t_1^n < t_2^n < t_3^n < t_4^n$ which are consecutive zeroes of u_n in the interval $(0, L)$ such that $t_4^n - t_1^n < 4L/n$. Let $I_n = [t_1^n, t_4^n]$. Then u_n has four zeroes in I_n, and hence by Rolle's theorem each of the derivatives u_n', u_n'', and u_n''' has at least one zero in I_n. Each u_n is a solution of the equation

$$\gamma u_n^{(4)} = \beta u_n'' - u_n^3 + u_n.$$

By integrating from a zero of u_n''' to any point $t \in I_n$ and using $|u_n| \leqslant 1$, we obtain the estimate

$$\gamma\, |u_n'''(t)| \leqslant \int_{I_n} |\beta u_n'' - u_n^3 + u_n|\, ds \leqslant \frac{4L}{n} (\beta\, \|u_n''\|_{L^\infty(I_n)} + 2\, \|u_n\|_{L^\infty(I_n)}).$$

Integrating again we can estimate

$$|u_n''(t)| \leqslant \int_{I_n} |u_n'''|\, ds \leqslant \frac{16L^2}{\gamma n^2} (\beta\, \|u_n''\|_{L^\infty(I_n)} + 2\, \|u_n\|_{L^\infty(I_n)}).$$

If n is sufficiently large, it follows that

$$|u_n''(t)| \leqslant \frac{C}{n^2} \|u_n\|_{L^\infty(I_n)} \leqslant \frac{C}{n^2},$$

and integrating two more times yields

$$|u_n'(t)| \leqslant \frac{C}{n^3} \qquad \text{and} \qquad |u_n(t)| \leqslant \frac{C}{n^4}.$$

Therefore u_n and all its derivatives through third-order become arbitrarily small on the interval I_n as $n \to \infty$. From the Hamiltonian, for $t \in I_n$ we have

$$\begin{aligned}\frac{1}{4} &= -\gamma u_n' u_n''' + \frac{\gamma}{2}(u_n'')^2 + \frac{\beta}{2}(u_n')^2 + \frac{1}{2}u_n^2 - \frac{1}{4}u_n^4 \\ &\leqslant \gamma\, |u_n'|\, |u_n'''| + \frac{\gamma}{2}(u_n'')^2 + \frac{\beta}{2}(u_n')^2 + \frac{1}{2}u_n^2 - \frac{1}{4}u_n^4 \\ &\leqslant \frac{C}{n^4}.\end{aligned}$$

This contradiction proves the claim that $L_n \to \infty$ as $n \to \infty$.

We now state a weaker version of a proposition by Buffoni and Séré [5]. Their original result applies to homoclinic solutions and requires both the concentration compactness principle (see Coti Zelati *et al.* [1990]) and the analyticity of F. The following result only applies to minimizing heteroclinics and therefore does not require concentration compactness.

THEOREM 2.9. *Either all minimizing heteroclinic orbits are isolated in M or $W^u(-1)$ and $W^s(1)$ coincide and the time for any heteroclinic to go from $W^u_{\mathrm{loc}}(-1)$ to $W^s_{\mathrm{loc}}(1)$ is uniformly bounded.*

We have just shown that there exist heteroclinic orbits u_n for which the time of flight from $W^u_{\mathrm{loc}}(-1)$ to $W^s_{\mathrm{loc}}(1)$ is at least $2L_n$ which becomes arbitrarily large, which excludes the second possibility in the above proposition and proves the theorem. ∎

Proof of Theorem 2.9. From the local analysis of the flow near a saddle-focus it follows that W^s_{loc} can be identified with $\mathbb{R}^+ \times S^1$. In particular every heteroclinic $u \in M$ can be associated with a unique $\theta = \Theta(u) \in S^1$. Define

$$\mathscr{C} = \{\theta \in S^1 : J[u_\theta] = \operatorname{Min} J \text{ and } u_\theta \in M \text{ is not isolated up to time translations in the class of all heteroclinics}\}.$$

Following Buffoni and Séré [5], $\mathscr{C} \subset S^1$ is open by analyticity. To show that $\mathscr{C}$ is also closed we argue as follows. Let $\{\theta_n\} \subset \mathscr{C}$ so that $J[u_{\theta_n}] = \operatorname{Min} J$. From Theorem 2.7 we know that all minimizing heteroclinics are odd and positive on the half-line. Hence we conclude the weak convergence of u_{θ_n} to u_θ which is again a minimizer of J in M using the analogue of Lemma 2.3 for the double-well case. Thus $\theta_n \to \theta \in \mathscr{C}$, and $\mathscr{C}$ is closed. Therefore $\mathscr{C}$ is either empty or all of S^1. In the former case all the minimizing heteroclinics are isolated, and in the latter $W^u(-1)$ and $W^s(1)$ coincide. ∎

3. MORSE REDUCTION

In this section we describe a reduction of the functional J from functions on the unbounded domain $\mathbb{R}$ to those on the finite interval $[-T, T]$. If T is sufficiently large, the minimizers of J found in the previous section restricted to the finite interval will be isolated minimizers of the reduced functional. This topological information will allow us to glue copies of these primary heteroclinic orbits to obtain multitransition solutions. Here we follow the method of Buffoni and Séré [5] adapted to our setting.

Equation (1) generates a four-dimensional flow, and $z = (u, u', u'', u''') \in \mathbb{R}^4$ will represent a point in the phase space in the usual way. Consider the local flow near the equilibrium points, and let $A_\varepsilon^\pm$ be a neighborhood of $(\pm 1, 0, 0, 0)$ of the form $B_\varepsilon(\pm 1, 0) \times B_\varepsilon(0, 0) \in \mathbb{R}^2 \times \mathbb{R}^2$. Then $\varepsilon > 0$ can be chosen small enough such that for any $\mathbf{x} = (x_1, x_2) \in B_\varepsilon(\pm 1, 0)$, the hyperplane $\{z\colon u = x_1, u' = x_2\}$ is transverse to the local stable and unstable manifolds, $W^s_{\mathrm{loc}}(\pm 1)$ and $W^u_{\mathrm{loc}}(\pm 1)$, and hence there exist unique solutions $\alpha(t, \mathbf{x}) \in W^u_{\mathrm{loc}}(\pm 1)$ for $t \geqslant 0$ and $\omega(t, \mathbf{x}) \in W^s_{\mathrm{loc}}(\pm 1)$ such that

$$(\alpha(0, \mathbf{x}), \alpha'(0, \mathbf{x})) = \mathbf{x} \qquad \text{and} \qquad (\omega(0, \mathbf{x}), \omega'(0, \mathbf{x})) = \mathbf{x}.$$

In the sequel, for the sake of simplicity we will not specify to which equilibrium point α and ω are asymptotic, but it should be clear from context.

Let $\hat{u} \in M$ be a minimizer of J, then $\hat{u} \in C^\infty(\mathbb{R})$ and satisfies

$$\begin{cases} \gamma \hat{u}^{(4)} - \beta \hat{u}'' + F'(\hat{u}) = 0 \\ \hat{u}(0) = \hat{u}''(0) = 0, \hat{u}'(0) \neq 0 \\ \lim_{t \to \pm\infty} \hat{u}(t) = \pm 1. \end{cases}$$

Define $\hat{z}(t) = (\hat{u}(t), \hat{u}'(t), \hat{u}''(t), \hat{u}'''(t))$, and $H^2_c(-T, T) = \{u \in H^2(-T, T)\colon u(0) = 0\}$. We perform a reduction by choosing $T > 0$ large enough so that $\hat{z}(T) \in W^s_{\mathrm{loc}}(1)$ and $\hat{z}(-T) \in W^u_{\mathrm{loc}}(-1)$. By standard Sobolev embeddings $H^2_c \hookrightarrow W^{1,\infty}$. Hence there exists a neighborhood N_ε of $\hat{u}$ in H^2_c such that for any $u \in N_\varepsilon$ we have $(u(\pm T), u'(\pm T)) \in B_\varepsilon(\pm 1, 0)$. Therefore we can define a continuous extension map $E_1\colon N_\varepsilon \to M$ by

$$E_1[u](t) = \begin{cases} \alpha(t + T, (u(-T), u'(-T))) & \text{if} \quad t \in (-\infty, -T] \\ u(t) & \text{if} \quad t \in [-T, T] \\ \omega(t - T, (u(T), u'(T))) & \text{if} \quad t \in [T, \infty). \end{cases}$$

We introduce a new functional $J_T = J \circ E_1$ on N_ε, and compute its derivative in two steps

$$dE_1[u]\,\varphi = \begin{cases} \dfrac{\partial}{\partial x_1}\alpha(t+T,(u(-T),u'(-T)))\,\varphi(-T) & \\ \quad +\dfrac{\partial}{\partial x_2}\alpha(t+T,(u(-T),u'(-T)))\,\varphi'(-T) & \text{if } t\in(-\infty,-T] \\ \varphi(t) & \text{if } t\in[-T,T] \\ \dfrac{\partial}{\partial x_1}\omega(t-T,(u(T),u'(T)))\,\varphi(T) & \\ \quad +\dfrac{\partial}{\partial x_2}\omega(t-T,(u(T),u'(T)))\,\varphi'(T) & \text{if } t\in[T,\infty) \end{cases}$$

for $\varphi\in H_c^2(-T,T)$ and

$$\begin{aligned} dJ_T[u]\,\varphi &= dJ[E_1[u]]\,dE_1[u]\,\varphi \\ &= \int_{-T}^{T}(\gamma u''\varphi''+\beta u'\varphi'+F'(u)\,\varphi)\,dt \\ &\quad+\int_{-\infty}^{-T}(\gamma E_1[u]''\,(dE_1[u]\,\varphi)''+\beta E_1[u]'\,(dE_1[u]\,\varphi)' \\ &\quad+F'(E_1[u])\,dE_1[u]\,\varphi)\,dt+\int_T^{\infty}(\gamma E_1[u]''\,(dE_1[u]\,\varphi)'' \\ &\quad+\beta E_1[u]'\,(dE_1[u]\,\varphi)'+F'(E_1[u])\,dE_1[u]\,\varphi)\,dt \\ &= \int_{-T}^{T}(\gamma u''\varphi''+\beta u'\varphi'+F'(u)\,\varphi)\,dt \\ &\quad+\gamma\alpha''(0)\,\varphi'(-T)-\gamma\alpha'''(0)\,\varphi(-T)+\beta\alpha'(0)\,\varphi(-T) \\ &\quad-\gamma\omega''(0)\,\varphi'(T)+\gamma\omega'''(0)\,\varphi(T)-\beta\omega'(0)\,\varphi(T). \end{aligned} \tag{6}$$

The last equality was obtained by integrating by parts and using the fact that α and ω are solutions. Here we have suppressed the dependence of α and ω on the parameters $u(-T)$ and $u(T)$, i.e., $\alpha(0)=\alpha(0,(u(-T),u(T)))$, and we will use this convention throughout the sequel.

LEMMA 3.1. *Suppose $u\in N_\varepsilon\subset H_c^2(-T,T)$ is a critical point of J_T. Then $E_1[u]\in M$ and satisfies* (1).

Proof. For all $\varphi\in H_c^2(-T,T)$, $dJ_T[u]\,\varphi=0$. First we consider variations such that $\varphi(t)\equiv 0$ for $t\in(-T,0)$ and $\varphi\in H_0^2(0,T)=\{\varphi\in H^2(0,T)$: $\varphi(0)=\varphi'(0)=\varphi(T)=\varphi'(T)=0\}$ for which

$$\int_0^T[\gamma u''\varphi''+\beta u'\varphi'+F'(u)\,\varphi]\,dt=0, \tag{7}$$

using (6). Therefore from regularity theory $u \in C^4(0, T)$ and at the endpoints $\lim_{t \to 0^+} z(t)$ and $\lim_{t \to T^-} z(t)$ exist. Similarly, $u \in C^4(-T, 0)$ and $\lim_{t \to 0^-} z(t)$ and $\lim_{t \to -T^+} z(t)$ exist. Now consider variations $\varphi \in H_c^2(-T, T) \cap H_0^2(-T, T)$ for which (7) holds integrating over the entire interval $(-T, T)$. Since u is a solution of (1) on $(-T, 0)$ and $(0, T)$, integration by parts yields $(u''(0^-) - u''(0^+))\, \varphi'(0) = 0$ so that $u \in C^2[-T, T]$. Since $\|u - \hat{u}\|_{H^2}$ is small, $u'(t) \neq 0$ on some interval $(-\delta, \delta)$, and

$$u'''(t) = \left(\frac{\gamma}{2}(u''(t))^2 + \frac{\beta}{2}(u'(t))^2 - F(u)\right) \Big/ \gamma u'(t)$$

from the Hamiltonian. Therefore $u \in C^3[-T, T]$. The differential equation

$$\gamma u^{(4)} = \beta u'' - F'(u) \tag{8}$$

is satisfied near 0, and hence $u \in C^4[-T, T]$. From the computation (6), integration by parts yields that $E_1[u]$ is C^3 at the endpoints $\pm T$ using the definition of α and ω. Hence $E_1[u] \in C^3(\mathbb{R})$ and C^4 on the intervals $(-\infty, -T]$, $[-T, T]$, and $[T, \infty)$. Again the differential equation (8) is satisfied near $\pm T$ which implies that $E_1[u] \in C^4(\mathbb{R})$ and is a classical solution to (1). ∎

We have proven the following theorem.

THEOREM 3.2. *Let $\hat{u} \in M$ be a minimizer of J and $T > 0$ be sufficiently large. Then*

(i) *$\hat{u}$ is an isolated solution to* (1) *in M,*

(ii) *if $u \in H_c^2(-T, T)$ is a critical point of J_T in a sufficiently small neighborhood of $\hat{u}$, then $E_1[u]$ is a classical solution of* (1), *and*

(iii) *$\hat{u}|_{[-T, T]}$ is an isolated critical point of J_T with* $\deg(\nabla J_T, \hat{u}|_{[-T, T]}, 0) = 1$.

Proof. (iii) We have $E_1[\hat{u}|_{[-T, T]}] = \hat{u}$, and hence $\hat{u}|_{[-T, T]}$ is a minimizer of J_T. It is isolated by (i) and (ii). The gradient ∇J_T is computed from the derivative dJ_T by composing with the canonical duality mapping between $(H_c^2)^*$ and H_c^2 endowed with the inner product

$$(u, v) = \int_{-T}^{T} [\gamma u'' v'' + \beta u' v' + uv]\, dt,$$

then $\nabla J_T = I + K$ where K is compact. Hence $\deg(\nabla J_T, \hat{u}|_{[-T, T]}, 0) = 1$ since $\hat{u}|_{[-T, T]}$ is a minimizer of J_T, cf. Amann [1]. ∎

4. MULTITRANSITION HETEROCLINICS AND HOMOCLINICS

Let $\hat{u}(t)$ be a heteroclinic orbit of (1) from -1 to $+1$ as found in Section 2. Note that by Theorem 2.6 these primary orbits have only one transition. Due to symmetry $\hat{u}(-t)$ is a heteroclinic orbit from $+1$ to -1. Using these two connections we will construct an approximate homoclinic orbit to -1 by gluing translates of $\hat{u}(t)$ and $\hat{u}(-t)$ and prove that a true homoclinic solution exists in a small neighborhood of this approximate solution. These arguments will be readily extendable to produce homoclinics to $+1$ and multitransition homoclinics and heteroclinics with any number of transitions separated by arbitrarily large distances.

Fix $\varepsilon > 0$ and recall the neighborhoods $A_\varepsilon^\pm$ of the equilibria $(\pm 1, 0, 0, 0)$ described in the previous section. As in Theorem 3.2 we choose $T > 0$ large enough such that $\hat{z}(-T) \in W^u_{\text{loc}}(-1)$ and $\hat{z}(T) \in W^s_{\text{loc}}(1)$ and $\hat{u}|_{[-T,T]}$ is an isolated critical point of J_T. We consider small neighborhoods $N_\delta^\pm$ of $\hat{u}(\pm t)$ in $H_c^2(-T, T)$ and $B_\delta^\pm = B_\delta((\hat{u}(T), \pm \hat{u}'(T))) \subset \mathbb{R}^2$ for $\delta < \varepsilon$. For sufficiently small δ, we can find a family of solutions g of (1) depending on $s > 1/\delta$ which connect any $\mathbf{x} \in B_\delta^+$ to any $\mathbf{y} \in B_\delta^-$ in time s, i.e., $g(t, \mathbf{x}, \mathbf{y}, s)$: $(0, s) \times B_\delta^+ \times B_\delta^- \times [1/\delta, \infty) \to \mathbb{R}$ is a family of solutions remaining in A_ε^+ which satisfy

$$(g(0, \mathbf{x}, \mathbf{y}, s), g'(0, \mathbf{x}, \mathbf{y}, s)) = \mathbf{x} \qquad \text{and} \qquad (g(s, \mathbf{x}, \mathbf{y}, s), g'(s, \mathbf{x}, \mathbf{y}, s)) = \mathbf{y}. \tag{9}$$

The existence of this family of solutions (proven in Lemma 12 of Buffoni and Séré [5]) is due to the fact that the equilibria ± 1 are hyperbolic and is in the spirit of the Lambda Lemma (cf. Guckenheimer and Holmes [13]), see Fig. 1.

Later we will need for g to satisfy some other properties which are characteristic of a saddle-focus, but first we define the extension map E_2: $N_\delta^+ \times N_\delta^- \times [1/\delta, \infty) \to -1 + H^2(\mathbb{R}) = \{w: 1 + w \in H^2(\mathbb{R})\}$ (see Fig. 2) by

$$E_2[u, v, s]$$

$$= \begin{cases} \alpha(t, (u(-T), u'(-T))) & \text{if } t \in (-\infty, 0] \\ u(t-T) & \text{if } t \in [0, 2T] \\ g(t-2T, (u(T), u'(T)), (v(-T), v'(-T)), s) & \text{if } t \in [2T, 2T+s] \\ v(t-3T-s) & \text{if } t \in [2T+s, 4T+s] \\ \omega(t-4T-s, (v(T), v'(T))) & \text{if } t \in [4T+s, \infty). \end{cases}$$

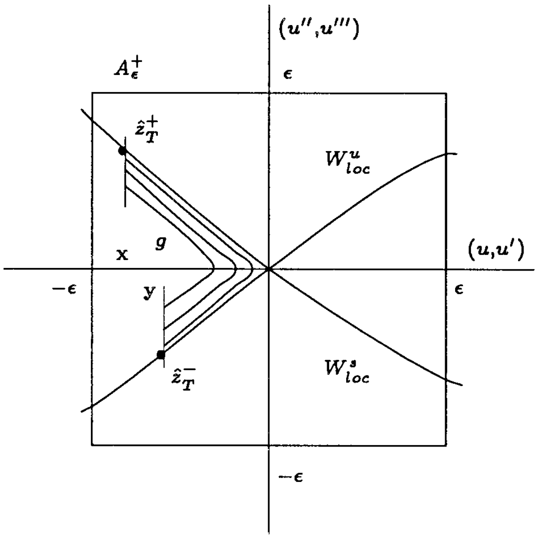

FIG. 1. A schematic of the local behavior near the equilibrium $(1, 0, 0, 0)$ and the family of solutions $g(t, \mathbf{x}, \mathbf{y}, s)$ with $\mathbf{x} = (\hat{u}(T), \hat{u}'(T))$ and $\mathbf{y} = (\hat{u}(T), -\hat{u}'(T))$. Also $\hat{z}_T^{\pm} = (\hat{u}(T), \pm\hat{u}'(T), \hat{u}''(T), \pm\hat{u}'''(T))$.

Proceeding as in the previous section, we consider the reduced functional $J \circ E_2 \colon N_\delta^+ \times N_\delta^- \times [1/\delta, \infty) \to \mathbb{R}$. Splitting up the integral and translating the intervals of integration gives

$$J \circ E_2[u, v, s] = \int_{-\infty}^{0} j(\alpha(t))\, dt + \int_{-T}^{T} j(u(t))\, dt + \int_{0}^{s} j(g(t; s))\, dt + \int_{-T}^{T} j(v(t))\, dt + \int_{0}^{\infty} j(\omega(t))\, dt.$$

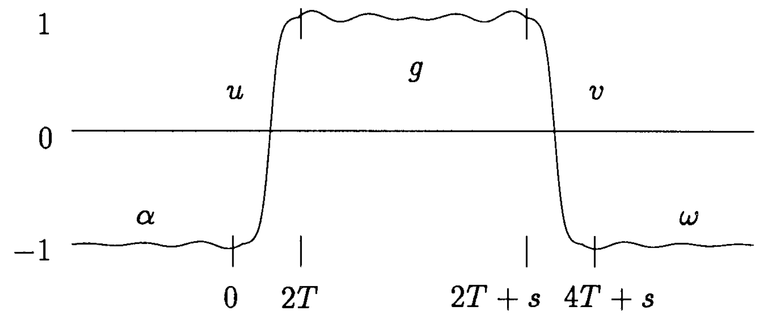

FIG. 2. The extension map $E_2[u, v, s]$.

As we did earlier for α and ω, we have suppressed the dependence of g on its middle two arguments, i.e., $g(t;s)=g(t,(u(T),u'(T)),(v(-T),v'(-T)),s)$. Differentiating yields

$$\begin{aligned}
d(J\circ E_2[u,v,s])(\varphi,\psi,\sigma) &= \int_{-T}^{T}[\gamma u''\varphi''+\beta u'\varphi'+F'(u)\,\varphi]\,dt \\
&\quad+\int_{-T}^{T}[\gamma v''\psi''+\beta v'\psi'+F'(v)\,\psi]\,dt \\
&\quad+\gamma\alpha''(0)\,\varphi'(-T)-\gamma\alpha'''(0)\,\varphi(-T) \\
&\quad-\gamma\omega''(0)\,\psi'(T)+\gamma\omega'''(0)\,\psi(T) \\
&\quad+\beta\alpha'(0)\,\varphi(-T)-\beta\omega'(0)\,\psi(T) \\
&\quad-\gamma g''(0;s)\,\varphi'(T)+\gamma g'''(0;s)\,\varphi(T) \\
&\quad+\gamma g''(s;s)\,\psi'(-T)-\gamma g'''(s;s)\,\psi(-T) \\
&\quad-\beta g'(0;s)\,\varphi(T)+\beta g'(s;s)\,\psi(-T) \\
&\quad+\frac{\partial}{\partial s}\int_0^s j(g(t;s))\,dt\,\sigma. \qquad (10)
\end{aligned}$$

Following the same arguments as in the Proof of Lemma 3.1, it can be verified that if $(u,v,s)\in N_\delta^+\times N_\delta^-\times[1/\delta,\infty)$ is a critical point of $J\circ E_2$ with δ sufficiently small, then $E_2[u,v,s]$ is a smooth solution to the differential equation. To find a multitransition solution homoclinic to -1 in a neighborhood of $E_2[\hat{u}(t),\hat{u}(-t),s]$ in $-1+H^2(\mathbb{R})$, we look for a critical point of the reduced functional $J\circ E_2$. To establish the existence of such a critical point, we will show that the degree of $\nabla(J\circ E_2)$ is nontrivial in some neighborhood.

The last term in (10) can be computed as follows. The partial derivatives of $j(u)=j(u,u',u'')$ as a function of three variables will be denoted by $\partial j/\partial u$, $\partial j/\partial u'$, and $\partial j/\partial u''$. From Leibnitz' rule

$$\begin{aligned}
\frac{\partial}{\partial s}\int_0^s j(g(t;s))\,dt &= j(g(s;s))+\int_0^s\left[\frac{\partial j}{\partial u}\frac{\partial g}{\partial s}+\frac{\partial j}{\partial u'}\frac{\partial g'}{\partial s}+\frac{\partial j}{\partial u''}\frac{\partial g''}{\partial s}\right]dt \\
&= \int_0^s\left[\frac{\partial j}{\partial u}-\frac{d}{dt}\frac{\partial j}{\partial u'}+\frac{d^2}{dt^2}\frac{\partial j}{\partial u''}\right]\frac{\partial g}{\partial s}\,dt \\
&\quad+\left[\frac{\partial j}{\partial u'}\frac{\partial g}{\partial s}+\frac{\partial j}{\partial u''}\frac{\partial g'}{\partial s}-\frac{d}{dt}\frac{\partial j}{\partial u''}\frac{\partial g}{\partial s}\right]_{t=0}^{t=s}+j(g(s;s)),
\end{aligned}$$

after integrating by parts. Furthermore

$$
\begin{aligned}
&\frac{\partial}{\partial s}\int_0^s j(g(t;s))\,dt\\
&\qquad=\frac{\partial j}{\partial u'}\frac{\partial g}{\partial s}(s;s)+\frac{\partial j}{\partial u''}\frac{\partial g'}{\partial s}(s;s)-\frac{d}{dt}\frac{\partial j}{\partial u''}\frac{\partial g}{\partial s}(s;s)+j(g(s;s))\\
&\qquad=-\frac{\partial j}{\partial u'}g'(s;s)-\frac{\partial j}{\partial u''}g''(s;s)+\frac{d}{dt}\frac{\partial j}{\partial u''}g'(s;s)+j(g(s;s)),
\end{aligned}
\tag{11}
$$

using the properties of the family of solutions g which satisfy

$$
\frac{\partial g}{\partial s}(0;s)=0 \qquad \text{and} \qquad \frac{\partial g}{\partial s}(s;s)=-g'(s;s),
$$

which follow from the definition of g in (9). Finally

$$
\frac{\partial}{\partial s}\int_0^s j(g(t;s))\,dt=H(g(\cdot;s),g'(\cdot;s),g''(\cdot;s),g'''(\cdot;s))
$$

which follows from a general relationship in fourth-order Hamiltonian systems coming from Noether theory which equates (11) and H for any solution g. This formula can be found in Logan [16].

Combining the above computations with (6), we see that

$$
\begin{aligned}
&d(J\circ E_2[u,v,s])(\varphi,\psi,\sigma)\\
&\quad=dJ_T[u]\,\varphi+dJ_T[v]\,\psi\\
&\qquad+H(g(\cdot;s),g'(\cdot;s),g''(\cdot;s),g'''(\cdot;s))\,\sigma\\
&\qquad-\gamma(g''(0;s)-\omega''(0,(u(T),u'(T))))\,\varphi'(T)\\
&\qquad+\gamma(g'''(0;s)-\omega'''(0,(u(T),u'(T))))\,\varphi(T)\\
&\qquad+\gamma(g''(s;s)-\alpha''(0,(v(-T),v'(-T))))\,\psi'(-T)\\
&\qquad-\gamma(g'''(s;s)-\alpha'''(0,(v(-T),v'(-T))))\,\psi(-T)\\
&\qquad-\beta(g'(0;s)-\omega'(0,(u(T),u'(T))))\,\varphi(T)\\
&\qquad+\beta(g'(s;s)-\alpha'(0,(v(-T),v'(-T))))\,\psi(-T).
\end{aligned}
\tag{12}
$$

The last two terms in (12) are zero from the definitions of α, ω, and g.

So far we have not used in any strong way the fact that the equilibria ± 1 are saddle-foci. To compute the degree of the above map, we would like to know first that we can ignore the boundary terms in (12) and also the behavior of the Hamiltonian on the family of solutions $g(t; s)$. Both of these come from a local analysis of the flow near a saddle-focus. In particular, $g(t; s)$ can be chosen such that

$$(g''(0, \mathbf{x}, \mathbf{y}, s), g'''(0, \mathbf{x}, \mathbf{y}, s)) \to (\alpha''(0, \mathbf{x}), \alpha'''(0, \mathbf{x})) \qquad \text{and}$$
$$(g''(s, \mathbf{x}, \mathbf{y}, s), g'''(s, \mathbf{x}, \mathbf{y}, s)) \to (\omega''(0, \mathbf{y}), \omega'''(0, \mathbf{y})) \tag{13}$$

uniformly in $\mathbf{x}$ and $\mathbf{y}$ as $s \to \infty$. Finally there exists a $\theta \in [0, 2\pi/\mu)$ such that the sequence $s_n = 2n\pi/\mu + \theta$ satisfies the following property: for a sufficiently small $v > 0$ independent of n and all $s_n \geqslant 1/\delta$

$$H(g(\cdot; s_n - v\varepsilon)) < 0 \qquad \text{and} \qquad H(g(\cdot; s_n + v\varepsilon)) > 0 \tag{14}$$

where $\mu = \tan^{-1}(\sqrt{8\gamma\beta^{-2} - 1})/2$ (the period of rotation of the linearized saddle-focus is $2\pi/\mu$). The fact that there exists a family $g(\cdot; s)$ of solutions which satisfy (13) and (14) follows from a local analysis of a saddle-focus equilibrium and is proven in Buffoni and Séré [5; Lemma 6].

Let $U = N_\delta^+ \times N_\delta^- \times [s_n - v\varepsilon, s_n + v\varepsilon]$. From (13) we see that the boundary terms in (12) become small as $n \to \infty$ which allows us to compute the degree of $\nabla(J \circ E_2)$ in this neighborhood by

$$\deg(\nabla(J \circ E_2), U, 0) = \deg(\nabla J_T[u] \oplus \nabla J_T[v] \oplus H(g(\cdot; s))\, I, U, 0).$$

From Theorem 3.2, $\deg(\nabla J_T, N_\delta^\pm, 0) = 1$. Since $H(g(\cdot; s))$ changes sign from negative to positive on the interval $[s_n - v\varepsilon, s_n + v\varepsilon]$, we have that $\deg(H(g(\cdot; s))\, I, [s_n - v\varepsilon, s_n + v\varepsilon], 0) = 1$ as well. Therefore $\deg(\nabla(J \circ E_2), U, 0) = 1$ from the product formula for the degree of a direct sum, and thus $J \circ E_2$ has a critical point $(u^*, v^*, s^*) \in U$. Using arguments similar to those in the proof of Lemma 3.1 and Theorem 3.2, we conclude that $w = E_2[u^*, v^*, s^*]$ is a critical point of J in $-1 + H^2(\mathbb{R})$ which is close to $E_2[\hat{u}(t)|_{[-T, T]}, \hat{u}(-t)|_{[-T, T]}, s_n]$. Hence w is an orbit homoclinic to -1 with one bump, i.e. two transition layers.

The above construction immediately generalizes to gluing any finite number of alternating copies of the primary heteroclinics $\hat{u}(t)$ and $\hat{u}(-t)$ together to produce homoclinics to both ± 1 with any (even) number of zeroes and heteroclinics between ± 1 with any (odd) number of zeroes. There is a lower bound on the time between consecutive zeroes, but these intervals can be arbitrarily large. This proves Theorem 1.1 stated in the introduction.

5. CONCLUDING REMARKS

We investigate a class of four-dimensional Hamiltonian systems which have a heteroclinic loop between two saddle-focus equilibria. Under the assumption that this loop is the result of the transverse intersection of the stable and unstable manifolds of these equilibria, Devaney [10] shows that for any small neighborhood of this loop and any Poincaré section transverse to it, the Poincaré map has an invariant set which is conjugate to a shift on N symbols. Moreover the transversality condition is difficult to check for specific systems. For the extended Fisher–Kolmogorov and periodic systems described in this paper, the transversality requirement can be replaced by the weaker condition of Buffoni and Séré [5] that the stable and unstable manifolds do not coincide in the phase space. This condition is automatically satisfied by the periodic system, but the double-well case is more delicate, and so far we have checked it only for Eq. (5) specifically.

Another approach would be to find homoclinic orbits to ± 1 directly through minimization in a restricted class of functions using the properties of saddle-focus equilibria. This would immediately imply that the above construction can be applied to the entire class of equations of the form (1). This approach is currently under investigation by Kalies, Kwapisz, and VanderVorst. Also Rabinowitz [22] has proven the existence of heteroclinic solutions for second-order Hamiltonian systems by minimizing an appropriate functional. These methods can possibly be modified to study the fourth-order problems considered in this paper which would allow both multi-well and non-symmetric potentials.

In this paper we construct a countable family of multitransition homoclinic and heteroclinic solutions. The gluing method applied above can also be used to obtain periodic orbits which are in an arbitrarily small neighborhood of the heteroclinic loop and are structurally similar to those which would be found using dynamical systems methods of Devaney. Also the existence of the family of homoclinic and heteroclinic solutions described in Theorem 1.1 is sufficient to show that the dynamics are chaotic. In particular the topological entropy can be estimated. Let U be any tubular neighborhood of the primary heteroclinic loop, and define

$$r_{\mathrm{sep}}(U, t, \varepsilon) = \max\{\#(V)\colon V \subset C \text{ is } (t, \varepsilon)\text{-separated}\}.$$

Recall that V is (t, ε)-separated with respect to the flow Φ generated by (1) if for any z_1 and z_2 in V we have that $|\Phi(\tau, z_1) - \Phi(\tau, z_2)| > \varepsilon$ for some $\tau \in [0, t]$, cf. Guckenheimer and Holmes [13]. Fix $\tau_0 = 2\pi/\mu + C + \varepsilon$ which is approximately the minimum time between transition layers in the multitransition solutions constructed above. Then for any $\varepsilon \ll 1$ and on any time interval $[0, k\tau_0]$, $k>0$ one can find at least 2^{k+1} orbits which are 2ε-separated on this interval. Therefore $r_{\text{sep}}(U, k\tau_0, 2\varepsilon) \geqslant 2^{k+1}$ for any $\varepsilon < \varepsilon_0$. The topological entropy is defined as

$$h(U) = \lim_{\varepsilon \to 0} \limsup_{t \to \infty} \frac{1}{t} \log r_{\text{sep}}(U, t, \varepsilon),$$

and hence $h(U) \geqslant \log 2/\tau_0 > 0$.

The dynamics of the extended Fisher–Kolmogorov equation are much richer than what can be deduced from Theorem 1.1. Peletier and Troy [17, 18] have found a countable family of heteroclinics (used in Theorem 2.8) and a class of chaotic solutions neither of which are contained in small neighborhoods of the primary loop. Their results also indicate another interesting phenomenon. The minimization argument above does not rely on the type of the equilibria, i.e., the heteroclinic loop exists for all $\gamma, \beta > 0$. When $\gamma < \beta^2/8$, the equilibria are saddle-nodes, and as γ crosses $\beta^2/8$ they become saddle-foci. Moreover this local change of the type of the equilibria triggers a global bifurcation of at least two branches of periodic orbits off of the heteroclinic loop, cf. Peletier and Troy [19].

Buffoni [3] describes how the gluing technique used above can be applied to the equation

$$u^{(4)} + Pu'' + u - u^2 = 0,$$

which has an orbit homoclinic to the origin which is a saddle-focus for certain parameter values. For a description of the dynamics of this equation see Buffoni *et al.* [6]. For the extended Fisher–Kolmogorov equation the origin is a saddle-center for all $\gamma > 0$ and $\beta \in \mathbb{R}$, and one can easily show that no solutions homoclinic to the origin exist when either $\beta \geqslant 0$ or $\gamma > 9\beta^2/28$ and $\beta < 0$. Indeed homoclinics must satisfy

$$\int_{\mathbb{R}} [7\gamma(u'')^2 + 3\beta(u')^2 + u^2]\, dt = 0$$

which follows from integration by parts and the Hamiltonian. The statement follows from interpolation inequalities. For futher results about homoclinics to saddle-centers in fourth-order problems see Grotta-Ragazzo [11, 12] and the references therein.

For further results on the use of variational techniques for finding homoclinic and heteroclinic orbits in fourth-order Hamiltonian systems and results about related equations cf., Buffoni and Toland [4], Hofer and Toland [14], and Leizarowitz and Mizel [15].

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