BRAIDS AND PARABOLIC DYNAMICS*

Robert Ghrist	*University of Illinois, Urbana, USA*
	CDSNS, Georgia Institute of Technology, Atlanta, USA
Robert Vandervorst	*VU Amsterdam, Netherlands*
	CDSNS, Georgia Institute of Technology, Atlanta, USA

1 Perspective

1.1 the comparison principle

Consider a scalar uniformly parabolic PDE,

$$u_t = f(x, u, u_x, u_{xx}) \quad ; \text{ where} \quad 0 < \lambda \leq \partial_{u_{xx}} f \leq \lambda^{-1} \text{ uniformly.} \tag{1}$$

Assume that f is of smoothness class C^2. For simplicity we use periodic boundary conditions; hence $x \in S^1$. We view (1) as an evolution equation on the curve $u(\cdot, t)$: as t increases, the graph of u evolves in the (x, u) plane.

It is a well-known fact (going back to Sturm, but revived and extended considerably by Matano [18], Brunovsky and Fiedler [6], Angenent [2], and others) that there is a **comparison principle** for (1). Specifically, let $u^1(t)$ and $u^2(t)$ be solutions to (1). Then the number of intersections of the graphs of u^1 and u^2,

$$z(t) := \# \left\{ x : u^1(x, t) = u^2(x, t) \right\}, \tag{2}$$

is a weak Lyapunov function for the dynamics: z is non-increasing. Furthermore, at those particular times t for which the graphs of $u^1(t)$ and $u^2(t)$ are tangent, the function z decreases strictly, even in the case where the tangencies are of arbitrarily high order [2]. These facts are all at heart an application of classical maximum principle arguments which have a geometric interpretation.

Parabolic dynamics separates tangencies monotonically.

Using this comparison principle (also known as *lap number* or *zero crossing* techniques), numerous authors have analyzed the dynamics of (1) in varying degrees of generality. We note in particular the paper of Fiedler and Mallet-Paret [9], in which the comparison principle is used to show that the dynamics of (1) is often Morse-Smale, and also the paper of Fiedler and Rocha [10], in which the global attractor for the dynamics is roughly classified.

*Supported in part by National Science Foundation Grant DMS-0134408 [RG].

1.2 A GLOBALIZATION VIA BRAIDS

Our contribution is a globalization of the comparison principle using topological braid theory. For a motivating example, consider again a pair of evolving curves $u^1(t)$ and $u^2(t)$ in the (x, u) plane. If we lift these curves to the three-dimensional (x, u, u_x) space, we no longer have intersecting curves, unless t is such that the planar graphs of u^1 and u^2 intersect tangentially. The graphs of u^1 and u^2 in the (x, u, u_x) space are instead an example of a **closed braid** on two strands. What was the intersection number of their projections is now the **linking number** of the pair of strands.

We therefore see that the comparison principle takes on a linking number interpretation (a fact utilized in a discrete setting by LeCalvez [16]). After lifting solutions u^1 and u^2 to the (x, u, u_x) space, the comparison principle says that the linking number is a nonincreasing function of time which decreases strictly at those times at which the curves are tangent. This two-strand example is merely motivation for adopting a braid-theoretic perspective on multiple strands, as in Figure 1.

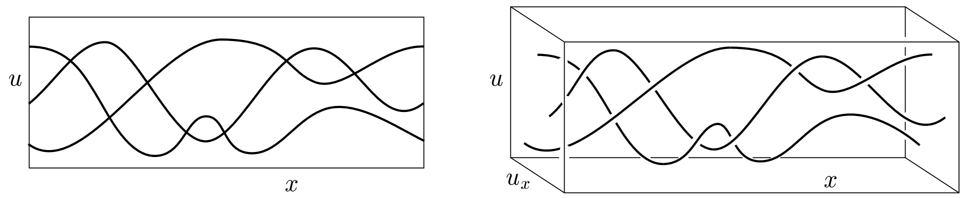

FIGURE 1: Multiple solutions lift to a braid in (x, u, u_x).

The key observation is that the comparison principle passes from a local statement ("linking number decreases at a tangency") to a global statement ("algebraic length in the braid group decreases at a tangency").

1.3 GOAL: FORCING

Our goal is to produce a forcing theory for the dynamics of (1), and, as we shall relate, more general discrete systems. For simplicity, we focus on forcing stationary solutions, though periodic and connecting orbits are likewise accessible. Say that one has found a **skeleton** of stationary curves $\{v^1, v^2, \ldots, v^m\}$ for a particular representative of (1). How many and which types of other stationary curves are forced to be present?

Since the skeleton of known fixed curves $\mathbf{v} = \{v^1\}_{i=1}^m$ lifts to a braid, the problem is naturally couched in braid-theoretic terms: given a braid $\mathbf{v}$ fixed by a particular uniform parabolic PDE, which other classes of braids $\mathbf{u}$ are forced to exist as stationary curves? In this context, the forcing theory is reminiscent of Boyland's theory of "braid types" for periodic orbits of two-dimensional homeomorphisms [5]: a braid-theoretic version of the Nielsen-Thurston theory for surface homeomorphisms.

The spirit of our forcing theory is as follows:

1. Given a fixed braid $\mathbf{v}$, construct the configuration space of all n-strand braids $\mathbf{u}$ which have $\mathbf{v}$ as a sub-braid.

2. Use the braid-theoretic comparison principle to decompose this space into isolating blocks for the parabolic dynamics corresponding to distinct braid classes.

3. Define a Conley index for these relative braid classes which depends only on the topology of the braids, and not on the analytic details of the dynamics.

4. Prove Morse-type inequalities for forcing stationary curves in the PDE from a nontrivial braid index.

This is the basic recipe, modulo the frequent discretization needed to ensure the compactness necessary for the Conley index. Section 2 gives a rough outline of the index definition. Topological and dynamical features are outlined in Section 3. Section 4 gives applications to PDEs as well as to second-order Lagrangian dynamics.

2 Definitions and discretizations

To define the homotopy index for braid classes requires an unfortunately large array of definitions. For clarity, we distill the details of [11] down to the core concepts.

2.1 braids

There are two types of braids to consider: **topological** and **discretized**. Roughly speaking, a **topological braid** on n strands is an embedding of n disjoint arcs into $D^2 \times [0,1]$ transverse to the discs $D^2 \times \{x\}$ for all x: see Figure 1. Given a braid β, its **braid class** $\{\beta\}$ is the equivalence class of isotopic braids. Braid classes possess a group structure for which generators are strand crossings in a planar projection and concatenation of the braids forms the group operation [4].

The class of **discretized braids** are best visualized as piecewise-linear braid diagrams, as in Figure 2[left]. A discretized braid, $\mathbf{u}$, on n strands of period p, is determined by np **anchor points**: $\mathbf{u} = \{u_i^\alpha\}$. Superscripts $\alpha = 1...n$ refer to strand numbers, and subscripts $i = 1...p$ refer to spatial discretizations. One connects the anchor point u_i^α to u_{i-1}^α and u_{i+1}^α via straight lines. Since "height" is determined by slope, all crossings in the braid diagram are of the same type.[1] Since we employ periodic boundary conditions on the x variable, all of the braids are **closed**: left and right hand endpoints of strands are abstractly identified (perhaps by a nontrivial permutation of the strands). Denote by $\mathcal{D}_p^n$ the set of all n-strand period p discretized braids.

For topological braids, a **singular braid** arises when any strands intersect. Since all of the braids we consider are lifts of graphs u, the only possible intersection is that which occurs when two strands are tangent in the projection. For a discretized braid $\mathbf{u}$, the singular braids are defined to be those braids at which anchor points on two different strands coincide in a

[1] These are examples of **Legendrian** braids from contact geometry.

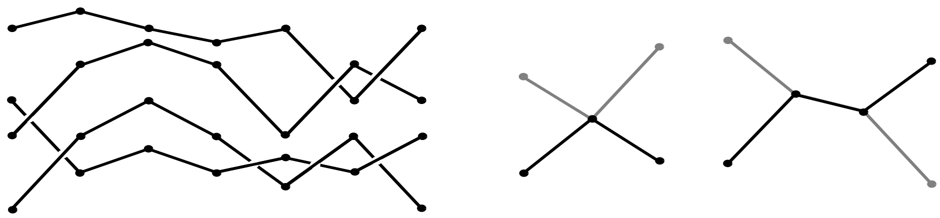

FIGURE 2: [left] A discretized braid in $\mathcal{D}^4_6$ (note: left and right hand sides are identified); [right] Two types of singular discretized braids: a simple tangency, and a high-order contact.

non-transverse fashion (looking at neighboring points): specifically,

$$\Sigma := \left\{ \mathbf{u} : u_i^\alpha = u_i^\beta \text{ for some } i \text{ and } \alpha \neq \beta, \text{ and } (u_{i-1}^\alpha - u_{i-1}^\beta)(u_{i+1}^\alpha - u_{i+1}^\beta) \geq 0 \right\} \tag{3}$$

The set Σ carves $\mathcal{D}^n_p$ into components: these are the discretized braid classes, denoted $[\mathbf{u}]$.

2.2 DYNAMICS

Discretizing Equation (1) in the standard way would yield a family of nearest neighbor equations of the form $\frac{d}{dt}u_i = f_i(u_{i-1}, u_i, u_{i+1})$ in which uniform parabolicity would manifest itself in terms of the derivatives of f_i with respect to the first and third variables. Instead of explicitly discretizing the PDE itself, we use the broadest possible category of nearest neighbor equations for which a comparison principle holds: these are related to the **monotone systems** of, *e.g.*, [19, 13, 9] and others.

A **parabolic relation** of period p is a sequence of maps $\mathcal{R} = \{\mathcal{R}_i : \mathbb{R}^3 \to \mathbb{R}\}$, $i = 1...p$, such that $\partial_1 \mathcal{R}_i > 0$ and $\partial_3 \mathcal{R}_i \geq 0$ for every i. These include discretizations of uniform parabolic PDE's, as well as a variety of other discrete systems [17], including monotone twist maps [16]. The small amount of degeneracy permitted ($\partial_3 \mathcal{R}_i = 0$) does not affect the manifestation of a comparison principle. Given a discretized braid $\mathbf{u} = \{u_i^\alpha\}$ and a parabolic relation $\mathcal{R}$, one evolves the braid according to the equation

$$\frac{d}{dt}(u_i^\alpha) = \mathcal{R}_i(u_{i-1}^\alpha, u_i^\alpha, u_{i+1}^\alpha). \tag{4}$$

Stationary curves for (4) correspond to a braid $\mathbf{u}$ such that $\mathcal{R}_i(\mathbf{u}) = 0$ for all i. The parabolic relation $\mathcal{R}$ induces a flow on $\mathcal{D}^n_p$ which respects a braid-theoretic comparison principle.

Lemma 2.1 *Let $\mathcal{R}$ be any parabolic relation and $\mathbf{u} \in \Sigma$ any singular braid. Then the flowline $\mathbf{u}(t)$ of $\mathcal{R}$ passing through $\mathbf{u} = \mathbf{u}(0)$ leaves a neighborhood of Σ in forward and backward time so as to strictly decrease the algebraic length of $\mathbf{u}(t)$ in the braid group as t passes through zero.*

2.3 THE INDEX

Given the discretizations of the braid classes and the dynamics, one is left with a conveniently finite dimensional problem. For purposes of a forcing theory, we use relative braids. Given a period p braid $\mathbf{v}$, denote by $\mathcal{D}_p^n$ REL $\mathbf{v}$ the space of all n strand, period p discretized braids which have $\mathbf{v}$ as a sub-braid.

In this context, the simplest version of Conley's index can be defined for braid classes (see [7] for an introduction to the Conley index). To do so, it must be shown that the braid classes $[\mathbf{u} \text{ REL } \mathbf{v}]$ are **isolated** in the sense that (informally) no flowlines within $[\mathbf{u} \text{ REL } \mathbf{v}]$ are tangent to the boundary of this set. It follows from Lemma 2.1 that $[\mathbf{u} \text{ REL } \mathbf{v}]$ is isolated for the flow of (4) assuming that the braid class is **proper**, *i.e.*, no free strands of $\mathbf{u}$ can "collapse" onto $\mathbf{v}$ or onto each other. Furthermore, to ensure compactness, we assume that the braid class $[\mathbf{u} \text{ REL } \mathbf{v}]$ is bounded.

The **homotopy braid index** of $[\mathbf{u} \text{ REL } \mathbf{v}]$ is defined as the Conley index of the braid class, computed as follows. Choose any $\mathcal{R}$ which fixes $\mathbf{v}$. Define $\mathcal{E}$ to be those braids on the boundary of $[\mathbf{u} \text{ REL } \mathbf{v}]$ along which evolution under the flow of $\mathcal{R}$ exits the braid class. The homotopy braid index is defined to be the pointed homotopy class[2]

$$\mathbf{h}([\mathbf{u} \text{ REL } \mathbf{v}]) := ([\mathbf{u} \text{ REL } \mathbf{v}]/\mathcal{E}, \{\mathcal{E}\}) . \tag{5}$$

As this is simply the Conley index of the isolating block $[\mathbf{u} \text{ REL } \mathbf{v}]$ under the flow of $\mathcal{R}$, it is easy to show that $\mathbf{h}([\mathbf{u} \text{ REL } \mathbf{v}])$ is well-defined and independent of the choice of $\mathcal{R}$ (so long as it is parabolic and fixes $\mathbf{v}$) as well as the choice of $\mathbf{v}$ within its braid class $[\mathbf{v}]$.

Although the homotopy type of a quotient of a braid class seems difficult to compute, the homology $CH_*([\mathbf{u} \text{ REL } \mathbf{v}])$ is both efficacious and computable.[3] The braid pair of Figure 3[right] has index $\mathbf{h} \simeq S^4 \vee S^5$; the pair on the left has trivial index, even though the linking numbers and periods of all strands are identical. This exemplifies the extra information carried by the braid classes.

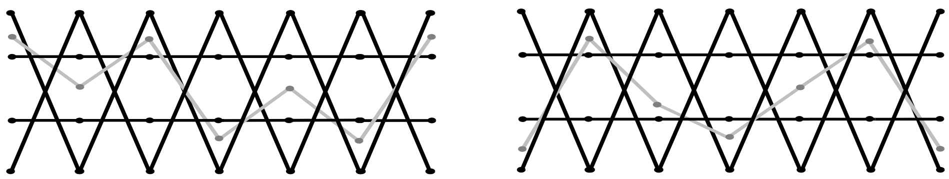

FIGURE 3: Discretized braid pairs with trivial [left] and nontrivial [right] homotopy index. The fixed strands $\mathbf{v}$ are black; the free strand is grey.

3 A FEW THEOREMS

3.1 TOPOLOGICAL CONTENT

The homotopy braid index has both topological and dynamical implications. The most important result about the index is the following invariance theorem:

[2]We omit a few technicalities concerning the rare cases in which the discretized braids get "locked" because of too-coarse discretization: see [11] for details.

[3]All of the theorems about the index $\mathbf{h}$ were predicated by rigorous computer experiments of M. Allili.

Theorem 3.1 ([11]) *The homotopy braid index is an invariant of topological braid pairs.*

Otherwise said, any two discretizations of a topological braid pair have identical homotopy indices, regardless of the period of the discretization used. The proof of this theorem involves a singular perturbation argument applied to a "stabilization operator" on discretized braids.

While the precise topological content of the index is as yet unclear, a duality operator on braids was discovered and analyzed in [11]. This has the pleasant corollary that, roughly speaking, adding a full twist to a braid class shifts the homotopy index up two dimensions (a dimension shift on the homology level; a double suspension on the homotopy level).

3.2 A FORCING THEOREM

The dynamical consequences of the index are forcing results. A simple example: given any parabolic relation $\mathcal{R}$ which has as a stationary solution the braid of Figure 3[right, black], then, since adding the grey strand yields a nontrivial braid index, there must be some invariant set for $\mathcal{R}$ within this braid class. At this point, one uses Morse theory ideas: if $\mathcal{R}$ is a gradient flow, then there must be a stationary solution of the form of the grey strand. If the flow is not of gradient type, then finer information can still detect stationary and/or periodic curves. By iterating the process of adding free strands and computing a nontrivial index, one can go quite far.

The following forcing theorem (for gradient-type $\mathcal{R}$) is a very general forcing theorem:

Theorem 3.2 ([11]) *Let $\mathcal{R}$ be a parabolic recurrence relation which* (i) *is of gradient type, and* (ii) *is dissipative (roughly, that large solutions are "repelled" from infinity). If $\mathcal{R}$ fixes a discretized braid $\mathbf{v}$ which is not in the trivial braid class (that is, if it has any crossings whatsoever), then there are an infinite number of distinct braid classes for multiple periods which arise as stationary solutions of $\mathcal{R}$.*

4 APPLICATIONS

4.1 PDEs

Though inspired by smooth curves and PDEs, all of the theorems about the homotopy braid index are constructed for discretized systems. It remains to connect back to the infinite-dimensional setting. Fortunately, Theorem 3.1 hints that passing to a continuum limit is permissible. Indeed, one can show:

Theorem 4.1 ([12]) *Assume that f in Equation (1) satisfies* (i) *the growth condition that for each $r > 0$ there exists a constant $C = C(r) > 0$ such that*

$$|\partial_{u_x} f(x, u, v, w)| \leq C(1 + |v|^{\gamma-1}) \tag{6}$$

for all $|u| \leq r$, all v, all w, and some $\gamma < 2$; and (ii) *that f is dissipative in the sense that $\mathrm{SIGN}(u) f(x, u, 0, 0) < 0$ as $|u| \to \infty$. Then the existence of any stationary curves of nontrivial braid class forces an infinity of stationary curves of different braid classes.*

In this context, Theorem 3.2 becomes a valuable tool for forcing stationary solutions. More refined results [12] yield homological lower bounds on the number of stationary solutions within a given braid class.

4.2 SECOND-ORDER LAGRANGIANS

The second major set of applications is to a class of fourth-order equations coming from a Lagrangian setting. Consider a **second order Lagrangian**, $L(u, u_x, u_{xx})$, such as is found in Swift-Hohenberg-type equations:

$$L = \frac{1}{2}(u_{xx})^2 - (u_x)^2 + \frac{1-\alpha}{2}u^2 + \frac{u^4}{4}. \tag{7}$$

We assume always the standard convexity assumption that $\partial^2_{u_{xx}} L \geq \delta > 0$. The Euler-Lagrange equations yield in this context a fourth-order ODE.

The problem of finding periodic orbits in a Lagrangian setting can of course be translated into the Hamiltonian context. In the case of a second order Lagrangian, the Hamiltonian flow is of two degrees of freedom. Hence, the problem translates to finding periodic orbits of a Hamiltonian flow on an energy-level three-manifold. This problem has been shown to be extremely delicate, requiring of late such refined techniques as pseudoholomorphic curves (see, *e.g.*, Hofer's seminal paper [14]) and the contact homology of Eliashberg, Givental, and Hofer [8]. Such techniques are strongly dependent upon compactness of the energy-level and upon a topological convexity (an invariant contact structure). Neither of these conditions are guaranteed for a general second-order Lagrangian: indeed, compactness is always lost.

Nevertheless, the homotopy braid index provides a very effective means of forcing periodic orbits. By restricting to systems which satisfy a **twist condition** (a mild variational hypothesis [20]), one can employ a "broken geodesics" construction which yields a restricted form of parabolic relation. By retooling the homotopy braid index to this setting, one extracts very general forcing theorems, a simple example of which is the following:

Theorem 4.2 ([11]) *Let $L(u, u_x, u_{xx})$ be a Lagrangian which is dissipative (infinity is repelling) and twist. Then, at any regular energy level, the existence of a single periodic orbit which traces out a nontrivial curve in the (u, u_x) plane implies the existence of infinitely many others at this energy level.*

Additional results are available for singular energy levels, as well as for non-dissipative systems [11]. Recent progress [15] indicates that the twist hypothesis may be unnecessary.

REFERENCES

[1] M. Allili, R. Ghrist and R.C. Vandervorst, in preparation.

[2] S. Angenent, *The zero set of a solution of a parabolic equation*, J. Reine Angew. Math. **390** (1988), 79-96.

[3] S. Angenent, J.-B. Van den Berg and R.C. Vandervorst, *Contact and noncontact energy hypersurfaces in second order Lagrangian systems*, in preparation.

[4] J. Birman, *Braids, Links and the Mapping Class Group*, Annals of Mathematics Studies vol. 82, Princeton Press, (1975).

[5] P. Boyland, *Topological methods in surface dynamics*, Topology & its Applications **58**:3, (1994), 223–298.

[6] P. Brunovský and B. Fiedler, *Connecting orbits in scalar reaction-diffusion equations.* Dynamics Reported **1**, (1988), 57-89.

[7] C. Conley, *Isolated invariant sets and the Morse index*, C.B.M.S. Reg. Conf. Ser. Math. **38**, (1978), published by the AMS.

[8] Y. Eliashberg, A. Givental, and H. Hofer. *An introduction to symplectic field theory.* Geom. Funct. Anal. Special Volume, Part II (2000), 560–673. `ArXiV` preprint `math.SG/0010059`.

[9] B. Fiedler and J. Mallet-Paret, *A Poincaré-Bendixson theorem for scalar reaction diffusion equations*, Arch. Rat. Mech. **107** (1989), 325-345.

[10] B. Fiedler and C. Rocha, *Orbit equivalence of global attractors of semilinear parabolic differential equations*, Trans. Amer. Math. Soc. **352**(1) (2000), 257-284.

[11] R. Ghrist, J.-B. Van den Berg, and R.C. Vandervorst, *Morse theory on spaces of braids and Lagrangian dynamics*, ArXiv preprint `math.DS/0105082`.

[12] R. Ghrist and R.C. Vandervorst, *Braids and scalar parabolic PDE's*, in preparation.

[13] M. Hirsch, *Systems of differential equations which are competitive or cooperative, I: Limit sets*, SIAM J. Math. Anal. **13** (1982), 167-179.

[14] H. Hofer, *Pseudo-holomorphic curves and the Weinstein conjecture in dimension three*, Invent. Math. **114** (1993), 515-563.

[15] W. Kalies and R.C. Vandervorst, in preparation.

[16] P. LeCalvez, *Propriété dynamique des diffeomorphismes de l'anneau et du tore*, Astérique **204**, (1991).

[17] J. Mallet-Paret and H. Smith, *The Poincaré-Bendixson theorem for monotone cyclic feedback systems*, J. Dyn. Diff. Eqns. **2** (1990), 367-421.

[18] H. Matano, *Nonincrease of the lap-number of a solution for a one dimensional semilinear parabolic equation*, J. Fac.Sci. Univ. Tokyo **29** (1982), 401-441.

[19] J. Smillie, *Competitive and cooperative tridiagonal systems of differential equations*, SIAM J. Math. Anal. **15** (1984), 531-534.

[20] J.-B. Van den Berg and R.C. Vandervorst, *Second order Lagrangian twist systems: simple closed characteristics*, Trans. Amer. Math. Soc., **354**(4) (2001), 1393-1420.