EXISTENCE AND NON-EXISTENCE OF POSITIVE SOLUTIONS OF NON-LINEAR ELLIPTIC SYSTEMS AND THE BIHARMONIC EQUATION

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(Submitted by F.V. Atkinson)

1. Introduction. Our main objective in this paper is to give some a priori estimates and to prove existence and non-existence theorems for particular elliptic systems, which were already briefly studied in [14], and the biharmonic equation. We shall restrict ourselves here to radially symmetric solutions so that we can apply ODE-methods. It should be noted that, by a result of W.C. Troy [13], solutions of Problem (I) are automatically radially symmetric if $f$ and $g$ are non-decreasing functions of the dependent variables $u$ and $v$. However, our results will not be restricted to such functions $f$ and $g$. We consider the problem

\[
\begin{cases}
-\Delta u = g(v), & v > 0 \quad \text{in } B_R, \\
-\Delta v = f(u), & u > 0 \quad \text{in } B_R, \\
u = 0, & v = 0 \quad \text{on } \partial B_R,
\end{cases}
\]

where $(u, v) \in C^2(\overline{B}_R) \times C^2(\overline{B}_R)$, with $B_R$ a ball in $\mathbb{R}^N \ (N \geq 4)$ of radius $R$. For functions $f$ and $g$, the following properties are assumed to hold:

\[(H1) \quad \begin{cases}
f, g \in C(\mathbb{R}), \\
f(0) = 0, \quad g(0) = 0.
\end{cases}\]

The system described by (I) appears in all kinds of problems in physics and chemistry because in many systems of reaction-diffusion equations, the steady-states are solutions of Problem (I). Actually, we can generalize Problem (I) to more general functions $f$ and $g$; for instance, if $f(u, v)$ and $g(u, v)$ are both positive functions in $u$ and $v$ with certain growth-conditions. In this paper, we shall concentrate on Problem (I) in order to develop a method for proving the existence of solutions.

The idea of studying Problem (I) came from the study of the biharmonic equation with the boundary conditions $u = 0$ and $\Delta u = 0$. This problem is easily put in the form of Problem (I) when we set $g(v) = v$. This initial study brought up the idea of families of critical exponents of Problem (I), which is described in [14] in great detail. We also refer to [15] for related results on biharmonic problems. Thus in this paper we shall introduce the critical exponents $p^*$ and $q^*$ defined by

\[p^* = \frac{N - \xi}{\xi}, \quad q^* = \frac{2 + \xi}{N - 2 - \xi},\]

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where
\[ \xi \in \left( \frac{N - 4}{2}, \frac{N}{2} \right), \quad N \geq 4. \]

Note that \( p^* \) and \( q^* \) satisfy the relation
\[ \frac{1}{1 + p^*} + \frac{1}{1 + q^*} = \frac{N - 2}{N}. \] (1.4)

The region in which
\[ \frac{1}{1 + p} + \frac{1}{1 + q} \leq \frac{N - 2}{N} \quad \text{and} \quad p, q \geq 1 \] (1.5)

will be denoted by \( S^+(N) \) and the region in which
\[ \frac{1}{1 + p} + \frac{1}{1 + q} > \frac{N - 2}{N} \quad \text{and} \quad p, q \geq 1, \] (1.6)

by \( S^-(N) \). These regions are sketched in Figure 1.

\[ \text{Figure 1. The regions } S^+(N) \text{ and } S^-(N). \]

To formulate our results, we shall need the linear eigenvalue problem
\[ \begin{cases} 
-\Delta u = \lambda_2 v, & v > 0 \quad \text{in } B_R, \\
-\Delta v = \lambda_1 u, & u > 0 \quad \text{in } B_R, \\
u = 0, & v = 0 \quad \text{on } \partial B_R.
\end{cases} \] (1.7-1.9)

As was shown in [14], there exists a curve \( \mathcal{C} \) of eigenvalues for which this problem has a solution. This curve is given by
\[ \mathcal{C} = \{(\lambda_1, \lambda_2) : \lambda_1 > 0, \lambda_2 > 0 \quad \text{and} \quad \lambda_1 \lambda_2 = \mu_1^2\}, \]

where \( \mu_1 \) is the principal eigenvalue of \(-\Delta\) on a ball with radius \( R \).

Finally, we set
\[ F(s) = \int_0^s f(\sigma) \, d\sigma, \quad G(s) = \int_0^s g(\sigma) \, d\sigma. \] (1.10)

The main result of this paper is the following \( L^\infty \)-bound for solutions of Problem (I).
**Theorem 1.** Let $f$ and $g$ satisfy hypothesis (H1) and let $(\lambda_1, \lambda_2) \in \mathcal{C}$. Suppose

(a) $\liminf_{s \to -\infty} f(s)s^{-1} > \lambda_1$, $\liminf_{s \to -\infty} g(s)s^{-1} > \lambda_2$,

(b) $NF(s) - \alpha s f(s) \geq -\theta_1 + \theta_2 s |f(s)|$, $s > 0$, for some $\theta_1 \geq 0$, $\theta_2 > 0$,

$NG(s) - \beta s g(s) \geq -\theta_3 + \theta_4 s |g(s)|$, $s > 0$, for some $\theta_3 \geq 0$, $\theta_4 > 0$,

where $\alpha$ and $\beta$ are positive numbers which satisfy $\alpha + \beta = N - 2$. Then for all positive radially symmetric solutions $(u, v) \in \left( C^2(\overline{B}_R) \right)^2$ of Problem (I)

$$\|u\|_{\infty} \leq C_1 \quad \text{and} \quad \|v\|_{\infty} \leq C_2,$$

where the constants $C_1$ and $C_2$ do not depend on $(u, v)$.

**Remark.** As we shall show in Section 2, the conditions (a) and (b) of Theorem 1 imply that

$$\lim_{s \to -\infty} f(s)s^{-p^*} = 0 \quad \text{and} \quad \lim_{s \to -\infty} g(s)s^{-q^*} = 0,$$

where $p^*$ and $q^*$ satisfy

$$\frac{1}{1 + p^*} + \frac{1}{1 + q^*} = \frac{N - 2}{N}, \quad p^*, q^* \geq 1.$$

As an application of Theorem 1, we study the following problem:

$$\begin{cases}
-\Delta u = \mu v + v^q, & v > 0 \quad \text{in} \ B_R, \\
-\Delta v = \lambda u + u^p, & u > 0 \quad \text{in} \ B_R, \\
u = 0, & v = 0 \quad \text{on} \ \partial B_R,
\end{cases} \quad (1.13)$$

(II)

(1.14)

(1.15)

where $\lambda, \mu \in \mathbb{R}$, $(p, q) \in S^-(N)$ and $(u, v) \in C^2(\overline{B}_R) \times C^2(\overline{B}_R)$. For this problem, we have the following uniform bounds.

**Theorem 2.** Suppose that in Problem (II),

$$\lambda < \lambda_1, \quad \mu < \lambda_2 \quad \text{and} \quad (p, q) \in S^-(N),$$

where $(\lambda_1, \lambda_2)$ is a pair of eigenvalues on the curve $\mathcal{C}$. Then for all positive radially symmetric solutions of Problem (II),

$$\|u\|_{\infty} \leq C_1(\lambda, \mu), \quad \|v\|_{\infty} \leq C_2(\lambda, \mu),$$
where the constants $C_1$ and $C_2$ do not depend on $(u, v)$.

With some additional assumptions on the functions $f$ and $g$, we can use Theorem 1 to establish the existence of a solution of Problem (I). We shall require

\[(H2) \quad f(s) > 0, \quad g(s) > 0 \quad \text{if} \quad s > 0\]

as well as some conditions near the origin.

**Theorem 3.** (Existence) Let $f$ and $g$ satisfy hypotheses (H1) and (H2) and let $(\lambda_1, \lambda_2) \in C$. Suppose

\[(a) \liminf_{s \to -\infty} f(s)s^{-1} > \lambda_1 \quad \text{and} \quad \limsup_{s \to -0} f(s)s^{-1} < \lambda_1,\]
\[\liminf_{s \to -\infty} g(s)s^{-1} > \lambda_2 \quad \text{and} \quad \limsup_{s \to -0} g(s)s^{-1} < \lambda_2,\]

\[(b) \quad NF(s) - \alpha s f(s) \geq \theta_1 s f(s) \quad \text{and} \quad NG(s) - \beta g(s) \geq \theta_2 g(s) \quad \text{for} \quad s > 0, \quad \text{where} \quad \theta_1 \quad \text{and} \quad \theta_2 \quad \text{are non-negative numbers such that} \quad \theta_1 + \theta_2 > 0 \quad \text{and} \quad \alpha \quad \text{and} \quad \beta \quad \text{are positive numbers such that} \quad \alpha + \beta = N - 2.\]

Then Problem (I) has a (positive) radially symmetric solution $(u, v) \in (C^2(\overline{B}_R))^2$.

For all positive radially symmetric solutions of Problem (I),

\[\|u\|_\infty \leq C_1 \quad \text{and} \quad \|v\|_\infty \leq C_2,\]

where the constants $C_1$ and $C_2$ do not depend on $(u, v)$.

Thus, if $f$ or $g$ satisfies a certain subcritical growth condition and at most one of the functions $f$ or $g$ has critical growth, we obtain the existence of at least one solution.

When $f$ and $g$ are pure powers, we have an even stronger result.

**Theorem 4.** Let $f(s) = s^p$ and $g(s) = s^q$. Then if $(p, q) \in S^-(N)$, Problem (I) has a unique (positive) solution, which is necessarily radially symmetric. If $(p, q) \in S^+(N)$, Problem (I) has no (positive) solutions.

Theorem 3 allows us to obtain an existence theorem for the biharmonic equation with a nonlinear source term and certain Dirichlet boundary conditions. Consider the problem

\[
\begin{align*}
\Delta^2 u &= f(u), \quad u > 0 \quad \text{in} \ B_R \\
\Delta u &= 0 \quad \text{in} \ \partial B_R.
\end{align*}
\]

(11.1)

(11.12)

As a simple consequence, we can then state the following existence theorem for Problem (B):

**Theorem 5.** Let $f$ satisfy hypotheses (H1) and (H2) and suppose

\[(a) \liminf_{s \to -\infty} f(s)s^{-1} > \mu_1^2 \quad \text{and} \quad \limsup_{s \to -0} f(s)s^{-1} < \mu_1^2,\]

\[(b) \quad NF(s) - \frac{N-4}{2} s f(s) \geq \theta s f(s), \quad s > 0,\]

where $\mu_1$ is the principal eigenvalue of the Laplacian on the ball with radius $R$ and $\theta > 0$. Then Problem (B) has at least one (positive) radially symmetric solution $u \in C^4(\overline{B}_R)$.

We refer to [12] for an existence theorem similar to the one obtained here but only for the biharmonic equation with the Dirichlet boundary conditions

\[u = 0, \quad \frac{\partial u}{\partial n} = 0 \quad \text{on} \quad \partial B_R.

To complement the existence theorems given above, we state a nonexistence theorem from [14].
Theorem 6. (Non-Existence) Assume that one of the following conditions is satisfied:

(a) \( NF(s) - \alpha sf(s) \leq 0 \), \( s \geq 0 \), and \( NG(s) - \beta sg(s) \leq 0 \) for \( s > 0 \), where \( \alpha, \beta > 0 \) and \( \alpha + \beta \leq N - 2 \);

(b) \( f(s) > \lambda_1 s \) and \( g(s) > \lambda_2 s \) for \( s > 0 \).

Then Problem (I) has no (positive) solution \( (u, v) \in (C^2(\Omega))^2 \).

Remark. Problem (I) possesses a weak maximum-principle. This follows from the simple observation that with the boundary conditions \( u = 0 \) and \( v = 0 \) on \( \partial \Omega \) and the positivity of the left hand sides of the equations, we can apply the weak maximum principle for the Laplace-operator.

Remark. If Problem (II) is considered, we can observe an additional property concerning the bifurcation diagram. The branch of positive radially symmetric solutions bifurcates from the principal curve \( C \) of positive eigenvalues of the linear problem.

Remark. After this paper was completed, we learned from Ph. Clement, D. de Figueiredo and E. Mitidieri [5] that they had obtained results similar to those in this paper. They considered \( \Omega \), in Problem (I), to be a convex domain. The conditions on the functions \( f \) and \( g \) are, however, more restrictive than the ones in this paper.

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2. Preliminaries. If we consider Problem (I) for radially symmetric solutions, writing \( u = u(r) \) and \( v = v(r) \), where \( r = |x| \), the equations (1.1) and (1.2) become

\[
\begin{aligned}
-u'' - \frac{N-1}{r} u' &= g(v), \quad v(r) > 0, \quad r \in [0, R), \\
-v'' - \frac{N-1}{r} v' &= f(u), \quad u(r) > 0, \quad r \in [0, R).
\end{aligned}
\]

From [14], we now recall the main identity for this problem, formulated for radially symmetric solutions:

\[
\int_0^R \left( \{ NF(u) - \alpha uf(u) \} + \{ NG(v) - \beta vg(v) \} \right) - (N - 2 - \alpha - \beta) u' v' r^{N-1} \, dr = u'(R)v'(R)R^{N},
\]  

(2.1)
where $\alpha + \beta \geq N - 2$. Rather than studying the equations for $u(r)$ and $v(r)$ in their present form, we proceed as in [1] to transform them to a system of generalized Emden-Fowler equations [6, 7]. We set

$$ t = \left(\frac{N - 2}{r}\right)^{N-2} \quad \text{and} \quad y(t) = u(r), \quad z(t) = v(r). \quad (2.2) $$

Then $y$ and $z$ solve the problem

$$
\begin{align*}
- y'' &= t^{-k}g(z), \quad z(t) > 0, \quad t \in (T, \infty), \\
- z'' &= t^{-k}f(y), \quad y(t) > 0, \quad t \in (T, \infty), \quad (2.3) \\
y(T) &= 0, \quad z(T) = 0, \quad y'(\infty) = 0, \quad z'(\infty) = 0, \quad (2.5)
\end{align*}
$$

where

$$
\begin{align*}
k &= \frac{2N - 2}{N - 2} \quad \text{and} \quad T = \left(\frac{N - 2}{R}\right)^{N-2}.
\end{align*}
$$

Of course we have additional conditions for smooth solutions:

$$
\lim_{t \to \infty} y(t) = \gamma \quad \text{and} \quad \lim_{t \to \infty} z(t) = \sigma, \quad \gamma, \sigma > 0. \quad (2.6)
$$

It will be convenient to formulate Problem (Ia) as a pair of integral equations, using the Green function $G(t, s)$. Because of the simplicity of the differential operator in (2.3) and (2.4), this function is easily obtained. We find that

$$
G(t, s) = \begin{cases} 
    t - T & \text{if } T \leq t \leq s < \infty \\
    s - T & \text{if } T < s \leq t < \infty,
\end{cases}
$$

and Problem (Ia) is transformed into

$$
\begin{align*}
(y(t) &= \int_{T}^{s} G(t, s) s^{-k} g(z(s)) \, ds \quad (2.7) \\
z(t) &= \int_{T}^{s} G(t, s) s^{-k} f(y(s)) \, ds. \quad (2.8)
\end{align*}
$$

It is readily verified that $(y, z)$ is a solution of Problem (Ia) if and only if it is a solution of Problem (Ib). Thus we have:

**Lemma 2.1.** Problem (Ia) and Problem (Ib) are equivalent.

In terms of the new variables, the linear Problem (L) becomes

$$
\begin{align*}
- \phi'' &= \lambda_2 t^{-k} \psi, \quad \psi(t) > 0, \quad t \in (T, \infty), \quad (2.9) \\
- \psi'' &= \lambda_1 t^{-k} \phi, \quad \phi(t) > 0, \quad t \in (T, \infty), \quad (2.10) \\
\phi(T) &= 0, \quad \psi(T) = 0, \quad \phi'(\infty) = 0, \quad \psi'(\infty) = 0. \quad (2.11)
\end{align*}
$$

We shall normalize $\phi$ and $\psi$ by requiring that

$$
\lim_{t \to \infty} \phi(t) = 1 \quad \text{and} \quad \lim_{t \to \infty} \psi(t) = 1. \quad (2.12)
$$
For a more extensive discussion of Problems (L) and (La), we refer to [14].

Note that $\phi$ and $\psi$ are both concave and that $0 < \phi(t) < 1$ and $0 < \psi(t) < 1$ for all $t > T$. Hence,

$$
\phi'(T) = \lambda_1 \int_T^{\infty} s^{-k} \psi(s) \, ds < \lambda_1 \int_T^{\infty} s^{-k} \, ds < \frac{\lambda_1}{k-1} T^{1-k}
$$

and similarly for $\psi'(T)$. This allows us to obtain simple upper and lower bounds for $\phi$ and $\psi$.

**Lemma 2.2.** Let $(\lambda_1, \lambda_2) \in \mathcal{C}$ and let $\phi$ and $\psi$ be the corresponding normalized eigenfunctions. Then

$$
\phi'(t) > 0 \quad \text{and} \quad \psi'(t) > 0 \quad \text{for all} \quad t \geq T,
$$

$$
\phi''(t) < 0 \quad \text{and} \quad \psi''(t) < 0 \quad \text{for all} \quad t \geq T.
$$

Similarly, if $(y, z)$ is a solution of Problem (Ia), then $y$ and $z$ are concave functions; so, we have, in view of the conditions at infinity,

**Lemma 2.3.** Let $(y, z)$ be a solution of Problem (Ib), and consequently of Problem (Ia), where $f$ and $g$ satisfy (H1) and (H2). Then

$$
y'(t) > 0 \quad \text{and} \quad z'(t) > 0 \quad \text{for all} \quad t \geq T,
$$

$$
y''(t) < 0 \quad \text{and} \quad z''(t) < 0 \quad \text{for all} \quad t \geq T.
$$

We conclude this section with a lemma which shows that conditions (a) and (b) of Theorem 1 imply certain growth restrictions on $f$ and $g$.

**Lemma 2.4.** Suppose that $f$ and $g$ satisfy conditions (a) and (b) of Theorem 1. Then

$$
\lim_{s \to \infty} f(s) s^{-p^*} = 0 \quad \text{and} \quad \lim_{s \to \infty} g(s) s^{-q^*} = 0,
$$

where $p^*$ and $q^*$ satisfy

$$
\frac{1}{1 + p^*} + \frac{1}{1 + q^*} = \frac{N - 2}{N}, \quad p^*, q^* \geq 1.
$$

**Proof.** By condition (a), $f(s) > 0$ and $g(s) > 0$ for $s$ large enough, say $s > s_0$. Hence, for $s > s_0$, condition (b) implies

$$
NF(s) \geq -\theta_1 + \eta sf(s), \quad \eta = \alpha + \theta_2,
$$

and

$$
NG(s) \geq -\theta_3 + \zeta sg(s), \quad \zeta = \beta + \theta_4.
$$

Because $\theta_2, \theta_4 > 0$,

$$
\eta + \zeta > \alpha + \beta = N - 2 \quad \text{if} \quad N \geq 4
$$

and so either $\eta$ or $\zeta$ must be positive.
Suppose $\eta > 0$. Then the inequality for $F$ reads
\[
\frac{d}{ds}\left\{s^{-N/\eta}F(s)\right\} \leq \frac{\theta_1}{\eta} s^{-1-(N/\eta)}
\]
and so
\[
F(s) \leq K_1 s^{N/\eta},
\]
where $K_1$ is a positive constant. Remembering condition (a), we conclude that
\[
\frac{N}{\eta} \geq 2
\]
and so
\[
\eta \leq \frac{N}{2} \leq N - 2 \quad \text{if} \quad N \geq 4.
\]
This means that $\zeta > N - 2 - \eta \geq 0$ and so, as before,
\[
G(s) \leq K_2 s^{N/\zeta},
\]
where $K_2$ is another positive constant. Using condition (b) again we find that there exist constants $s_1 > s_0$ and $C > 0$ such that
\[
f(s) \leq Cs^{(N/\eta)-1} \quad \text{and} \quad g(s) \leq Cs^{(N/\zeta)-1} \quad \text{for} \quad s > s_1.
\]
Since $\eta + \zeta > N - 2$, we can choose $\varepsilon_1 > 0$ and $\varepsilon_2 > 0$ so that with
\[
p^* = \frac{N}{\eta - \varepsilon_1} - 1 \quad \text{and} \quad q^* = \frac{N}{\zeta - \varepsilon_2} - 1,
\]
we obtain
\[
\frac{1}{p^* + 1} + \frac{1}{q^* + 1} = \frac{1}{N} \{(\eta + \zeta) - (\varepsilon_1 + \varepsilon_1)\} = \frac{N - 2}{N}
\]
and, clearly,
\[
f(s)s^{-p^*} \to 0 \quad \text{and} \quad g(s)s^{-q^*} \to 0 \quad \text{as} \quad s \to \infty.
\]

3. A priori estimates for positive solutions. In this section, we shall establish a priori bounds for solutions $(u, v)$ of Problem (I) when the non-linearities $f$ and $g$ satisfy hypothesis (H1) and thus may change sign. We give the proofs in a series of lemmas.

Lemma 3.1. If condition (a) of Theorem 1 is fulfilled, then there exist positive constants $C_1$ to $C_4$ such that if $(y, z)$ is a solution of Problem (Ia), then
\[
\int_T^\infty t^{-k}|f(y)||\phi| dt \leq C_1, \quad \int_T^\infty t^{-k}|g(z)||\psi| dt \leq C_2,
\]
\[
\int_T^\infty t^{-k}y|\phi| dt \leq C_3, \quad \int_T^\infty t^{-k}z|\psi| dt \leq C_4.
\]
**Proof.** As before, we deduce from condition (a) that there exist constants \( k_1 > \lambda_1 \), \( k_2 > \lambda_2 \) and \( A, B > 0 \) such that

\[
f(y) \geq k_1 y - A, \quad g(z) \geq k_2 z - B \quad \text{for} \ s \geq 0.
\]

Hence,

\[
|f(s)| \leq C + f(s), \quad |g(s)| \leq C + g(s) \quad \text{for} \ s \geq 0,
\]

where \( C \) is a generic positive constant. Thus, if \((y, z)\) is a solution of Problem (Ia), then

\[
\int_T^\infty t^{-k} |g(z)| \psi \, dt \leq C + \int_T^\infty t^{-k} g(z) \psi \, dt = C - \int_T^\infty y'' \psi \, dt
\]

\[
= C - \int_T^\infty \lambda_1 \psi'' \, dt = C + \lambda_1 \int_T^\infty t^{-k} y \phi \, dt
\]

\[
\leq C + a \int_T^\infty t^{-k} f(y) \phi \, dt \leq C + a \int_T^\infty t^{-k} |f(y)| \phi \, dt, \quad a = \frac{\lambda_1}{k_1},
\]

\[
\int_T^\infty t^{-k} |f(y)| \phi \, dt \leq C + \int_T^\infty t^{-k} f(y) \phi \, dt = C - \int_T^\infty z'' \phi \, dt
\]

\[
= C - \int_T^\infty \lambda_2 \phi'' \, dt = C + \lambda_2 \int_T^\infty t^{-k} z \psi \, dt
\]

\[
\leq C + b \int_T^\infty t^{-k} g(z) \psi \, dt \leq C + b \int_T^\infty t^{-k} |g(z)| \psi \, dt, \quad b = \frac{\lambda_2}{k_2},
\]

where \( C \) is again a generic positive constant. Combining these estimates, we finally obtain

\[
\int_T^\infty t^{-k} |g(z)| \psi \, dt \leq C + ab \int_T^\infty t^{-k} |g(z)| \psi \, dt,
\]

\[
\int_T^\infty t^{-k} |f(y)| \phi \, dt \leq C + ab \int_T^\infty t^{-k} |f(y)| \phi \, dt.
\]

Because \( ab < 1 \), the assertion is proved.

**Lemma 3.2.** If condition (a) of Theorem 1 is satisfied, then there exist positive constants \( C_1, \ldots, C_4 \) and \( \varepsilon \) such that if \((y, z)\) is a solution of Problem (Ia), then

\[
y(t) \leq C_1 \quad \text{for} \ t \in [T, T + \varepsilon] \quad \text{and} \quad y'(T) \leq C_2, \tag{3.4}
\]

\[
z(t) \leq C_3 \quad \text{for} \ t \in [T, T + \varepsilon] \quad \text{and} \quad z'(T) \leq C_4. \tag{3.5}
\]

**Proof.** Observe that

\[
y(t) \leq y^+(t) = \int_T^\infty s^{-k} G(t, s)|g(z(s))| \, ds.
\]
On the other hand, by (3.3),

\[ y^+(t) \leq C \int_T^\infty s^{-k}G(t, s)\, ds + \int_T^\infty s^{-k}G(t, s)g(z(s))\, ds \leq C + y(t). \tag{3.6} \]

Because \( G(t, s) \) is increasing with respect to \( t \) if \( s > t \) and independent of \( t \) if \( s < t \), \( y^+ \) is increasing. Therefore, if \( T \leq t \leq T + \varepsilon \),

\[ y^+(t) \leq y^+(T + \varepsilon) \leq \frac{1}{\varepsilon} \int_{T+\varepsilon}^{T+2\varepsilon} y^+(s)\, ds. \]

Hence,

\[ y(t) \leq \frac{1}{\varepsilon} \int_{T+\varepsilon}^{T+2\varepsilon} \left( \frac{s}{T + 2\varepsilon} \right)^{-k} \frac{\phi(s)}{\phi(T + \varepsilon)} y^+(s)\, ds \]

\[ \leq C \int_T^\infty s^{-k}y^+(s)\phi(s)\, ds \leq C + C \int_T^\infty s^{-k}y(s)\phi(s)\, ds \leq C, \]

where we have used (3.6) and the upper bound from Lemma 3.1. This proves the boundary estimate for \( y \). The one for \( z \) is proved similarly.

To prove the boundary estimates for \( y' \) and \( z' \), we integrate the equations (2.3) and (2.4) for \( y \) and \( z \). Since solutions are bounded and the integrals therefore converge, we obtain

\[ y'(T) = \int_T^\infty s^{-k}g(z)\, ds \quad \text{and} \quad z'(T) = \int_T^\infty s^{-k}f(y)\, ds. \]

Using the bound we derived for \( z \) on \([T, T + \varepsilon]\), we obtain

\[ \int_T^{T+\varepsilon} s^{-k}|g(z)|\, ds \leq C_{1,\varepsilon}, \]

and because \( \psi(t) \) is bounded away from zero and increasing on \([T + \varepsilon, \infty)\), we can estimate the remaining part of the integral over \((T, \infty)\) by

\[ \int_{T+\varepsilon}^\infty s^{-k}|g(z)|\, ds \leq C_{2,\varepsilon} \int_{T+\varepsilon}^\infty s^{-k}|g(z)| \frac{\psi(s)}{\psi(T + \varepsilon)}\, ds \]

\[ \leq C_{3,\varepsilon} \int_{T+\varepsilon}^\infty s^{-k}\psi|g(z)|\, ds \leq C_{3,\varepsilon} \int_T^\infty s^{-k}\psi|g(z)|\, ds \leq C_{4,\varepsilon}. \]

Finally, this gives that

\[ y'(T) = \int_T^\infty s^{-k}g(z)\, ds \leq \int_T^\infty s^{-k}|g(z)|\, ds \]

\[ = \int_T^{T+\varepsilon} s^{-k}|g(z)|\, ds + \int_{T+\varepsilon}^\infty s^{-k}|g(z)|\, ds = C_{1,\varepsilon} + C_{4,\varepsilon} \leq C_2. \]

The bound for \( z'(T) \) is obtained in a similar way.
Lemma 3.3. If conditions (a) and (b) of Theorem 1 are satisfied, then there exist positive constants $C_1$–$C_4$ such that if $(y, z)$ is a solution of Problem (Ia), then

\begin{align*}
\int_T^\infty s^{-k} y(s)|f(y(s))|\,ds &\leq C_1, \quad \int_T^\infty s^{-k} z(s)|g(z(s))|\,ds \leq C_2, \\
\int_T^\infty s^{-k} |f(y(s))|\,ds &\leq C_3, \quad \int_T^\infty s^{-k} |g(z(s))|\,ds \leq C_4.
\end{align*}

(3.7) \hspace{1cm} (3.8)

Proof. In analogy with the Pohozaev functional $H$, which was used extensively in [1], we introduce the functional

\[ H(t) = ty'z' - \xi yz' - \zeta y'z + t^{1-k}\{F(y) + G(z)\}, \quad \xi + \zeta = 1, \quad (3.9) \]

where $(y, z)$ is understood to be a solution of Problem (Ia). Differentiation yields

\[ H'(t) = -(k-1)t^{-k}\left(\{F(y) - \frac{\xi}{k-1} y f(y)\} + \{G(z) - \frac{\zeta}{k-1} z g(z)\}\right). \]

Because $y(t) \to \gamma$ and $z(t) \to \sigma$ as $t \to \infty$, we deduce from equations (2.3) and (2.4) that

\[ y'(t), \quad z'(t) = O(t^{1-k}) \quad \text{as} \quad t \to \infty, \]

so that

\[ H(\infty) = 0 \]

and

\[ H(t) = (k-1) \int_t^\infty s^{-k}\{F(y) - \hat{\alpha} y f(y)\} + \{G(z) - \hat{\beta} z g(z)\}\,ds, \]

where

\[ \hat{\alpha} = \frac{\xi}{k-1}, \quad \hat{\beta} = \frac{\zeta}{k-1} \quad \text{and so} \quad \hat{\alpha} + \hat{\beta} = \frac{1}{k-1}. \]

In particular, we find for $t = T$

\[ \int_T^\infty \{F(y) - \hat{\alpha} y f(y)\} + \{G(z) - \hat{\beta} z g(z)\}\,dt = \frac{1}{k-1} Ty'(T)z'(T), \quad (3.10) \]

where $\hat{\alpha} + \hat{\beta} = \frac{1}{k-1}$.

Using (b) and the boundary estimates for $y'$ and $z'$ in (3.10), we obtain

\[ -\theta_1 \int_T^\infty s^{-k} \,ds - \theta_3 \int_T^\infty s^{-k} \,ds + \theta_2 \int_T^\infty s^{-k} y(s)|f(y(s))|\,ds \]

\[ + \theta_4 \int_T^\infty s^{-k} z(s)|g(z(s))|\,ds \leq C. \]

Rearranging the terms in this inequality yields

\[ \theta_2 \int_T^\infty s^{-k} y(s)|f(y(s))|\,ds + \theta_4 \int_T^\infty s^{-k} z(s)|g(z(s))|\,ds \leq C. \]
Because both terms on the left hand side are positive, (3.7) is proved.

For the proof of (3.8), we refer to the estimates for \( y' \) and \( z' \) proved in Lemma 3.2.

**Remark.** The identity (3.10) used in the proof of Lemma 3.3 also follows from [14] if we employ the variational structure of Problem (Ia). We have the following Lagrangian density:

\[
L = y'z' - t^{-k}F(y) - t^{-k}G(z).
\]

If we use the boundary conditions at infinity and at \( T \), we can apply Theorem 2.1 of [14] to yield (3.10).

**Lemma 3.4.** (A priori estimates) Suppose the conditions (a) and (b) of Theorem 1 are satisfied. Then there exist positive constants \( C_1 \) and \( C_2 \) such that for any (positive) solution \( (y, z) \) of Problem (Ia),

\[
\|y\|_\infty \leq C_1 \quad \text{and} \quad \|z\|_\infty \leq C_2.
\]

**Proof.** We shall start by estimating \( y(t) \), employing the integral equations (2.7) and (2.8):

\[
\|y\|_\infty \leq \lim_{t \to -\infty} y^+(t) = \lim_{t \to -\infty} \int_T^\infty s^{-k}G(t, s)|g(z(s))| \, ds
\]

\[
= \lim_{t \to -\infty} \int_T^t (s - T)s^{-k}|g(z(s))| \, ds + \lim_{t \to -\infty} \int_t^\infty (t - T)s^{-k}|g(z(s))| \, ds
\]

\[
= \int_T^\infty (s - T)s^{-k}|g(z(s))| \, ds. \tag{3.11}
\]

In a similar way, we find for \( z \),

\[
\|z\|_\infty \leq \int_T^\infty (s - T)s^{-k}|f(y(s))| \, ds. \tag{3.12}
\]

We now continue with (3.12), which can be simplified a little,

\[
\|z\|_\infty \leq \int_T^\infty (s - T)s^{-k}|f(y(s))| \, ds < \int_T^\infty s^{1-k}|f(y(s))| \, ds,
\]

and split the interval of integration;

\[
\|z\|_\infty < \int_T^\infty s^{1-k}|f(y(s))| \, ds
\]

\[
= \left( \int_T^t + \int_t^\infty \right) s^{1-k}|f(y(s))| \, ds = J_1(t) + J_2(t). \tag{3.13}
\]

The second integral is simply estimated by

\[
J_2(t) \leq f^+(\|y\|_\infty) \int_t^\infty s^{1-k} \, ds = \frac{1}{k - 2}t^{2-k}f^+(\|y\|_\infty), \tag{3.14}
\]
where \( f^+(s) = \max \{ f(y) : 0 \leq y \leq s \} \). To estimate the first integral, we use Hölder’s inequality to factor out the integrals which were bounded in Lemma 3.3. We set
\[
\delta = 1 - k + k \frac{p^*}{p^* + 1}.
\]
Then
\[
J_1(t) = \int_T^t s^\delta s^{-k} s^{\frac{p^*}{p^* + 1}} |f(y(s))| \, ds 
\leq \left( \int_T^t s^{\delta(p^* + 1)} \, ds \right)^{\frac{1}{p^* + 1}} \left( \int_T^t s^{-k} |f(y(s))|^{\frac{p^* + 1}{p^*}} \, ds \right)^{\frac{p^*}{p^* + 1}}.
\] (3.15)

By Lemma 2.4, there exists a positive constant \( M \) such that
\[
|f(y)| < M(1 + y)^{p^*} \quad \text{for all} \quad y \geq 0.
\]
Hence,
\[
|f(y)|^{\frac{p^* + 1}{p^*}} < M^{\frac{1}{p^*}} (1 + y) |f(y)| \quad \text{for all} \quad y \geq 0,
\]
so that the last integral in (3.15) can be estimated by
\[
M^{\frac{1}{p^*}} \left( \int_T^\infty s^{-k} |f(y(s))| \, ds + \int_T^\infty s^{-k} |f(y(s))| y(s) \, ds \right).
\] (3.16)

Because both integrals in (3.16) are uniformly bounded by Lemma 3.3, we conclude that
\[
J_1(t) = C \left( \int_T^t s^{p^* + 1 - k} \, ds \right)^{\frac{1}{p^* + 1}} \leq C t^{\frac{p^* - 2 - k}{p^* + 1}}.
\] (3.17)

Thus, putting (3.14) and (3.17) into (3.13), we obtain
\[
\| z \|_\infty < \frac{f^+(\| y \|_\infty)}{k - 2} t^{2 - k} + C z t^{\frac{p^* + 2 - k}{p^* + 1}} \quad \text{for any} \quad t \geq T.
\] (3.18)

In exactly the same manner, we find that
\[
\| y \|_\infty < \frac{g^+(\| z \|_\infty)}{k - 2} t^{2 - k} + C y t^{\frac{p^* + 2 - k}{p^* + 1}} \quad \text{for any} \quad t \geq T,
\] (3.19)

where \( g^+(s) = \max \{ g(z) : 0 \leq z \leq s \} \).

We shall combine (3.18) and (3.19) to obtain the required bounds. Observe that there exist positive constants \( K \) and \( C \) such that
\[
g^+(z) \leq \begin{cases} 
K & \text{for } 0 \leq z \leq 1 \\
C z^{q^*} & \text{for } z \geq 1.
\end{cases}
\]

Note that if \( g^+ \) is bounded, (3.19) immediately yields an a priori bound for \( \| y \|_\infty \).
However, if \( g^+(z) \) is not bounded, we obtain from (3.19) for \( z > 1 \),
\[
\| y \|_\infty < C_1 t^{2 - k} \| z \|_\infty^{q^*} + C_y t^{\frac{p^* + 2 - k}{p^* + 1}}.
\]
Together with (3.18) this yields
\[
\|y\|_\infty < C_1 2^{q^*-1} \left\{ C_2 t^{q^*(2-k)} (f^+(\|y\|_\infty))^{q^*} + C_2 t^{q^* \frac{q^*+2-k}{q^*+1}} \right\} + C_y t^{\frac{q^*+2-k}{q^*+1}},
\]
(3.20)
where we used the inequality \((a + b)^r \leq 2^{r-1}(a^r + b^r)\) if \(a, b \geq 0\) and \(r \geq 1\). Since
\[
\frac{1}{p^* + 1} + \frac{1}{q^* + 1} = \frac{1}{k-1},
\]
one finds after some rearrangement that the two exponents of \(t\) in (3.20) are equal;
\[
2 - k + q^* \frac{p^* + 2 - k}{p^* + 1} = q^* + 2 - k \quad \frac{q^* + 2 - k}{q^* + 1}.
\]
Hence, we can write (3.20) as
\[
\|y\|_\infty < C_3 t^{(q^*+1)(2-k)} (f^+(\|y\|_\infty))^{q^*} + C_5 t^{\frac{q^*+2-k}{q^*+1}}.
\]
(3.21)
This inequality holds for all \(t \geq T\). Thus, to get the optimal estimate for \(\|y\|_\infty\), we take the infimum of the right hand side. After some manipulations, we find that it is attained at the point
\[
t_0 = C_6 (f^+(\|y\|_\infty))^{\frac{q^*+1}{(k-3)(k-2)}}.
\]
and so we obtain upon substitution of \(t_0\) into (3.21),
\[
\|y\|_\infty < C_7 + C_8 (f^+(\|y\|_\infty))^{\frac{q^*+2-k}{q^*+k+2}} = C_7 + C_8 (f^+(\|y\|_\infty))^{1/p^*}
\]
(3.22)
because
\[
p^* = \frac{(k-2)q^* + 2k - 3}{q^* - k + 2}.
\]
Remembering condition (b) of Theorem 1 and Lemma 2.4, which states that
\[
f^+(y) = o(y^{p^*}) \quad \text{as} \quad y \to \infty,
\]
the desired bound on \(\|y\|_\infty\) follows. Substitution of the bound on \(\|y\|_\infty\) into (3.18) yields the bound on \(\|z\|_\infty\).

**Remark.** The a priori estimates obtained in this section using the conditions (a) and (b) of Theorem 1 can be slightly improved by allowing one of the numbers \(\theta_2\) or \(\theta_4\) to be zero. One of the functions \(f\) or \(g\) can then attain critical growth. In the above proof, one then has to proceed by choosing the function with subcritical growth in a manner similar to the way the function \(f\) was chosen above.

**Proof of Theorem 2.** We need to check that the functions
\[
\left\{ \begin{array}{l}
f(s) = \lambda s + s^{p^*}, \\
g(s) = \lambda s + s^{q},
\end{array} \right.
\]

where \((p,q) \in S^-(N)\), satisfy the conditions (a) and (b) of Theorem 1. The condition on \(\lambda\) and \(\mu\) of Theorem 2 is obviously needed in order to satisfy condition (a) of Theorem 1. If it is violated, Theorem 6 will yield non-existence of positive solutions (see Figure 2).

The condition on \(p\) and \(q\) is required in order to satisfy condition (b) of Theorem 1. It remains to check condition (b) for the functions \(f\) and \(g\) as given above. A straightforward calculation shows that

\[
NF(s) - \alpha sf(s) = \lambda \left(\frac{N}{2} - \alpha\right)s^2 + \left(\frac{N}{p + 1} - \alpha\right)s^{p+1}, \tag{3.24}
\]

\[
NG(s) - \beta sg(s) = \mu \left(\frac{N}{2} - \beta\right)s^2 + \left(\frac{N}{q + 1} - \beta\right)s^{q+1}, \tag{3.25}
\]

where \(\alpha\) and \(\beta\) satisfy

\[
\alpha + \beta = N - 2. \tag{3.26}
\]

Because \((p,q) \in S^-(N)\),

\[
\frac{1}{p + 1} + \frac{1}{q + 1} = \delta \frac{N - 2}{N}, \quad \delta > 1. \tag{3.27}
\]

We choose

\[
\alpha = -\frac{N}{q + 1} + \gamma(1 + \delta)(N - 2),
\]

\[
\beta = -\frac{N}{p + 1} + (1 - \gamma)(1 + \delta)(N - 2)
\]

and

\[
\gamma = \frac{\theta}{1 + \delta} \quad \text{with} \quad 1 < \theta < \delta.
\]

Then \(\alpha + \beta = N - 2\) and

\[
\frac{N}{p + 1} - \alpha = (N - 2)(\delta - \theta)^{\text{def}}C_1 > 0,
\]

\[
\frac{N}{q + 1} - \beta = (N - 2)(\theta - 1)^{\text{def}}C_2 > 0.
\]

Now set

\[
Q(s) = \{NF(s) - \alpha sf(s)\} - \xi sf(s), \quad \xi \in \mathbb{R}.
\]

Then by (3.24),

\[
Q(s) = \lambda \left(\frac{N}{2} - \alpha - \xi\right)s^2 + (C_1 - \xi)s^{p+1}.
\]

Hence, if we choose \(\xi < C_1\), there exists a constant \(M \geq 0\) such that

\[
Q(s) \geq -M \quad \text{for all} \quad s \geq 0. \tag{3.28}
\]

This gives

\[
NF(s) - \alpha sf(s) \geq -M + \xi sf(s). \tag{3.29}
\]
As before, there exists a constant $C \geq 0$ such that for all $s \geq 0$,

$$|sf(s)| \geq C + sf(s).$$

Consequently,

$$sf(s) \geq -C + s|f(s)|, \quad s \geq 0. \quad (3.30)$$

Combining (3.23) and (3.24) yields the estimate

$$NF(s) - \alpha sf(s) \geq -M - \xi C + \xi s|f(s)|, \quad (3.31)$$

where $\theta_1 = M + \xi C$ and $\theta_2 = \xi$. So condition (b) is verified for $f(s)$. The verification of (b) for $g(s)$ is similar, using the same choices for $\alpha$ and $\beta$. This of course proves the lemma.

4. Existence of positive solutions. The a priori estimates we established in the previous sections for positive solutions of Problem (I) and (Ia) can be used to establish existence theorems for Problems (I) and (Ia). For these existence theorems, additional requirements are needed on the functions $f$ and $g$ concerning their behaviour near $(u, v) = (0, 0)$. The precise conditions are formulated in Theorem 3. The proof of this theorem is based on a fixed point theorem which is originally due to Krasnosel’skii [9] and Benjamin [2]. Here we use a modified version due to de Figueiredo, Lions and Nussbaum [8].

**Proposition 4.1.** Let $C$ be a cone in a Banach space $X$ and $\Phi: C \rightarrow C$ a compact map such that $\Phi(0) = 0$. Assume that there exist numbers $0 < r < R$ such that

(i) $x \neq \lambda \Phi(x)$ for $0 \leq \lambda \leq 1$ and $\|x\| = r$,

(ii) there exists a compact map $F: \overline{B}_R \times [0, \infty) \rightarrow C$ such that

$$F(x, 0) = \Phi(x) \quad \text{if} \quad \|x\| = R,$$

$$F(x, \mu) \neq x \quad \text{if} \quad \|x\| = R \quad \text{and} \quad 0 \leq \mu < \infty,$$

$$F(x, \mu) \neq x \quad \text{if} \quad x \in \overline{B}_R \quad \text{and} \quad \mu \geq \mu_0.$$

Then if

$$U = \{x \in C: r < \|x\| < R\} \quad \text{and} \quad B_\rho = \{x \in C: \|x\| < \rho\},$$

one has

$$i_C(\Phi, B_R) = 0, \quad i_C(\Phi, B_r) = 1, \quad i_C(\Phi, U) = -1,$$

where $i_C(\Phi, \Omega)$ denotes the index of $\Phi$ with respect to $\Omega$. In particular, $\Phi$ has a fixed point in $U$.

Condition (i) is satisfied if there exists a bounded linear operator $A: X \rightarrow X$ such that $A(C) \subset C$, $A$ has spectral radius strictly less than 1 and $\Phi(x) \leq A(x)$ for $x \in C$ and $\|x\| = r$.

For the proof of this proposition we refer to [8].

**Proof of Theorem 3.** We shall now apply Proposition 4.1 to Problem (Ia). Let $C^*([T, \infty))$ denote the space of continuous bounded functions defined on $[T, \infty)$, endowed with the norm

$$\|u\| = \sup\{|u(t)|: T \leq t < \infty\}.$$
We introduce the Banach space $X = (C^0([T, \infty)))^2$ and we define the norm of $x = (y, z)$ by

$$\|x\|_X = \max\{\|y\|, \|z\|\}.$$ 

We shall usually omit the subscript $X$. The closed cone of positive functions $C$ is now defined by

$$C = \{x \in X: x(t) \geq 0 \quad \text{for all } t \geq T\}, \quad (4.1)$$

where $x = (y, z) \geq 0$ means that $y \geq 0$ and $z \geq 0$. Rewriting Problem (Ib), we define the compact map $\Phi: X \to X$ by

$$\Phi(x)(t) = \int_T^\infty s^{-k}G(t, s)h(x(s)) \, ds, \quad h(x) = (g(z), f(y)), \quad (4.2)$$

so that any non-zero solution of the fixed point equation

$$x(t) = \Phi(x(t)) \quad \text{and} \quad x \in C$$

is a solution of the Problem (Ia).

(1) Verification of condition (i). Fix a constant $r > 0$ small enough and take an $x \in C$ with $\|x\| = r$. Then we have by condition (a) of Theorem 3 that $f(y(t)) \leq q_1 \lambda_1 y(t)$ and $g(z(t)) \leq q_2 \lambda_2 z(t)$, where $q_1, q_2 < 1$. Thus, because

$$\lambda_2 \int_T^\infty t^{-k}z \psi \, dt = - \int_T^\infty z\phi'' \, dt = - \int_T^\infty z'' \phi \, dt = \int_T^\infty t^{-k}f(y)\phi \, dt,$$

we have

$$\lambda_2 \int_T^\infty t^{-k}z \psi \, dt \leq q_1 \lambda_1 \int_T^\infty t^{-k}y \phi \, dt.$$ 

Similarly, because

$$\lambda_1 \int_T^\infty t^{-k}y \phi \, dt = - \int_T^\infty y\psi'' \, dt = - \int_T^\infty y'' \psi \, dt = \int_T^\infty t^{-k}g(z)\psi \, dt,$$

we have

$$\lambda_1 \int_T^\infty t^{-k}y \phi \, dt \leq q_2 \lambda_2 \int_T^\infty t^{-k}z \psi \, dt.$$ 

Combining these two inequalities gives that

$$\lambda_2 \int_T^\infty t^{-k}z \psi \, dt \leq q_1 \lambda_1 \int_T^\infty t^{-k}y \phi \, dt \leq q_1 q_2 \lambda_2 \int_T^\infty t^{-k}z \psi \, dt, \quad (4.3)$$

$$\lambda_1 \int_T^\infty t^{-k}y \phi \, dt \leq q_2 \lambda_2 \int_T^\infty t^{-k}z \psi \, dt \leq q_1 q_2 \lambda_1 \int_T^\infty t^{-k}y \phi \, dt. \quad (4.4)$$

Because $q_1 q_2 < 1$ and the integrals are nonzero, (4.3) and (4.4) yield a contradiction. Moreover, if $h(x)$ in (4.2) is replaced by $\lambda h(x)$, for $\lambda \in [0, 1]$, a contradiction also follows and therefore

$$x(t) \neq \lambda \Phi(x(t)) \quad \text{with} \quad \lambda \in [0, 1], \quad \|x\| = r, \quad x \in C.$$
This conclusion can also be reached if we use the last remark of Proposition 4.1. 

(2) Verification of condition (ii). Define the compact mapping $F: \mathcal{C} \times [0, \infty) \to \mathcal{C}$ by

$$F(x, \mu)(t) = \int_T^\infty s^{-k}G(t, s)h(x(s) + \mu) \, ds.$$  (4.5)

Clearly $F(x, 0) = \Phi(x)$. To prove the second condition on $F$, we observe that by condition (a) of Theorem 3, there exist constants $k_1 > \lambda_1$, $k_2 > \lambda_2$ and $\mu_0 > 0$ such that

$$f(y + \mu) \geq k_1 y \text{ and } g(z + \mu) \geq k_2 z \text{ if } \mu \geq \mu_0$$

for all $(y, z) \geq (0, 0)$. If we carry out the same steps as in (1), we obtain

$$\lambda_2 \int_T^\infty t^{-k}z\psi \, dt \geq k_2 \int_T^\infty t^{-k}z\psi \, dt,$$  (4.6)

$$\lambda_1 \int_T^\infty t^{-k}y\phi \, dt \geq k_1 \int_T^\infty t^{-k}y\phi \, dt.$$  (4.7)

The inequalities (4.6) and (4.7) yield a contradiction because the integrals are both nonzero and $k_1 > \lambda_1$, $k_2 > \lambda_2$. Thus, there exists a constant $\mu_0 > 0$ such that

$$x(t) \neq F(x, \mu)(t) \text{ for all } x \in \mathcal{C} \text{ and } \mu \geq \mu_0.$$  (4.8)

We proceed using the a priori estimates of Theorem 1. We choose the family of nonlinearities $(f(y + \mu), g(z + \mu))$ for $\mu \in [0, \mu_0]$, observing that the a priori estimates of Theorem 1 can be chosen independently of $\mu$ for these nonlinearities. Thus, choosing $R > r$ sufficiently large, we can ensure that

$$x(t) \neq F(x, \mu)(t), \quad \forall \mu \in [0, \mu_0], \quad x \in \mathcal{C}, \quad \|x\| = R.$$  (4.9)

We conclude that (4.8) and (4.9) prove the second condition of (ii). The third condition of (ii) is proved by (4.8). This completes the verification of condition (ii).

Thus, we may apply Proposition 4.1 to conclude the existence of at least one nontrivial nonnegative solution of Problem (Ib). It can readily be seen to be strictly positive on $[T, \infty)$. The latter also insures the existence of at least one positive solution of the Problems (I) and (Ia).

Remark. The application of Proposition 4.1 heavily depends on the compactness of the operators. It is easily verified that the maps $\Phi$ and $F$ are indeed compact maps. To insure that $\Phi$ and $F$ are cone-preserving, we regretfully need the positivity condition on $f$ and $g$. For the a priori, estimates this is irrelevant.

5. Proofs of Theorems 4, 5 and 6. In this section, we shall briefly discuss the proofs of the Theorems 4, 5 and 6. The basic ingredients of the proofs were given in Section 3. With some additional tools, Theorems 4, 5 and 6 follow.

Proof of Theorem 4. We shall prove Theorem 4 by using the Emden-Fowler system given by Problem (Ia). When $f$ and $g$ are pure powers it becomes
\[
\begin{aligned}
- y'' &= t^{-k} z^q, \quad z(t) > 0, \quad t \in (T, \infty), \\
- z'' &= t^{-k} y^p, \quad y(t) > 0, \quad t \in (T, \infty), \\
y(T) = 0, \quad z(T) = 0, \quad y'(\infty) = 0, \quad z'(\infty) = 0.
\end{aligned}
\]

From previous arguments we already know about the shooting heights \( \gamma \) and \( \sigma \):

\[\|y\|_\infty = \gamma \leq C_1 \quad \text{and} \quad \|z\|_\infty = \sigma(\gamma) \leq C_2.\]

In the proof we shall use the following scale invariance of the equation (5.1) and (5.2). Set

\[s = \frac{t}{\ell}, \quad y_\ell(s) = \ell^{-a} y(t), \quad z_\ell(s) = \ell^{-b} z(t), \quad \ell > 0.\]

Then, if \((y, z)\) is a solution of (5.1) and (5.2), so is \((y_\ell, z_\ell)\) if

\[a = (k - 2) \frac{q + 1}{qp - 1}, \quad b = (k - 2) \frac{p + 1}{qp - 1}.\]

Note that

\[\frac{y_\ell(s)^{p+1}}{z_\ell(s)^{q+1}} = \frac{y(t)^{p+1}}{z(t)^{q+1}}.\]

In order to prove the uniqueness of a radial solution, we assume, to the contrary, that there exists a second solution \((\hat{y}, \hat{z})\) with limits \(\hat{\gamma}\) and \(\hat{\sigma}\). Choose \(\ell\) in (5.4) so that

\[\ell^{-a} \hat{\gamma} = \gamma \quad \text{or} \quad \ell = (\hat{\gamma}/\gamma)^{1/a}.\]

Then

\[\hat{y}_\ell(\infty) = \gamma \quad \text{and} \quad \hat{z}_\ell(\infty) = (\gamma/\hat{\gamma})^{b/a} \hat{\sigma}.\]

We assert that \(\hat{z}_\ell(\infty) = \sigma\). This follows from the next monotonicity lemma.

**Lemma 5.1.** Let \((y_1, z_1)\) and \((y_2, z_2)\) be solutions of (5.1) and (5.2) such that

\[y_i(\infty) = \gamma_i \quad \text{and} \quad z_i(\infty) = \sigma_i, \quad i = 1, 2,\]

where \(\gamma_i, \sigma_i \in \mathbb{R}^+\). Suppose

\[\gamma_1 \geq \gamma_2 \quad \text{and} \quad \sigma_1 < \sigma_2.\]

Then

\[y_1(t) > y_2(t), \quad z_1(t) < z_2(t) \quad \text{for} \quad t \geq \tau,\]

where \(\tau = \inf\{t > 0: y_2(t) > 0 \quad \text{and} \quad z_1(t) > 0\}\).

**Proof.** Since \(\sigma_1 < \sigma_2\), it follows that \(z_1(t) < z_2(t)\) for \(t\) large enough. Let

\[t_0 = \inf\{t > 0: z_1 < z_2 \text{ on } (t, \infty)\}\]

and suppose \(t_0 \geq \tau\). Then

\[z_1(t_0) = z_2(t_0) \quad \text{and} \quad z'_1(t_0) \leq z'_2(t_0). \quad (5.5)\]
However, if we integrate (5.1) over \((t_0, \infty)\), then, when \(t > t_0\),

\[ y'_1(t) = \int_t^\infty s^{-k} z_1^q(s) \, ds < \int_t^\infty s^{-k} z_2^q(s) \, ds = y'_2(t), \]

and so

\[ y_1(t) > \gamma_1 - \gamma_2 + y_2(t) \geq y_2(t) \quad \text{for} \quad t \geq t_0. \]

This yields in turn, when we integrate (5.2) over \((t_0, \infty)\),

\[ z'_1(t_0) = \int_{t_0}^\infty s^{-k} y_1^q(s) \, ds > \int_{t_0}^\infty s^{-k} y_2^q(s) \, ds = z'_2(t_0), \]

which contradicts (5.5).

We return to the proof of uniqueness. Writing \(\hat{\gamma}(\infty) = \hat{\sigma}_\ell\), we assert that \(\hat{\sigma}_\ell = \sigma\). Suppose to the contrary that

\[ \hat{\sigma}_\ell > \sigma. \]

Then, because \(\hat{\gamma}_\ell = \hat{y}_\ell(\infty) = \gamma\), it follows from Lemma 5.1 that

\[ y(t) > \hat{y}_\ell(t) \quad \text{and} \quad z(t) < \hat{z}_\ell(t) \]

as long as these functions are nonnegative. The first inequality implies that \(T > T/\ell\) and the second one that \(T < T/\ell\). This is impossible and so we must conclude that \(\hat{\sigma}_\ell \leq \sigma\). However, proceeding similarly, we can equally well show that \(\hat{\sigma}_\ell \geq \sigma\), so that we may conclude that \(\hat{\sigma}_\ell = \sigma\), as asserted.

Thus, by the uniqueness of the solutions of (2.3) and (2.4) with prescribed limits at infinity [11], \(\hat{y}_\ell = y\) and \(\hat{z}_\ell = z\) and hence \(T/\ell = T\); so, we must have \(\ell = 1\). This means that \(\hat{\gamma} = \gamma\) and by the argument based on Lemma 5.1, \(\hat{\sigma} = \sigma\). Therefore, \(\hat{y} = y\) and \(\hat{z} = z\), a contradiction. This completes the uniqueness proof.

The non-existence part of Theorem 4 can be found in [14]. Also, see [10] for a similar non-existence theorem.

**Proof of Theorem 5.** Theorem 5 is simply proved by substituting

\[ g(v) = v \]

in the equations (1.1) and (1.2). Problem (I) with this substitution is then exactly Problem (B). We notice that for this substitution, the exponent \(q^* = 1\) for \(g(v)\) is a critical exponent corresponding to \(p^* = \frac{N+4}{N-4}\) for \(f(u)\). This critical pair is a limit point in Figure 1. If we now take \(\beta = \frac{N}{2}\), and consequently \(\alpha = \frac{N-4}{2}\), we can apply Theorem 3 to conclude the proof.

**Proof of Theorem 6.** Part (a) of Theorem 6 follows directly from [14]. This is based on variational identities. For similar results, also see [10]. Part (b) can be proved by integrating the equations (1.1) and (1.2) using arguments such as those used in the proof of Lemma 3.1. Denoting the normalized solution of Problem (L) by \((\phi, \psi)\) and using the properties (b), we obtain

\[ \int_{B_R} g(v) \psi \, dx = - \int_{B_R} \psi \Delta u \, dx = - \int_{B_R} u \Delta \psi \, dx = \lambda_1 \int_{B_R} u \phi \, dx < \int_{B_R} f(u) \phi \, dx \]
and
\[ \int_{B_R} f(u)\phi\,dx = -\int_{B_R} \phi \Delta v\,dx = -\int_{B_R} v \Delta \phi\,dx = \lambda_2 \int_{B_R} v \psi\,dx < \int_{B_R} g(v)\psi\,dx. \]

Combining these two inequalities, we find that
\[ \int_{B_R} g(v)\psi\,dx < \int_{B_R} g(v)\psi\,dx \quad \text{and} \quad \int_{B_R} f(u)\phi\,dx < \int_{B_R} f(u)\phi\,dx, \]

which is of course a contradiction. This gives the required non-existence result.

REFERENCES