

Interpolation spaces for ∂_T -systems and applications to critical point theory

by

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1. Introduction

The main motivation of this paper is to introduce a canonical functional analytic framework for a class of variational equations which can be seen as generalizations of Hamiltonian systems. We consider the following system of semilinear equations

$$(I) \begin{cases} T^*v = \mathcal{H}_u(u, v), \\ Tu = \mathcal{H}_v(u, v), \end{cases} \quad (1.1)$$

$$(1.2)$$

where $T : D(T) \subset H_1 \rightarrow H_2$ is a closed and densely defined operator and H_1, H_2 are real Hilbert spaces. The functional $\mathcal{H}$ is called the Hamiltonian and is a C^1 -function on an appropriate Banach space X . Observe that the equations (1.1) and (1.2) can formally be viewed as Euler equations of the functional

$$\mathcal{L}(z) = (Tu, v)_{H_2} - \mathcal{H}(z) = (u, T^*v)_{H_1} - \mathcal{H}(z), \quad (1.3)$$

where $z = (u, v)^t$ and $(\cdot, \cdot)_{H_i}$ are the innerproducts in H_i , $i = 1, 2$. We want to find Hilbert spaces on which $\mathcal{L}$ is a C^1 -function. It will appear that the quadratic part of $\mathcal{L}$ is well-defined on the 1-parameter family of Hilbert spaces $E_\alpha = [E_0, E_1]_\alpha$, $\alpha \in [0, 1]$, where $E_0 = H_1 \times D(T^*)$ and $E_1 = D(T) \times H_2$. The spaces $[E_0, E_1]_\alpha$, $\alpha \in [0, 1]$, are the real ($p = 2$) or complex interpolations between E_0 and E_1 . The choice of a particular space E_α for some $\alpha \in [0, 1]$ will depend on the Hamiltonian $\mathcal{H}$, which, in general, is only well-defined for a subset of parameters α .

In the Sections 2, 3 and 4 we study the quadratic form Q , which can be written as

$$Q(z) = \frac{1}{2}(\partial_T z, z)_H = (Tu, v)_{H_2} = (u, T^*v)_{H_1}, \quad (1.4)$$

with

$$\partial_T = \begin{pmatrix} 0 & T^* \\ T & 0 \end{pmatrix} : D(T) \times D(T^*) \subset H \rightarrow H$$

and we examine the extensions of Q to the 1-parameter family of Hilbert spaces E_α , $\alpha \in [0, 1]$.

In the Sections 5 and 6 we shall give an abstract existence theorem for critical points of $\mathcal{L}$ and we discuss some examples of ∂_T -systems. The example about reaction-diffusion systems that we give in Section 6 extends results of [FF, HV1] in the case that $\Omega = B_R$.

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Let us now give some examples of physical systems which can be described by Problem (I). Consider the equations of electrostatics:

$$(II) \begin{cases} -\operatorname{div} D = \rho, & \text{in } \Omega, \\ \epsilon E = D, & \text{in } \Omega, \\ \nabla \phi = E, & \text{in } \Omega, \\ \phi = 0, & \text{on } \partial\Omega, \end{cases} \quad \begin{aligned} (1.5) \\ (1.6) \\ (1.7) \\ (1.8) \end{aligned}$$

where $\Omega \subset \mathbf{R}^3$ is a bounded domain, ϕ the electric potential, E the electric field intensity, D the displacement flux density, ϵ the diëlectric constant and ρ the charge density. Setting $E = \epsilon^{-1}D$ and

$$\mathcal{H}(\phi, D) = \frac{\epsilon^{-1}}{2} \int_{\Omega} |D|^2 dx + \frac{1}{2} \int_{\Omega} \rho \phi dx, \quad (1.9)$$

Problem (II) is of the same type as Problem (I), by taking $T = \nabla$, $u = \phi$, $v = D$ and the Hamiltonian as in (1.9). The precise formulation of this problem is carried out in Section 6 in the example about multiparameter Hamiltonian systems. In that particular example more general Hamiltonians are considered and an existence and non-existence theorem is given. In magnetostatics a system similar to Problem (II) occurs (see e.g. [A1,A2,OR]). In that case $T = \operatorname{curl}$ in $\mathbf{R}^3$.

A second example of a ∂_T -system is given by the Maxwell equations. For simplicity we set the diëlectric constant ϵ and magnetic permeability μ equal to one. We then use the variables: E the electric field intensity, B the magnetic fields intensity, ϕ the electric potential, A the magnetic potential, ρ the charge density and j the current density. The Maxwell equations are (skipping the boundary data for simplicity)

$$(III) \begin{cases} -\frac{\partial E}{\partial t} + \operatorname{curl} B = j, & \text{in } \Omega \times \mathbf{R}, \\ -\operatorname{div} E = -\rho, & \text{in } \Omega \times \mathbf{R}, \\ \frac{\partial A}{\partial t} + \nabla \phi = -E, & \text{in } \Omega \times \mathbf{R}, \\ \operatorname{curl} A = B, & \text{in } \Omega \times \mathbf{R}. \end{cases} \quad \begin{aligned} (1.10) \\ (1.11) \\ (1.12) \\ (1.13) \end{aligned}$$

By taking

$$T = \begin{pmatrix} \frac{\partial}{\partial t} & \nabla \\ \operatorname{curl} & 0 \end{pmatrix}, \quad u = \begin{pmatrix} A \\ \phi \end{pmatrix}, \quad v = \begin{pmatrix} E \\ B \end{pmatrix}$$

and

$$\mathcal{H}(A, \phi, E, B) = \int_{\mathbf{R}} \int_{\Omega} \left\{ j \cdot A - \rho \phi + \frac{1}{2}(|B|^2 - \frac{1}{2}|E|^2) \right\} dx dt,$$

Problem (III) reduces to Problem (I). We do not discuss this example further in the paper. It is important to observe that in the steady state case, i.e. when the equations are time independent, the operator T has again a special form, it has additional symmetries. In Section 7 we shall discuss this extensively and explain the additional structure these systems have.

2. $\mathcal{J}$ -Selfadjoint operators and the operator ∂_T

Let H_1 and H_2 be real Hilbert spaces equipped with the inner products $(\cdot, \cdot)_{H_1}$ and $(\cdot, \cdot)_{H_2}$ respectively and let T be a linear operator (in general unbounded) from H_1 to H_2 . Let $D(T) \subset H_1$ denote the domain of T , $N(T) \subset H_1$ the null space of T and $R(T) \subset H_2$ the range of T . If $D(T)$ is dense in H_1 , then the adjoint T^* is a closed operator from H_2 to H_1 , which is densely defined if and only if T is closable; moreover if T is closed and densely defined, then $T^{**} = T$ [see e.g. Kato [K] Thm. 5.28, 5.29, p. 168]. The operator T satisfies the assumption (H_T) if

$$(H_T) \begin{cases} T : D(T) \subset H_1 \longrightarrow H_2 \text{ is closed, and} \\ \overline{D(T)} = H_1. \end{cases}$$

Given an operator T satisfying (H_T) , we are interested in the bilinear form

$$\mathbf{b} : D(T) \times D(T^*) \longrightarrow \mathbf{R},$$

defined by

$$\mathbf{b}(u, v) = (Tu, v)_{H_2} = (u, T^*v)_{H_1},$$

for $u \in D(T)$ and $v \in D(T^*)$. We introduce several inner products on $D(T) \times D(T^*)$ for which the bilinear form $\mathbf{b}$ is continuous. It will be useful for later applications to critical point theory (Section 5 and 6) to consider the bilinear form $\mathbf{b}$ as a *quadratic form* on $D(T) \times D(T^*)$ associated with a selfadjoint operator ∂_T , acting on the Hilbert space $H = H_1 \times H_2$.

Definition 2.1. Let $H = H_1 \times H_2$ (Hilbert space) be equipped with the inner product $(\cdot, \cdot)_H = (\cdot, \cdot)_{H_1} + (\cdot, \cdot)_{H_2}$ and let T be a densely defined operator from H_1 to H_2 . Then we define

$$\partial_T = \begin{pmatrix} 0 & T^* \\ T & 0 \end{pmatrix} : D(\partial_T) \subset H \longrightarrow H, \quad (2.1)$$

with $D(\partial_T) = D(T) \times D(T^*)$.

We denote by $z = (u, v)^t$ the elements of the cartesian product H . For the sake of completeness we recall

Proposition 2.2. Let T be a closed and densely defined linear operator from H_1 into H_2 and S be a closed and densely defined linear operator from H_2 into H_1 . Further let A be a linear operator from H into H given by

$$A = \begin{pmatrix} 0 & S \\ T & 0 \end{pmatrix},$$

with domain $D(A) = D(T) \times D(S)$. Then the operator A is closed and densely defined, moreover A is selfadjoint if and only if $S = T^*$. In particular if $S = T^*$ then $A = \partial_T$ and

$$(Tu, v)_{H_2} = (u, T^*v)_{H_1} = \frac{1}{2}(\partial_T z, z)_H, \quad (2.2)$$

for all $z \in D(\partial_T)$.

Remark 2.3. In the same spirit as before one can introduce the operator $\delta_T : H \rightarrow H$ as follows: $D(\delta_T) = D(T) \times D(T^*)$ and

$$\delta_T = \begin{pmatrix} 0 & T^* \\ -T & 0 \end{pmatrix} : D(\delta_T) \subset H \longrightarrow H. \quad (2.3)$$

This operator δ_T is skew-adjoint, i.e. $\delta_T^* = -\delta_T$, if T satisfies (H_T) .

Clearly

$$N(\partial_T) = N(T) \times N(T^*), \quad R(\partial_T) = R(T) \times R(T^*)$$

and $R(\partial_T)$ is closed if and only if $R(T)$ is closed, because the latter is closed if and only if $R(T^*)$ is closed [see Kato [K], Thm. 5.13, p. 234].

It is interesting to characterize selfadjoint operators A in Hilbert space H , which are of the form $A = \partial_T$ for some operator T satisfying (H_T) , with $H = H_1 \times H_2$. For this purpose let us introduce the operator

$$\mathcal{J} = \begin{pmatrix} I_{H_1} & 0 \\ 0 & -I_{H_2} \end{pmatrix}. \quad (2.4)$$

Clearly the operator $\mathcal{J}$ satisfies

$$\mathcal{J}^* = \mathcal{J}^{-1} = \mathcal{J}, \quad \mathcal{J}^2 = I, \quad (2.5)$$

and the operator ∂_T anticommutes with $\mathcal{J}$. We have

$$\mathcal{J}\partial_T = \begin{pmatrix} 0 & T^* \\ -T & 0 \end{pmatrix} = \delta_T, \quad (2.6)$$

$$\partial_T\mathcal{J} = \begin{pmatrix} 0 & -T^* \\ T & 0 \end{pmatrix} = -\delta_T, \quad (2.7)$$

$$\delta_T^* = (\mathcal{J}\partial_T)^* = \partial_T\mathcal{J} = -\delta_T. \quad (2.8)$$

We shall prove now that these properties of the operator ∂_T , when they are imposed on a selfadjoint operator A in H , characterize the operators A we seek.

Let $(H, (\cdot, \cdot)_H)$ be a real Hilbert space. An operator $\mathcal{J} : H \rightarrow H$, satisfying (2.5) will be called a *conjugation* on H . We associate with $\mathcal{J}$ the orthogonal projections

$$P_1 = \frac{1}{2}(I + \mathcal{J}), \quad (2.9)$$

$$P_2 = \frac{1}{2}(I - \mathcal{J}), \quad (2.10)$$

and denote by H_1 (resp. H_2) the range of P_1 (resp. P_2). We identify H with $H_1 \times H_2$ by setting $(u, v)^t = u + v$, where $u \in H_1$ and $v \in H_2$. Note that if $\mathcal{J} = I$, then $H = H_1$, and if $\mathcal{J} = -I$, then $H = H_2$.

Conversely we can associate a conjugation $\mathcal{J}$ with an orthogonal projection P by setting

$$\mathcal{J} = 2P - I. \quad (2.11)$$

Suppose now that we have an operator A which is selfadjoint in a Hilbert space H equipped with a conjugation $\mathcal{J}$ such that

$$\mathcal{J}A = -A\mathcal{J} \quad (2.12)$$

(in particular $D(A) = D(\mathcal{J}A) = D(A\mathcal{J})$). Set $H_i = R(P_i)$, $i = 1, 2$, where P_i are defined in (2.9) and (2.10). We want to define an operator T from H_1 to H_2 satisfying (H_T) and such that $A = \partial_T$. By definition $D(A) = D(\mathcal{J}A) = D(A\mathcal{J}) = D(\mathcal{J}A\mathcal{J})$ and

$$\mathcal{J}A\mathcal{J} = -A. \quad (2.13)$$

Set $D(T) = H_1 \cap D(A)$ and $D(S) = H_2 \cap D(A)$. Define

$$\begin{aligned} Tu &= Au \quad \text{for } u \in D(T), \\ Sv &= Av \quad \text{for } v \in D(S). \end{aligned}$$

We have now $P_1Au = P_1AP_1u = \frac{1}{4}(I + \mathcal{J})A(I + \mathcal{J})u = \frac{1}{4}(A + \mathcal{J}A + A\mathcal{J} + \mathcal{J}A\mathcal{J})u = 0$, for $u \in D(T)$; hence $R(T) \subseteq H_2$. Similarly one verifies that $R(S) \subseteq H_1$. Clearly the operators $T : D(T) \subset H_1 \rightarrow H_2$ and $S : D(S) \subset H_2 \rightarrow H_1$ are closed. Moreover for $z = (u, v)^t$, $u \in D(T)$; $v \in D(S)$ we have

$$Az = (Sv, Tu)^t = \begin{pmatrix} 0 & S \\ T & 0 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}.$$

Finally, since A is selfadjoint it follows from Proposition 2.2 that $S = T^*$. Summarizing we have proved the following proposition;

Proposition 2.4. *Let A be a selfadjoint operator in a Hilbert space H . Then the operator A is of the form $A = \partial_T$ defined in (2.1), for some operator T from H_1 to H_2 satisfying (H_T) , if and only if there exists a conjugation $\mathcal{J}$ on H which satisfies*

$$\mathcal{J}A = -A\mathcal{J}, \quad (2.12)$$

If this is the case we have

$$H_1 = R(I + \mathcal{J}), \quad H_2 = R(I - \mathcal{J}),$$

and $D(T) = H_1 \cap D(A)$, $D(T^) = H_2 \cap D(A)$; $Tu = Au$, $T^*v = Av$, for $u \in D(T)$, $v \in D(T^*)$.*

Remark 2.5. Observe that if the conjugation $\mathcal{J}$ is plus or minus the identity and A satisfies (2.12), then $A = 0$.

Definition 2.6. A selfadjoint operator A on a Hilbert space H with a conjugation $\mathcal{J}$ is called $\mathcal{J}$ -selfadjoint if (2.12) holds.

Remark 2.7. *The relation with supersymmetry.* Following Witten [W] (see also [CFKS]) we observe that by the above considerations, $\mathcal{J}$ -selfadjoint operators in complex Hilbert space H posses a simple supersymmetric structure. We can write the Hilbert space H as

$$H = H_1 \oplus H_2, \quad (2.14)$$

where we identified $H_1 \sim H_1 \times \{0\}$, $H_2 \sim \{0\} \times H_2$. Introduce the closed and densely defined operators;

$$d = \begin{pmatrix} 0 & 0 \\ T & 0 \end{pmatrix}, \quad d^* = \begin{pmatrix} 0 & T^* \\ 0 & 0 \end{pmatrix}, \quad (2.15)$$

with $D(d) = D(T) \times H_2 = E_1$ and $D(d^*) = H_1 \times D(T^*) = E_0$. The $\mathcal{J}$ -selfadjoint operators $\partial_T = d + d^*$ and $i\delta_T = i(d - d^*)$, with domain $D(\partial_T) = D(i\delta_T) = E_0 \cap E_1$, will be called the *supersymmetry operators* and they act from H_1 to H_2 and vice versa. The conjugation $\mathcal{J}$ annihilates H , i.e. $\mathcal{J}z = z$, $z \in H_1$ and $\mathcal{J}z = -z$, $z \in H_2$. Further the supersymmetry operators satisfy the relations;

$$\mathcal{J}\partial_T + \partial_T\mathcal{J} = 0 \quad \text{on } E_0 \cap E_1, \quad (2.16)$$

$$\mathcal{J}i\delta_T + i\delta_T\mathcal{J} = 0 \quad \text{on } E_0 \cap E_1. \quad (2.17)$$

Next we introduce the *Laplacian*;

$$\Delta_T = d^*d + dd^* = |\partial_T|^2.$$

Then

$$\partial_T\Delta_T - \Delta_T\partial_T = 0, \quad (2.18)$$

$$i\delta_T\Delta_T - \Delta_T i\delta_T = 0. \quad (2.19)$$

Also

$$\partial_T^2 = \Delta_T, \quad (2.20)$$

$$(i\delta_T)^2 = \Delta_T. \quad (2.21)$$

Finally

$$i\delta_T\partial_T + \partial_T i\delta_T = 0, \quad \text{on } D(T^*T) \times D(TT^*). \quad (2.22)$$

The above described supersymmetry algebra is characteristic for the class of $\mathcal{J}$ -selfadjoint operators.

3. Polar decomposition of $\mathcal{J}$ -selfadjoint operators

Let T be an operator satisfying (H_T) , then the operator T^*T is non-negative selfadjoint in H_1 and $|T| = (T^*T)^{1/2}$ is also non-negative selfadjoint in H_1 and satisfies

$$D(|T|) = D(T), \quad N(|T|) = N(T), \quad \text{and} \quad R(|T|) = R(T^*). \quad (3.1)$$

The operator $|T|$ is called the *absolute value* of T . There exists a unique operator $U \in B(H_1, H_2)$ satisfying:

$$T = U|T|, \quad U|_{N(T)} = 0. \quad (3.2)$$

The decomposition $T = U|T|$ is called the *polar decomposition* of T (see e.g. Kato [K]). The operator U is a partial isometry, more precisely

$$U^*U = P_{N(T)^\perp}; \quad UU^* = P_{N(T^*)^\perp}, \quad (3.3)$$

where $P_{N(T)^\perp}$ (respectively $P_{N(T^*)^\perp}$) denotes the orthogonal projection onto $N(T)^\perp$ (respectively $N(T^*)^\perp$). We recall that the polar decomposition of T^* is

$$T^* = U^*|T^*|; \quad |T^*| = (TT^*)^{1/2}. \quad (3.4)$$

If A is selfadjoint in H and $A = Z|A|$ is the polar decomposition of A , then Z is selfadjoint and commutes with A and $|A|$. The operators $P_0 = I - Z^2$, $P_+ = \frac{1}{2}(Z^2 + Z)$ and $P_- = \frac{1}{2}(Z^2 - Z)$ are projections, mutually orthogonal, decomposing the space H into three subspaces:

$$H = H^+ \oplus H^0 \oplus H^-, \quad (3.5)$$

being orthogonal to one another. Finally, we define the positive and negative part of A by

$$A_+ = AP_+, \quad A_- = -AP_-. \quad (3.6)$$

We now consider the case in which the operator A is $\mathcal{J}$ -selfadjoint. By Proposition 2.4, $A = \partial_T$ for some operator T satisfying (H_T) and $\mathcal{J}$ as given in (4.2.4). An easy computation shows that

$$|A| = \begin{pmatrix} |T| & 0 \\ 0 & |T^*| \end{pmatrix}, \quad Z = \begin{pmatrix} 0 & U^* \\ U & 0 \end{pmatrix}. \quad (3.7)$$

Observe that $Z = \partial_U$ and is also $\mathcal{J}$ -selfadjoint. Furthermore $\partial_U^2 = I|_{N(\partial_T)^\perp}$ and $\partial_U = 0$ on $N(\partial_T)$. On the other hand $|A|$ commutes with $\mathcal{J}$. It follows that $\mathcal{J}Z = -Z\mathcal{J}$ and $\mathcal{J}Z^2 = Z^2\mathcal{J}$. Therefore we obtain

$$\mathcal{J}P_0 = \mathcal{J}(I - Z^2) = (I - Z^2)\mathcal{J} = P_0\mathcal{J}, \quad (3.9)$$

$$\mathcal{J}P_+ = \frac{1}{2}\mathcal{J}(Z^2 + Z) = \frac{1}{2}(Z^2 - Z)\mathcal{J} = P_-\mathcal{J}, \quad (3.10)$$

$$\mathcal{J}P_- = \frac{1}{2}\mathcal{J}(Z^2 - Z) = \frac{1}{2}(Z^2 + Z)\mathcal{J} = P_+\mathcal{J}, \quad (3.11)$$

$$\mathcal{J}A_+ = \mathcal{J}AP_+ = -A\mathcal{J}P_+ = -AP_-\mathcal{J} = A_-\mathcal{J}, \quad (3.12)$$

$$\mathcal{J}A_- = -\mathcal{J}AP_- = A\mathcal{J}P_- = AP_+\mathcal{J} = A_+\mathcal{J}. \quad (3.13)$$

Consequently a $\mathcal{J}$ -selfadjoint operator A is completely determined by its positive part and by the conjugation $\mathcal{J}$:

$$A = A_+ - \mathcal{J}A_+\mathcal{J}, \quad (3.14)$$

$$A_+(\mathcal{J}A_+\mathcal{J}) = (\mathcal{J}A_+\mathcal{J})A_+ = 0. \quad (3.15)$$

Conversely if a selfadjoint operator A in a Hilbert space H with conjugation $\mathcal{J}$ satisfies (3.12) (equivalently (3.13)) or

$$A_- = \mathcal{J}A_+\mathcal{J}, \quad (3.16)$$

the operator A is $\mathcal{J}$ -selfadjoint. Indeed we have:

$$\begin{aligned} \mathcal{J}A &= \mathcal{J}(A_+ - A_-) = \mathcal{J}A_+ - \mathcal{J}A_- \\ &= A_-\mathcal{J} - A_+\mathcal{J} = -A\mathcal{J}. \end{aligned}$$

Summarizing we obtain yet another characterization of $\mathcal{J}$ -selfadjoint operators.

Proposition 3.1. *Let $(H, \mathcal{J})$ be a Hilbert space with a conjugation. A selfadjoint operator A is $\mathcal{J}$ -selfadjoint if and only if $A_+ = \mathcal{J}A_-\mathcal{J}$, where A_+ and A_- denote the positive and negative parts of A .*

Remark 3.2. It follows from the previous considerations that the spectrum of a $\mathcal{J}$ -selfadjoint operator is symmetric with respect to the origin. Because of Proposition 2.4, $A = \partial_T$ for some T satisfying (H_T) and therefore we have the following identity for the spectrum:

$$\sigma(A) \cup \{0\} = \sigma(\partial_T) \cup \{0\} = \pm\sigma(|T|) \cup \{0\} = \pm\sigma(|T^*|) \cup \{0\}. \quad (3.17)$$

Example: Let Ω be a smooth bounded domain in $\mathbf{R}^N$ and let $-\Delta$ be Laplace operator in $L^2(\Omega)$, with domain $D(-\Delta) = H^2 \cap H_0^1(\Omega)$. We consider the operators

$$A_1 = \begin{pmatrix} 0 & -\Delta \\ -\Delta & 0 \end{pmatrix}, \quad A_2 = \begin{pmatrix} -\Delta & 0 \\ 0 & \Delta \end{pmatrix}, \quad (3.18)$$

in $L^2(\Omega) \times L^2(\Omega)$. See e.g. [BR], [FF], [HV1], [PV]. Then both operators A_1 and A_2 are $\mathcal{J}$ -selfadjoint with respect to the conjugations $\mathcal{J}_1$ and $\mathcal{J}_2$ respectively, given by

$$\mathcal{J}_1 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \quad \mathcal{J}_2 = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}. \quad (3.19)$$

Let us now consider the d'Alembertian, i.e. the wave operator,

$$A = \frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2}. \quad (3.20)$$

To be more precise we consider A as an unbounded operator from $L^2(\mathbf{R}^2)$ to $L^2(\mathbf{R}^2)$, with domain

$$D(A) = \{z \in L^2(\mathbf{R}^2); Az \in L^2(\mathbf{R}^2) \text{ (in the distributional sense)}\}. \quad (3.21)$$

Clearly A is symmetric and by using the Fourier transform $\mathcal{F}$ one deduces that A is selfadjoint. Furthermore one finds that $\mathcal{F}(Az(x, t)) = \mathcal{A}\tilde{z}(k, \omega) = (k^2 - \omega^2)\tilde{z}(k, \omega)$, $\tilde{z} \in L^2(\mathbf{R}^2)$, which gives that $\sigma(A) = \mathbf{R}$. Next we define the conjugation $\mathcal{J}$ by

$$\mathcal{J}z(x, t) = z(t, x), \quad \forall z \in L^2(\mathbf{R}^2), \quad (3.22)$$

which possesses the additional properties $\mathcal{J}^2 = I$, $\mathcal{J}^{-1} = \mathcal{J}$, $\mathcal{J}^* = \mathcal{J}$ and $\mathcal{F}\mathcal{J} = \mathcal{J}\mathcal{F}$. We have

$$\mathcal{A}\mathcal{J}\tilde{z}(k, \omega) = (k^2 - \omega^2)\tilde{z}(\omega, k) = \mathcal{J}((\omega^2 - k^2)\tilde{z}(k, \omega)) = -\mathcal{J}\mathcal{A}\tilde{z}(k, \omega),$$

which clearly proves A is $\mathcal{J}$ -selfadjoint with respect to conjugation $\mathcal{J}$.

4. Normalized extensions of $\mathcal{J}$ -quadratic forms and the Hilbert interpolation spaces E_α

Let $(H, \mathcal{J})$ be a Hilbert space with conjugation $\mathcal{J}$ and let $A \neq 0$ be a $\mathcal{J}$ -selfadjoint operator in H . We denote by Q the associated quadratic form defined by

$$Q(z) = \frac{1}{2}(Az, z)_H, \quad z \in D(A) \quad (4.1)$$

and Q is called a $\mathcal{J}$ -quadratic form. Observe that $\mathcal{J}$ -quadratic forms can also be characterized by the relation

$$Q(\mathcal{J}z) = -Q(z), \quad \forall z \in D(A). \quad (4.2)$$

If the operator A is bounded and everywhere defined, then the corresponding quadratic form Q is continuous on H and if P_0 , P_+ and P_- denote the orthogonal projections corresponding to the decomposition (4.3.5), then

$$|Q|(z) = Q(P_+z) - Q(P_-z), \quad (4.3)$$

is a non-negative quadratic form on H . Furthermore,

$$\|z\|_Q^2 = 2|Q|(z) + \|P_0z\|_H^2 \quad (4.4)$$

defines a norm on H which is weaker than the original norm. It is well-known that the norm $\|\cdot\|_Q$ is equivalent to the original norm on H if and only if the range of A is closed in H . Note that the quadratic form $|Q|(z)$ is the quadratic form associated with the operator $|A|$ and $R(|A|) = R(A)$. With the quadratic form Q one can associate a partial isometry L_Q as follows

$$Q(z) = \frac{1}{2}(L_Qz, z)_Q, \quad z \in H, \quad (4.5)$$

and

$$L_QP_0 = 0, \quad (4.6)$$

$$L_QP_+ = P_+, \quad (4.7)$$

$$L_QP_- = -P_-. \quad (4.8)$$

Such a quadratic form will be called a *normalized $\mathcal{J}$ -quadratic form*. Equivalently a normalized $\mathcal{J}$ -quadratic form on a Hilbert space $(H, \mathcal{J})$ is completely determined by the orthogonal projections P_0 and P_+ satisfying:

$$P_0^* = P_0, \quad P_0^2 = P_0; \quad P_+^* = P_+, \quad P_+^2 = P_+, \quad (4.9)$$

$$P_0 P_+ = P_+ P_0 = 0, \quad (4.10)$$

$$\mathcal{J} P_0 = P_0 \mathcal{J}, \quad (4.11)$$

$$I - P_0 - P_+ = \mathcal{J} P_+ \mathcal{J}, \quad (4.12)$$

since

$$Q(z) = \frac{1}{2} \|P_+ z\|_Q^2 - \frac{1}{2} \|P_- z\|_Q^2. \quad (4.13)$$

In this section we consider $\mathcal{J}$ -quadratic forms Q on a Hilbert space $(H, \mathcal{J})$, associated with an unbounded $\mathcal{J}$ -selfadjoint operator A , i.e. $Q(z) = \frac{1}{2}(Az, z)_H$, $z \in D(A)$ and study the normalized extensions of Q .

Let us first recall the classical case that $A = A^* \geq cI$, $c > 0$ (so not $\mathcal{J}$ -selfadjoint), then $Q(z) = \frac{1}{2}(Az, z)_H$ can be extended by continuity to $D(A^{1/2})$ and the extension is $Q(z) = \frac{1}{2}(A^{1/2}z, A^{1/2}z)_H$. The norm $\|\cdot\|_Q = \|A^{1/2} \cdot\|_H$ is an equivalent norm on $D(A^{1/2})$. By taking $\|\cdot\|_Q$ as the new norm on $D(A^{1/2})$, the quadratic form Q is normalized and is given by $Q(z) = \frac{1}{2}(z, z)_Q$. Furthermore, A can be extended to an isometrical isomorphism between $D(A^{1/2})$ and $D(A^{1/2})^*$ (e.g. [RN], [L], [LM]).

To get a feeling for the case that $Q(z) = \frac{1}{2}(Az, z)_H$, with A an unbounded $\mathcal{J}$ -selfadjoint operator, we assume that H is separable and A has a compact resolvent. By Proposition 2.4, $A = \partial_T$ for some T satisfying (H_T) . In this particular case we deduce from the spectral representation of ∂_T (see eg. [K], Thm 6.29, pg 187) that

$$Q(z) = \frac{1}{2}(\partial_T z, z)_H = \frac{1}{2} \sum_{\mathbf{N}} \lambda_k z_k^2 - \frac{1}{2} \sum_{\mathbf{N}} \lambda_k z_{-k}^2,$$

where $z_k = (\Phi_k, z)_H$, $z_{-k} = (\Phi_{-k}, z)_H$ and $\Phi_k = (\psi_k, \phi_k)^t$, $\Phi_{-k} = (\psi_k, -\phi_k)^t$, with ψ_k and ϕ_k the eigenvectors of T^*T and TT^* respectively. Furthermore $\lambda_k = \mu_k^{1/2}$, where the $\{\mu_k\}_{\mathbf{N}}$ are the positive eigenvalues of T^*T and thus also of TT^* . Observe that H^+ , in decomposition (3.5), is the closure of the linear space spanned by the vectors $\{\Phi_k\}_{\mathbf{N}}$ and H^- the closure of the linear space spanned by the vectors $\{\Phi_{-k}\}_{\mathbf{N}}$. Also $H^+ = \mathcal{J}H^-$, where $\mathcal{J}$ is given by (2.4). Because H^+ and H^- are $\mathcal{J}$ -isomorphic we can rewrite $Q(z)$ as (change of basis)

$$\begin{aligned} Q(z) &= \frac{1}{2} \sum_{\mathbf{N}} \lambda_k (z_k + z_{-k})(z_k - z_{-k}) \\ &= \sum_{\mathbf{N}} \mu_k^{1/2} u_k v_k = \sum_{\mathbf{N}} \mu_k^{\alpha/2} u_k \mu_k^{1/2 - \alpha/2} v_k, \end{aligned}$$

where $u_k = \frac{1}{\sqrt{2}}(z_k + z_{-k})$, $v_k = \frac{1}{\sqrt{2}}(z_k - z_{-k})$ and $u_k = \sqrt{2}(\psi_k, u)_{H_1}$, $v_k = \sqrt{2}(\phi_k, v)_{H_2}$. We can now make the estimate (compare [HV1])

$$\begin{aligned} |Q(z)| &\leq \left(\sum_N \mu_k^\alpha u_k^2 \right)^{1/2} \left(\sum_N \mu_k^{1-\alpha} v_k^2 \right)^{1/2} \\ &= \| |T|^\alpha u \|_{H_1} \| |T^*|^{1-\alpha} v \|_{H_2} \\ &\leq \frac{1}{2} \| |T|^\alpha u \|_{H_1}^2 + \frac{1}{2} \| |T^*|^{1-\alpha} v \|_{H_2}^2. \end{aligned}$$

This implies that $Q(z)$ can be extended to the domains $D(|T|^\alpha) \times D(|T^*|^{1-\alpha})$, $\alpha \in [0, 1]$.

We show now that this procedure can also be performed in general for arbitrary operators T which satisfy hypotheses (H_T) .

Lemma 4.1. *There exist a continuous extensions of $Q(z) = \frac{1}{2}(\partial_T z, z)_H$ to the family of domains*

$$D(|T|^\alpha) \times D(|T^*|^{1-\alpha}), \quad \alpha \in [0, 1], \quad (4.14)$$

which is denoted by $Q_\alpha(z)$. Moreover

$$Q_\alpha(z) = \left(|T|^\alpha u, |T|^{1-\alpha} U^* v \right)_{H_1} = \left(|T^*|^{1-\alpha} v, |T^*|^\alpha U u \right)_{H_2}, \quad (4.15)$$

where U is defined in (3.2) and

$$|Q_\alpha(z)| \leq \| |T|^\alpha u \|_{H_1} \| |T^*|^{1-\alpha} v \|_{H_2} \quad (4.16)$$

$$\leq \frac{1}{2} \| |T|^\alpha u \|_{H_1}^2 + \frac{1}{2} \| |T^*|^{1-\alpha} v \|_{H_2}^2. \quad (4.17)$$

Proof of Lemma 4.1. Take $z \in D(\partial_T)$, then by using the polar decomposition of $\partial_T = Z|\partial_T|$ (Section 3) we rewrite $Q(z)$ as

$$\begin{aligned} Q(z) &= \frac{1}{2}(\partial_T z, z)_H = \frac{1}{2}(Z|\partial_T|z, z)_H \\ &= \frac{1}{2}(|\partial_T|z, Z^*z)_H = \frac{1}{2}(|\partial_T|z, Zz)_H. \end{aligned}$$

For $\alpha \in [0, 1]$, we introduce the selfadjoint operator

$$\Lambda_\alpha = \begin{pmatrix} |T|^\alpha & 0 \\ 0 & |T^*|^{1-\alpha} \end{pmatrix}, \quad (4.18)$$

which satisfies the identity $\Lambda_{1-\alpha}\Lambda_\alpha = |\partial_T|$ (compare $\lambda^{1-\alpha}\lambda^\alpha = \lambda$). Using the commutativity relation $[|\partial_T|^\beta, Z] = 0$, $\beta \in [0, 1]$, we deduce

$$\begin{aligned} \|\Lambda_{1-\alpha}Zz\|_H^2 &= \| |T|^{1-\alpha} U^* v \|_{H_1}^2 + \| |T^*|^\alpha U u \|_{H_2}^2 \\ &= \| U |T|^{1-\alpha} U^* v \|_{H_2}^2 + \| U^* |T^*|^\alpha U u \|_{H_1}^2 \\ &= \| |T^*|^{1-\alpha} v \|_{H_2}^2 + \| |T|^\alpha u \|_{H_1}^2 \\ &= \|\Lambda_\alpha z\|_H^2. \end{aligned} \quad (4.19)$$

As before we rewrite $Q(z)$ as

$$\begin{aligned} Q(z) &= \frac{1}{2}(|\partial_T|z, Zz)_H = \frac{1}{2}(\Lambda_{1-\alpha}\Lambda_\alpha z, Zz)_H \\ &= \frac{1}{2}(\Lambda_\alpha z, \Lambda_{1-\alpha}Zz)_H, \quad z \in D(\partial_T). \end{aligned} \quad (4.20)$$

Thus

$$\begin{aligned} |Q(z)| &= \frac{1}{2}|(\Lambda_\alpha z, \Lambda_{1-\alpha}Zz)_H| \leq \frac{1}{2}\|\Lambda_\alpha z\|_H \|\Lambda_{1-\alpha}Zz\|_H \\ &= \frac{1}{2}\|\Lambda_\alpha z\|_H^2 = \frac{1}{2}\||T|^\alpha u\|_{H_1}^2 + \frac{1}{2}\||T^*|^{1-\alpha} v\|_{H_2}^2, \end{aligned}$$

which proves (4.17). Expression (4.16) follows in a similar way, i.e.

$$\begin{aligned} Q(z) &= \frac{1}{2}\left(|T|^\alpha u, |T|^{1-\alpha}U^*v\right)_{H_1} + \frac{1}{2}\left(|T^*|^{1-\alpha}v, |T^*|^\alpha Uu\right)_{H_2} \\ &= \frac{1}{2}\left(|T|^\alpha u, |T|^{1-\alpha}U^*v\right)_{H_1} + \frac{1}{2}\left(U|T|^{1-\alpha}U^*v, U|T|^\alpha u\right)_{H_2} \\ &= \left(|T|^\alpha u, |T|^{1-\alpha}U^*v\right)_{H_1}, \end{aligned}$$

which proves (4.15) and (4.16) follows. $\square$

Using Lemma 4.1 we now define the intermediate spaces $E_\alpha = D(|T|^\alpha) \times D(|T^*|^{1-\alpha})$, $\alpha \in [0, 1]$. Equipped with the graph-norm $\|z\|_\alpha^2 = \|z\|_H^2 + \||T|^\alpha u\|_{H_1}^2 + \||T^*|^{1-\alpha} v\|_{H_2}^2$ the spaces E_α , $\alpha \in [0, 1]$ are again Hilbert spaces. We set

$$\Theta_1^\alpha = D(|T|^\alpha), \quad \Theta_2^{1-\alpha} = D(|T^*|^{1-\alpha}), \quad (4.21)$$

and introduce the product spaces

Definition 4.2. For $\alpha \in [0, 1]$

$$E_\alpha = D(\Lambda_\alpha) = \Theta_1^\alpha \times \Theta_2^{1-\alpha}, \quad (4.22)$$

with the norms

$$\|z\|_\alpha^2 = \|z\|_H^2 + \||T|^\alpha u\|_{H_1}^2 + \||T^*|^{1-\alpha} v\|_{H_2}^2. \quad (4.23)$$

The operators $|T|^\alpha$ and $|T^*|^{1-\alpha}$ are both closed and therefore we can yet define another norm on E_α . We define

$$\begin{aligned} \|z\|_{E_\alpha}^2 &= \||T|^\alpha u\|_{H_1}^2 + \||T^*|^{1-\alpha} v\|_{H_2}^2 + \|z^0\|_H^2 \\ &= \|\Lambda_\alpha z\|_H^2 + \|z^0\|_H^2, \end{aligned} \quad (4.24)$$

according to the decomposition given in (3.5). It is interesting to observe that the norms $\|\cdot\|_{E_\alpha}$ coincide with the real interpolation norms (see [LM], [BL])

$$\|z\|_{(\alpha)}^2 = c_\alpha^{-1} \int_0^\infty t^{-2\alpha} K(t, z, E_0, E_1)^2 dt/t, \alpha \in (0, 1), \quad (4.25)$$

where

$$K(t, z, E_0, E_1) = \inf_{z=z_0+z_1} \left(\|z_0\|_{E_0}^2 + t^2 \|z_1\|_{E_1}^2 \right)^{1/2}$$

and

$$c_\alpha = \int_0^\infty \frac{s^{2\alpha-1}}{1+s^2} ds = \frac{1}{2} \Gamma(\alpha) \Gamma(1-\alpha), \quad \alpha \in (0, 1).$$

We have

Lemma 4.3. *The norms $\|\cdot\|_{(\alpha)}$ and $\|\cdot\|_{E_\alpha}$ coincide on the domains E_α for all $\alpha \in (0, 1)$.*

Proof of Lemma 4.3. From definition (4.24) of the norms of E_0 and E_1 we have

$$\begin{aligned} K(t, z, E_0, E_1)^2 &= \inf_{z^0=z_0^0+z_1^0} \left(\|z_0^0\|_H^2 + t^2 \|z_1^0\|_H^2 \right) \\ &\quad + \inf_{z^\pm=z_0^\pm+z_1^\pm} \left(\|\Lambda_0 z_0^\pm\|_H^2 + t^2 \|\Lambda_1 z_1^\pm\|_H^2 \right). \end{aligned}$$

Evaluating the first minimum we deduce that z_0^0 satisfies the Euler equation

$$z_0^0 = \frac{t^2}{1+t^2} z^0$$

and the identity $(1+t^2)\|z_0^0\|_H^2 = t^2(z^0, z_0^0)_H$. Substitution of the latter yields

$$K(t, z, E_0, E_1)^2 = \frac{t^2}{1+t^2} \|z^0\|_H^2 + \inf_{z^\pm=z_0^\pm+z_1^\pm} \left(\|\Lambda_0 z_0^\pm\|_H^2 + t^2 \|\Lambda_1 z_1^\pm\|_H^2 \right).$$

We shall denote the latter term by $K^\perp(t, z, \tilde{E}_0, \tilde{E}_1)^2$, where $\tilde{E}_i = E_i \ominus N(\partial_T)$, $i = 1, 2$ ($\tilde{E}_i = E_i \ominus N(\partial_T)$ denotes the othogonal complement of $N(\partial_T)$ in E_i). We have

$$\begin{aligned} K^\perp(t, z, \tilde{E}_0, \tilde{E}_1)^2 &= \inf_{z=z_0+z_1} \left(\|u_0\|_{H_1}^2 + t^2 \| |T| u_1 \|^2_{H_1} + \| |T^*| v_0 \|^2_{H_2} + t^2 \|v_1\|_{H_2}^2 \right) \\ &= \inf_{z_0 \in \tilde{E}_0} \left(\|u_0\|_{H_1}^2 + t^2 \| |T| (u - u_0) \|^2_{H_1} + \| |T^*| v_0 \|^2_{H_2} + t^2 \|v - v_0\|_{H_2}^2 \right) \end{aligned}$$

Evaluating the minimum above, we obtain

$$K^\perp(t, z, \tilde{E}_0, \tilde{E}_1)^2 = (u, |T|^2(|T|^2 + t^{-2})^{-1}u)_{H_1} + (v, t^2|T^*|^2(|T^*|^2 + t^2)^{-1}v)_{H_2}$$

and the norm becomes

$$\begin{aligned} \|z\|_{(\alpha)}^2 &= \|z^0\|_H^2 + c_\alpha^{-1} \int_0^\infty t^{-2\alpha} (u, |T|^2(|T|^2 + t^{-2})^{-1}u)_{H_1} dt/t \\ &\quad + c_\alpha^{-1} \int_0^\infty t^{-2\alpha} (v, t^2 |T^*|^2(|T^*|^2 + t^2)^{-1}v)_{H_2} dt/t. \end{aligned} \quad (4.26)$$

Using the spectral representation of the selfadjoint operators $|T|$ and $|T^*|$ we get

$$\int_0^\infty t^{-2\alpha} |T|^2(|T|^2 + t^{-2})^{-1} u dt/t = c_\alpha |T|^{2\alpha} u, \quad \forall u \in D(|T|^2), \quad (4.27)$$

$$\int_0^\infty t^{-2\alpha} t^2 |T^*|^2(|T^*|^2 + t^2)^{-1} v dt/t = c_\alpha |T^*|^{2(1-\alpha)} v, \quad \forall v \in D(|T^*|^2). \quad (4.28)$$

We obtain from (4.26), (4.27) and (4.28)

$$\begin{aligned} \|z\|_{(\alpha)}^2 &= \|z^0\|_H^2 + (|T|^{2\alpha} u, u)_{H_1} + (|T^*|^{2(1-\alpha)} v, v)_{H_2} \\ &= \|z^0\|_H^2 + \||T|^\alpha u\|_{H_1}^2 + \||T^*|^{1-\alpha} v\|_{H_2}^2 \\ &= \|z^0\|_H^2 + \|\Lambda_\alpha z\|_H^2 = \|z\|_{E_\alpha}^2 \quad \forall z \in D(\partial_T^2). \end{aligned}$$

Since $D(\partial_T^2)$ is a core of Λ_α , the proof is complete.³ □

On the vector space E_α , the norm $\|\cdot\|_{E_\alpha}$ is a weaker than the norm $\|\cdot\|_\alpha$ and the normed linear space $(E_\alpha, \|\cdot\|_{E_\alpha})$ is not necessarily complete. As well-known $(E_\alpha, \|\cdot\|_{E_\alpha})$ is complete if and only if the norms $\|\cdot\|_\alpha$ and $\|\cdot\|_{E_\alpha}$ are equivalent.

Proposition 4.4. *The norms $\|\cdot\|_\alpha$ and $\|\cdot\|_{E_\alpha}$ are equivalent norms on E_α if and only if $R(\partial_T)$ is closed in H , or equivalently $R(T)$ is closed.*

Proof of Proposition 4.4. (i) In the case that $R(\partial_T)$ is closed we have that $R(T)$, $R(|T|)$ is closed and thus also $R(T^*)$, $R(|T^*|)$ is closed. It is well-known that $R(T)$ is closed if and only if $R(|T|^\alpha)$, $\alpha \in (0, 1)$, is closed (similar for $R(|T^*|)$). Thus $R(T)$ and $R(T^*)$ are closed if and only if

$$\text{dist}(u, N(T)) \leq C \||T|^\alpha u\|_{H_1}, \quad \forall u \in \Theta_1^\alpha, \quad \text{and}$$

$$\text{dist}(v, N(T^*)) \leq C \||T^*|^{1-\alpha} v\|_{H_2}, \quad \forall v \in \Theta_2^{1-\alpha}.$$

Combining the latter estimates we obtain

$$\text{dist}(z, N(\partial_T)) \leq C \|\Lambda_\alpha z\|_H, \quad \forall z \in \Theta_1^\alpha \times \Theta_2^{1-\alpha} = E_\alpha,$$

³ In the book by Lions & Magenes one can find the analogue of Lemma 4.3 when E_0 is continuously embedded in E_1 . This case is associated with a positive definite operator Λ such that $D(\Lambda) = E_0$ (see [LM] pages 98-99).

and

$$\|z\|_H^2 \leq \max(C, 1) \left(\|\Lambda_\alpha z\|_H^2 + \|z^0\|_H^2 \right), \quad \forall z \in E_\alpha.$$

This proves that the norms (4.23) and (4.24) are equivalent.

(ii) If the norms (4.23) and (4.24) are equivalent then there exists a constant $C > 1$ such that

$$\|z\|_{E_\alpha}^2 \leq \|z\|_\alpha^2 \leq C \|z\|_{E_\alpha}^2, \quad \forall z \in E_\alpha.$$

This yields, for $z \in N(\partial_T)^\perp \cap E_\alpha$,

$$\|\Lambda_\alpha(z^+ + z^-)\|_H^2 \geq \frac{1}{C-1} \|z^+ + z^-\|_H^2,$$

and

$$(C-1)\|\Lambda_\alpha z\|_H^2 + \|z^0\|_H^2 \geq \|z\|_H^2,$$

which proves that $R(\Lambda_\alpha)$ is closed and consequently $R(T)$ and $R(T^*)$ are closed. $\square$

Remark 4.5. If $R(\partial_T)$ is closed, Proposition 4.4 implies that the embedding

$$i : E_\alpha \longrightarrow H, \tag{4.29}$$

is continuous and dense not only with respect to the norms $\|\cdot\|_\alpha$ but also with respect to the norms $\|\cdot\|_{E_\alpha}$.

Assuming from now on that $R(T)$ is closed we take $\|\cdot\|_{E_\alpha}$ as the norm in E_α and we have the estimate $|Q(z)| \leq \frac{1}{2} \|z\|_{E_\alpha}^2$ from Lemma 4.1. We now state the main result concerning normalized extensions of Q .

Theorem 4.6. *Let Q be a $\mathcal{J}$ -quadratic form on a Hilbert space $(H, \mathcal{J})$, corresponding with an unbounded $\mathcal{J}$ -selfadjoint operator $A = \partial_T$, having closed range. Then there exists a family of Hilbert spaces $E_\alpha = D(|T|^\alpha) \times D(|T^*|^{1-\alpha})$, $\alpha \in [0, 1]$, equipped with the norms $\|\cdot\|_{E_\alpha}$, satisfying:*

(i) $\mathcal{J}_\alpha$, the restriction $\mathcal{J}$ to E_α (i.e. $D(\mathcal{J}_\alpha) = \{z \in E_\alpha; \mathcal{J}z \in E_\alpha\}$), is a conjugation in E_α ,

(ii) Q has a continuous extension Q_α to E_α , which is a normalized $\mathcal{J}_\alpha$ -quadratic form on E_α , given by $Q_\alpha(z) = \frac{1}{2}(L_\alpha z, z)_{E_\alpha}$, where

$$L_\alpha = \begin{pmatrix} 0 & |T|^{1-2\alpha} U^* \\ |T^*|^{2\alpha-1} U & 0 \end{pmatrix}, \tag{4.30}$$

(iii) $L_\alpha : E_\alpha \rightarrow E_\alpha$ is a selfadjoint partial isometry, with closed range, inducing the decomposition

$$E_\alpha = E^+ \oplus E^0 \oplus E^-, \tag{4.31}$$

on E_α , where

$$E^+ = \left\{ z \in E_\alpha \ominus N(\partial_T); |T|^{1-\alpha} U^* v = |T|^\alpha u \right\}, \quad E^- = \mathcal{J}_\alpha E^+, \tag{4.32}$$

(iv) the spectrum $\sigma(L_\alpha) = \{-1, 1\}$ if $0 \in \rho(\partial_T)$ and $\sigma(L_\alpha) = \{-1, 0, 1\}$ if $0 \in \sigma(\partial_T)$. Furthermore, E^+ and E^- are the eigenspaces of L_α corresponding to the eigenvalues 1 and -1 .

Proof of Theorem 4.6. Clearly (i) holds. As for (ii)-(iv) we argue as follows. Because $Q_\alpha(z)$ is a continuous quadratic form on E_α , there exists a unique bounded selfadjoint operator L_α such that $Q_\alpha(z) = \frac{1}{2}(L_\alpha z, z)_{E_\alpha}$. By Lemma 4.1 the extension of $\frac{1}{2}(\partial_T z, z)_H$ to E_α is given by $\frac{1}{2}(\Lambda_\alpha z, \Lambda_{1-\alpha} Z z)_H$. Thus if $z \in E_\alpha \ominus N(\partial_T)$

$$\begin{aligned} Q_\alpha(z) &= \frac{1}{2}(\Lambda_\alpha z, \Lambda_{1-\alpha} Z z)_H = \frac{1}{2}(\Lambda_\alpha z, \Lambda_\alpha \Lambda_\alpha^{-1} \Lambda_{1-\alpha} Z z)_H \\ &= \frac{1}{2}(\Lambda_\alpha^{-1} \Lambda_{1-\alpha} Z z, z)_{E_\alpha}. \end{aligned}$$

Therefore

$$L_\alpha = \Lambda_\alpha^{-1} \Lambda_{1-\alpha} Z = \Lambda_\alpha^{-2} \partial_T$$

and from (4.18), $\|L_\alpha z\|_{E_\alpha} = \|z\|_{E_\alpha}$ for all $z \in E_\alpha \ominus N(\partial_T)$. This shows L_α is a selfadjoint partial isometry on E_α . Let us consider the eigenvalue problem $L_\alpha z = \lambda z$ in E_α :

$$\begin{cases} |T|^{1-2\alpha} U^* v = \lambda u, \end{cases} \quad (4.33)$$

$$\begin{cases} |T^*|^{2\alpha-1} U u = \lambda v. \end{cases} \quad (4.34)$$

Because $u \in \Theta_1^\alpha$, (4.33) yields

$$U^* |T^*|^{1-\alpha} \lambda v = |T|^{1-\alpha} U^* \lambda v = \lambda^2 |T|^\alpha u, \quad (4.35)$$

where we used the commutation relation $[|\partial_T|^\alpha, Z] = 0$. Substituting the equation (4.34) in (4.35) we obtain

$$U^* |T^*|^\alpha U u = \lambda^2 |T|^\alpha u,$$

and by the commutativity of $|\partial_T|^\alpha$ and Z

$$|T|^\alpha U^* U u = |T|^\alpha u = \lambda^2 |T|^\alpha u,$$

which gives $\lambda^2 = 1$ and consequently $\lambda = \pm 1$, with the eigenspaces E^+ and E^- as given by (4.32). Furthermore, because $R(L_\alpha)$ is closed and L_α maps $R(L_\alpha)$ onto $R(L_\alpha)$, L_α is unitary on $R(L_\alpha)$. The spectrum of L_α on $E_\alpha \ominus N(\partial_T)$ consists only of the eigenvalues -1 and 1 . We already concluded that the values $-1, 1$ are eigenvalues of L_α and therefore $E_\alpha \cap N(\partial_T)^\perp = E^+ \oplus E^-$. If the kernel $N(\partial_T)$ is non-trivial, also 0 is an eigenvalue of L_α and then $E_\alpha = E^+ \oplus E^0 \oplus E^-$. This completes the proof. $\square$

By means of Theorem 4.6 we can also decompose an element $z \in E_\alpha$ as $z = z^+ + z^0 + z^-$, with $z^\pm \in E^\pm$ and $z^0 \in E^0 = N(\partial_T)$. This yields a decomposition of $Q(z)$: $Q(z) = Q(z^+) + Q(z^-)$ and an equivalent expression for the norm $\|\cdot\|_{E_\alpha}$:

$$\frac{1}{2} \|z\|_{E_\alpha}^2 = Q(z^+) - Q(z^-) + \frac{1}{2} \|z^0\|_H^2 = |Q|(z) + \frac{1}{2} \|z^0\|_H^2. \quad (4.36)$$

This alternative formulation will be convenient for applications in critical point theory (e.g. [HV1]).

The family of intermediate spaces E_α form a 'diagonal' of spaces between the 'end-points' $E_1 = D(T) \times H_2$ and $E_0 = H_1 \times D(T^*)$ (see Figure 1):

Fig 1: The 'diagonal' of intermediate spaces.

Along with the intermediate spaces E_α we have the intermediate forms between $(u, T^*v)_{H_1}$ and $(Tu, v)_{H_2}$:

$$\begin{aligned} (Tu, v)_{H_2} &= (|T|^\alpha u, |T|^{1-\alpha} U^* v)_{H_1} \\ &= (|T^*|^{1-\alpha} v, |T^*|^\alpha U u)_{H_2} = (u, T^* v)_{H_1}. \end{aligned} \quad (4.37)$$

We show now that if $R(A)$ is closed (equivalently when $R(T)$ is closed), the spaces E_α are proper interpolation spaces between E_0 and E_1 and they coincide with the real interpolation with $p = 2$ between E_0 and E_1 .

We recall that, because $R(T)$ is closed, the real interpolation spaces $[E_0, E_1]_{\alpha, 2}$ can be defined by

$$[E_0, E_1]_{\alpha, 2} = \left\{ z \in E_0 + E_1; \ t^{-\alpha} K(t, z, E_0, E_1) \in L^2(0, \infty; dt/t) \right\}, \quad (4.38)$$

with $K(t, z, E_0, E_1)$ as defined before (e.g. [BL, LM]). By Lemma 4.3

$$E_\alpha = [E_0, E_1]_{\alpha, 2}, \quad (4.39)$$

and the norms coincide. The real interpolation functor described above is an exact interpolation functor of exponent α , i.e. let F_0 and F_1 be Hilbert spaces and B a bounded linear operator from E_i to F_i , $i = 1, 2$, with

$$B : E_0 \longrightarrow F_0, \quad \|B\| = M_0,$$

$$B : E_1 \longrightarrow F_1, \quad \|B\| = M_1,$$

then

$$\|Bz\|_{[F_0, F_1]_{\alpha, 2}} \leq M_0^{1-\alpha} M_1^\alpha \|z\|_{E_\alpha}, \quad (4.40)$$

This proves that E_α are proper interpolation spaces.

Remark 4.7. From Proposition 2.4 we know that every $\mathcal{J}$ -selfadjoint operator A in $(H, \mathcal{J})$ is of the form $A = \partial_T$ for some operator T satisfying (H_T) . However using Remark 2.7 one can also write the operator $A = d + d^*$, with $d^2 = d^{*2} = 0$, instead of using the matrix representation.

Remark 4.8. As in the case of positive definite operators, one can extend the operator ∂_T in H to

$$\partial_T : E_\alpha \rightarrow E_\alpha^*, \quad (4.41)$$

which is a partial isometry. One can even give a more precise statement;

Corollary 4.9. *Suppose that T satisfies (H_T) . Then the operator*

$$\partial_T + P_0 : E_\alpha \longrightarrow E_\alpha^*, \quad (4.42)$$

is an isometric isomorphism between E_α and E_α^ , i.e. for every $f \in E_\alpha^*$, the equation $(\partial_T + P_0)z = f$ has a unique solution $u \in E_\alpha$, with $\|z\|_{E_\alpha} = \|f\|_{E_\alpha^*}$.*

In the event that $R(T)$ is not closed the domains $E_\alpha = D(|T|^\alpha) \times D(|T^*|^{1-\alpha})$ are not complete with respect to the norms $\|\cdot\|_{E_\alpha}$. So in order to work with the norms $\|\cdot\|_{E_\alpha}$ and Hilbert spaces we have to complete the domains E_α with respect to the norms $\|\cdot\|_{E_\alpha}$.

Definition 4.10. *Assume $R(T)$ is not closed. Then for all $\alpha \in [0, 1]$,*

$$\mathcal{E}_\alpha = \text{completion}(E_\alpha; \|\cdot\|_{E_\alpha}), \quad (4.43).$$

Theorem 4.6 is still valid for the spaces $\mathcal{E}_\alpha$, which is important for applications to critical point theory. As for the interpolation property we define

$$\mathcal{E}_\alpha = \{\mathcal{E}_0, \mathcal{E}_1\}_\alpha. \quad (4.44)$$

Clearly these spaces satisfy the interpolation property (4.40) by using a density argument.

5. Critical point theorems for ∂_T -systems

In this section we are interested in finding non-trivial solutions of the semilinear equation

$$\partial_T z - h(z) = 0, \quad (5.1)$$

where $h(\cdot)$ is a potential operator associated with the potential $\mathcal{H}$. Formally solutions of (5.1) are critical points of the functional

$$\mathcal{L}(z) = \frac{1}{2}(\partial_T z, z)_H - \mathcal{H}(z). \quad (5.2)$$

From the previous sections we already know that the quadratic part of $\mathcal{L}$ is a C^2 -function on the spaces E_α , for all $\alpha \in [0, 1]$. As mentioned in the introduction the Hamiltonian $\mathcal{H}$ need not be well-defined on E_α for all $\alpha \in [0, 1]$. A Hamiltonian $\mathcal{H}$ is admissible if

$$\mathcal{H} \in C^1(E_\alpha, \mathbf{R}), \text{ for some } \alpha \in [0, 1]. \quad (5.3)$$

In the case $\mathcal{H}$ is admissible we have that the functional $\mathcal{L}$ is a C^1 -function on E_α for some $\alpha \in [0, 1]$ depending on $\mathcal{H}$.

We shall prove a theorem now which ensures the existence of a non-trivial critical point of $\mathcal{L}$ (and a thus solution of (5.1)) under certain assumptions on the Hamiltonian $\mathcal{H}$.

In [FF,HV1] the Indefinite Functional Theorem of Benci & Rabinowitz [BR] was used to find critical points of $\mathcal{L}$. If we assume that also $\mathcal{H}'$ is a compact operator on E_α this theorem can also be applied to (5.2). We must however have that in the decomposition

$$E_\alpha = E^+ \oplus E^0 \oplus E^-, \quad \mathcal{J}_\alpha E^+ = E^-,$$

the space E^0 is finite dimensional. For the applications we have in mind this is too restrictive. We therefore extend an abstract critical point theorem by Benci & Fortunato [BF], which enables us to cope with an infinite dimensional kernel $N(\partial_T) = E^0$.

In the previous we assumed that $\mathcal{H}$ was everywhere defined on E_α , for some $\alpha \in [0, 1]$. In various situations however $\mathcal{H}$ need not be defined on all elements of the kernel $N(\partial_T) = E^0$. Therefore (5.3) needs to be modified slightly. Suppose that $\mathcal{H} \in C^1(X, \mathbf{R})$ for some real reflexive Banach space X . Furthermore we assume that there is a $\alpha \in [0, 1]$ such that⁴

- (s1) $E^+ \oplus E^- \hookrightarrow X$ compactly,
- (s2) $E^0 \cap X$ is closed in X ,
- (s3) $E^+ \oplus E^- \oplus (E^0 \cap X)$ is dense in X ,
- (s4) $E^0 \cap X$ is dense in E^0 (w.r.t. the H -topology).

Define

$$X^+ = \overline{E^+}, \quad X^- = \overline{E^-}, \quad (5.4)$$

with the closures taken in X and

$$X_1^0 = (N(T) \times \{0\}) \cap X, \quad X_2^0 = (\{0\} \times N(T^*)) \cap X. \quad (5.5)$$

Using the assumptions (s1)-(s4) we obtain

$$X = X^+ \oplus X^- \oplus X_1^0 \oplus X_2^0,$$

$$X_1 = P_1(X^+ \oplus X^-) \oplus X_1^0,$$

$$X_2 = P_2(X^+ \oplus X^-) \oplus X_2^0,$$

⁴ We use the decomposition $E_\alpha = E^+ \oplus E^- \oplus E^0$ according to (4.31).

where P_1 and P_2 are the extensions in X of the projections on $\{(u, 0)\}$ and $\{(0, v)\}$ in $E^+ \oplus E^-$ respectively. In $X = X_1 \times X_2$ we employ the equivalent norms

$$\begin{aligned} \|z\|_X^2 &= \|(u, 0)\|_X^2 + \|(0, v)\|_X^2 \\ &= \|u\|_{X_1}^2 + \|v\|_{X_2}^2, \end{aligned} \quad (5.6)$$

and similarly for the dual space X^* ,

$$\|w\|_{X^*}^2 = \|w_1\|_{X_1^*}^2 + \|w_2\|_{X_2^*}^2. \quad (5.7)$$

On $\mathcal{H}$ we impose the hypotheses:

- (k1) $\mathcal{H} \in C^1(X, \mathbf{R})$, is a strictly convex functional and $\mathcal{H}' = h$, with $h(0) = 0$, is a bounded homeomorphism from X to X^* .
- (k2) If $\{z_n\} \subset X$, $z \in X$ such that $\langle h(z_n) - h(z), z_n - z \rangle \rightarrow 0$ as $n \rightarrow \infty$, then $\{z_n\}$ has a convergent subsequence converging to z in X .
- (k3) There are strictly convex functions $\phi_1, \dots, \phi_4$, positive constants $c_1, \dots, c_4$, satisfying

$$\begin{aligned} \phi_1(t)/t^a &\rightarrow c_1, \quad \phi_2(t)/t^b \rightarrow c_2, \quad \text{as } t \rightarrow \infty, \quad \phi_1(0) = \phi_2(0) = 0, \\ \phi_3(t)/t^a &\rightarrow c_3, \quad \phi_4(t)/t^b \rightarrow c_4, \quad \text{as } t \rightarrow 0, \quad a, b > 0, \quad \frac{1}{a} + \frac{1}{b} < 1, \end{aligned}$$
 s.t. $\phi_1(\|u\|_{X_1}) + \phi_2(\|v\|_{X_2}) \leq \mathcal{H}(z) \leq \phi_3(\|u\|_{X_1}) + \phi_4(\|v\|_{X_2})$, $\forall z \in X$.
- (k4) $|\theta_1 \langle \mathcal{H}_u(z), u \rangle + \theta_2 \langle \mathcal{H}_v(z), v \rangle - \mathcal{H}(z)| \rightarrow \infty$ as $\|z\|_X \rightarrow \infty$, for some $\theta_1, \theta_2 > 0$, $\theta_1 + \theta_2 = 1$.

We have the following critical point theorem (compare Benci & Fortunato [BF]):

Theorem 5.1. *Let X be a real reflexive Banach space such that for some $\alpha \in [0, 1]$ Hypotheses (s1)-(s4) are satisfied. Suppose that Hamiltonian $\mathcal{H}$ satisfies (k1)-(k4), then $\mathcal{L}(z)$ has at least one non-trivial critical point $z \in E_\alpha \cap X$.*

Proof of Theorem 5.1: From Theorem 4.6 we have that $L_\alpha = L_\alpha^{-1} \in B(E_\alpha)$. Using the hypotheses (s1)-(s4) we can extend L_α to a bounded operator from W to W^* , where $W = (\overline{E^+})^* \oplus (\overline{E^-})^*$. Following [BF] we transform the functional $\mathcal{L}$ into the dual setting in the usual way, i.e.

$$\begin{aligned} f(w) &= b(w) - \frac{1}{2} \langle w, L_\alpha^{-1} w \rangle \\ &= b(w) - \frac{1}{2} \langle w, L_\alpha w \rangle, \quad w \in W \subset X, \end{aligned} \quad (5.8)$$

with $b(w) = \mathcal{H}^*(w)|_W$, where $\mathcal{H}^*$ is the Legendre-Fenchel transform of $\mathcal{H} : X \rightarrow \mathbf{R}$. From Section 4.4 (Theorem 4.4.6) we have that $\sigma(L_\alpha)|_{E^+ \oplus E^-} = \{-1, 1\}$ and we consider L_α

here as a operator from W to W^* . We shall use the Linking Theorem as given in [BBF] (Theorem 2.3) in order to obtain critical points of f , which can be translated then to critical points of $\mathcal{L}$. This version of the Linking Theorem admits a weaker compactness condition then the usual Palais-Smale condition. We recall

$$(C) \left\{ \begin{array}{l} \text{(i) every bounded sequence } \{w_k\} \subset f^{-1}((0, +\infty)), \text{ for which } \{f(w_k)\} \text{ is} \\ \text{bounded and } f'(w_k) \rightarrow 0, \text{ possesses a convergent subsequence;} \\ \text{(ii) for all } c \in (0, +\infty), \text{ there are } \epsilon, R, \beta > 0, \text{ such that for all} \\ w \in f^{-1}([c - \epsilon, c + \epsilon]), \|w\|_{X^*} \geq R : \|f'(w)\|_X \|w\|_{X^*} \geq \beta. \end{array} \right.$$

Proposition 5.2. (Linking Theorem, [BBF]). Suppose that $f \in C^1(W, \mathbf{R})$ satisfies the following properties

- (f1) f satisfies (C), $f(0) = 0$,
- (f2) there exists a closed subset $S \subset W$ and a Banach manifold $Q \subset W$ with boundary ∂Q which fulfil the following properties;
 - a) there exist two constants $\gamma > \omega \geq 0$ such that
$$f|_{\partial Q} \leq \omega, \quad f|_S \geq \gamma,$$
 - b) S and ∂Q link,
 - c) $f(0) = 0$ and $\sup_{w \in Q} f(w) < +\infty$.

Then f possesses a non-trivial critical value $c \geq \gamma$.

First we proceed with verifying condition (C) for the functional f on W . The first part of (C) is identical to the verification in [BF]. As for the second part of (C), let $c \in \mathbf{R}$ and consider $w \in f^{-1}([c - \epsilon, c + \epsilon])$. Choose $y = (2\sigma_1 w_1, 2\sigma_2 w_2)^t \in W$, with $1 > \sigma_1, \sigma_2 > 0$, then

$$\begin{aligned} 2\|w\|_{X^*} \|f'(w)\|_X &\geq \|y\|_{X^*} \|f'(w)\|_X \geq |\langle y, f'(w) \rangle| \\ &= |\langle 2\sigma_1 w_1, b_{w_1}(w) \rangle + \langle 2\sigma_2 w_2, b_{w_2}(w) \rangle - \langle w, L_\alpha w \rangle|, \end{aligned}$$

if we set $\sigma_1 + \sigma_2 = 1$ and use the special structure of L_α given by the ∂_T -operator. Now set

$$\begin{aligned} q(w) &= b(w) - \frac{1}{2} \langle w, L_\alpha w \rangle - c \\ p(w) &= \sigma_1 \langle w_1, b_{w_1}(w) \rangle + \sigma_2 \langle w_2, b_{w_2}(w) \rangle - b(w) - c. \end{aligned}$$

Then

$$\|y\|_{X^*} \|f'(w)\|_X \geq |2(q(w) + p(w)) + 4c|,$$

where $|q(w)| \leq \epsilon$ by assumption. Clearly, because $\mathcal{H}^*(w) = \langle w, z \rangle - \mathcal{H}(z)$, with $z = h^{-1}(w)$, we have $b(w) = \mathcal{H}^*(w) = \langle w, h^{-1}(w) \rangle - \mathcal{H}(h^{-1}(w))$, $w \in W$. Thus, setting

$\sigma_1 = 1 - \theta_1$ and $\sigma_2 = 1 - \theta_2$,

$$\begin{aligned} |\frac{1}{2} \langle y, b'(w) \rangle - b(w)| &= |\frac{1}{2} \langle y, h^{-1}(w) \rangle - \langle w, h^{-1}(w) \rangle + \mathcal{H}(h^{-1}(w))| \\ &= |(\sigma_1 - 1) \langle w_1, u \rangle + (\sigma_2 - 1) \langle w_2, v \rangle + \mathcal{H}(z)| \\ &= |\theta_1 \langle \mathcal{H}_u(z), u \rangle + \theta_2 \langle \mathcal{H}_v(z), v \rangle - \mathcal{H}(z)| \rightarrow \infty, \end{aligned}$$

as $\|z\|_X \rightarrow \infty$, by hypothesis (k4). This completes the verification of condition (C).

In order to check condition (f2) of Proposition 5.2 we proceed as follows; as in [BF] we obtain from (k3)

$$\Phi_1^*(w) \geq \mathcal{H}^*(w) \geq \Phi_2^*(w),$$

where $\Phi_1(z) = \phi_1(\|u\|_{X_1}) + \phi_2(\|v\|_{X_2})$ and $\Phi_2(z) = \phi_3(\|u\|_{X_1}) + \phi_4(\|v\|_{X_2})$ and its Legendre-Fenchel transforms

$$\Phi_1^*(w) = \phi_1^*(\|w_1\|_{X_1^*}) + \phi_2^*(\|w_2\|_{X_2^*}),$$

$$\Phi_2^*(w) = \phi_3^*(\|w_1\|_{X_1^*}) + \phi_4^*(\|w_2\|_{X_2^*}), \quad w \in X^*.$$

We consider the set (ellipsoid)

$$S = \{(\rho^{k-1}w_1, \rho^{l-1}w_2)^t; w = (w_1, w_2)^t \in W, \|w\|_{X^*} = \rho\},$$

where we choose k and l such that $\frac{1}{a'} > \frac{k}{k+l}$, $\frac{1}{b'} > \frac{l}{k+l}$, with a' and b' the conjugates of a and b .

$$\begin{aligned} f|_S &\geq \mathcal{H}^*|_S - \frac{1}{2}\rho^{k+l-2}\|L_\alpha\|_{\mathcal{L}(W, W^*)}\|w\|_{X^*}^2 \\ &\geq \phi_3^*(\rho^{k-1}\|w_1\|) + \phi_4^*(\rho^{l-1}\|w_2\|) - \frac{1}{2}\rho^{k+l}\|L_\alpha\|_{\mathcal{L}(W, W^*)} \end{aligned}$$

Using (k3) and assuming $\|w\|_{X^*}$ is small we obtain

$$\begin{aligned} f|_S &\geq C_1\rho^{a'(k-1)}\|w_1\|^{a'} + C_2\rho^{b'(l-1)}\|w_2\|^{b'} - \frac{1}{2}\rho^{k+l}\|L_\alpha\|_{\mathcal{L}(W, W^*)} \\ &\geq C_1\rho^{a'k}|\sin \phi|^{a'} + C_2\rho^{b'l}|\cos \phi|^{b'} - \frac{1}{2}\rho^{k+l}\|L_\alpha\|_{\mathcal{L}(W, W^*)} \\ &\geq \rho^{a'k}(C_1|\sin \phi|^{a'} + C_2\rho^{b'l-a'k}|\cos \phi|^{b'}) - \frac{1}{2}\rho^{k+l}\|L_\alpha\|_{\mathcal{L}(W, W^*)} \\ &\quad \text{assume now } \rho < 1 \text{ and } b'l < a'k, \\ &\geq C\rho^{a'k} - \frac{1}{2}\rho^{k+l}\|L_\alpha\|_{\mathcal{L}(W, W^*)} \geq \gamma > 0, \end{aligned}$$

for ρ small enough, because $a'k < k+l$. In the case $b'l > a'k$ we obtain a similar estimate. Next, using (s1)-(s4), we choose $w^+ \in (\overline{E^+})^*$, with $\|w^+\| = 1$. Set

$$Q = \{w = (r^k w_1^+, r^l w_2^+); r \in [0, R]\},$$

If r is sufficiently large we derive from (k3) that

$$\begin{aligned}
f(r^k w_1^+, r^l w_2^+) &= \mathcal{H}^*(r^k w_1^+, r^l w_2^+) - \frac{1}{2} C r^{k+l} \\
&\leq \phi_1^*(r^k \|w_1^+\|) + \phi_2^*(r^l \|w_2^+\|) - \frac{1}{2} C r^{k+l} \\
&\leq C_1 r^{a'k} \|w_1^+\|^{a'} + C_2 r^{b'l} \|w_2^+\|^{b'} - \frac{1}{2} C r^{k+l} \\
&\leq C_1 r^{a'k} + C_2 r^{b'l} - \frac{1}{2} C r^{k+l} \rightarrow -\infty,
\end{aligned}$$

as $r \rightarrow \infty$, because $a'k < k+l$ and $b'l < k+l$. Choosing $r = R$ large we have $f(R^k w_1^+, R^l w_2^+) < 0$, which gives that $f|_{\partial Q} \leq 0 = \omega$, because $f(0) = 0$. The particular choices of S and Q provide that S and ∂Q link in the usual sense (see [R2], [BBF]). It is also obvious that (f2) (c) is satisfied. This completes the verification of the conditions (f1) and (f2) of Proposition 5.2. We refer to [BF] for the completion of the proof, which, from this point onward, is identical to the proof of Theorem 1.1 in [BF]. $\square$

6. Applications

In this section we shall exhibit two example of ∂_T -systems.

1. Multiparameter Hamiltonian system.

A multiparameter Hamiltonian system is an extensions of the classical Hamiltonian system and is obtained by introducing the 'time'-variables $x_1, \dots, x_N$, $N \geq 1$. In the case $N = 1$ we have the classical Hamiltonian system. In the multidimensional case new phenomena occur, which were absent in the one-dimensional case. It will appear that the interpolation spaces as introduced in Section 4 will be crucial for the critical point theory of multiparameter Hamiltonian systems. These Hamiltonian systems can also serve to describe certain second-order quasi-linear elliptic equations. Also the equations of electrostatics (see introduction) form a multiparameter Hamiltonian system. We shall give an existence result for these systems, for a large class of Hamiltonians $H(u, v)$.

Let $\Omega \subset \mathbf{R}^N$ be a smooth bounded domain and let $z = (u, v)^t$, with $v = (v_1, \dots, v_N)^t$ and

$$z : \Omega \subset \mathbf{R}^N \longrightarrow \mathbf{R}^{N+1}. \quad (6.1)$$

The Euclidian norm in $\mathbf{R}^m$ will be denoted by $|\cdot|$. A multiparameter Hamiltonian system is given by equations

$$(II) \begin{cases} -\operatorname{div} v = H_u(u, v), & \text{in } \Omega, \\ \nabla u = H_v(u, v), & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases} \quad (6.2)$$

$$(6.3)$$

$$(6.4)$$

where $H \in C^1(\mathbf{R}^{N+1}, \mathbf{R})$ and $u \in C^1(\Omega) \cap C_0(\overline{\Omega})$, $v \in (C^1(\Omega))^N \cap (C(\overline{\Omega}))^N$. As before the system can be viewed, formally, as the Euler-Lagrange equations of the Lagrangian

$$\mathcal{L}(z) = \int_{\Omega} v \cdot \nabla u - \int_{\Omega} H(u, v). \quad (6.5)$$

First we derive a non-existence theorem for Problem (II), by using variational identities [Mi,PS,vdV1].

Theorem 6.1. *Let $z = (u, v)^t$, with $u \in C^1(\Omega) \cap C_0(\overline{\Omega})$, $v \in (C^1(\Omega))^N \cap (C^1(\Omega))^N$, be a solution of Problem (II). Suppose*

$$1) \quad \Omega \text{ is star-shaped}, \quad (6.6)$$

$$2) \quad H \in C^1(\mathbf{R}^{N+1}, \mathbf{R}), \quad H(0, 0) = 0, \quad (6.7)$$

$$3) \quad H \geq 0, \quad \forall z, \quad a + b = N - 1, \\ auH_u(z) + bv \cdot H_v(z) - NH(z) > 0, \quad \forall z \neq 0. \quad (6.8)$$

Then $z = (u, v)^t = (0, 0)^t$ in Ω .

Proof of Theorem 6.1. The Lagrangian density for Problem (II) is

$$L = v \cdot \nabla u - H(u, v). \quad (6.9)$$

We apply Theorem 2.1 of [vdV1] taking $h^i(x) = x^i$, $a_{kl}(x) = a_k$ if $k = l = 1, \dots, N + 1$ and $a_{kl}(x) = 0$ if $k \neq l$. We obtain, using the boundary condition $u = 0$ on $\partial\Omega$,

$$\begin{aligned} & \oint_{\partial\Omega} \left((x(v \cdot \nabla u - H(u, v)) - x^j \frac{\partial u}{\partial x^j} v), n \right) ds = \\ & - \oint_{\partial\Omega} H(0, v)(x, n) ds = \\ & \int_{\Omega} \left(Nv \cdot \nabla u - NH(u, v) - v \cdot \nabla u - a_1 v \cdot \nabla u + a_1 u H_u(u, v) \right. \\ & \quad \left. - \sum_{k=2}^{N+1} a_k v_{k-1} \frac{\partial u}{\partial x^{k-1}} + \sum_{k=2}^{N+1} a_k v_{k-1} H_{v_{k-1}}(u, v) \right) dx = \\ & \sum_{k=2}^{N+1} (N - 1 - a_1 - a_k) \int_{\Omega} v_{k-1} \frac{\partial u}{\partial x^{k-1}} dx + \sum_{k=2}^{N+1} \int_{\Omega} a_k v_{k-1} H_{v_{k-1}}(u, v) dx \\ & \quad + \int_{\Omega} (a_1 H_u(u, v) - NH(u, v)) dx. \end{aligned}$$

We want the parameters $a_1, \dots, a_{N+1}$ to satisfy the following N equations

$$N - 1 = a_1 + a_k, \quad k = 2, \dots, N + 1,$$

which yields $a_2 = a_3 = \dots = a_{N+1}$. Setting $a_1 = a$, $a_2 = b$ with $a + b = N - 1$, we finally obtain the identity

$$\begin{aligned} & - \oint_{\partial\Omega} H(0, v)(x, n) ds = \\ & \int_{\Omega} (auH_u(u, v) + bv \cdot H_v(u, v) - NH(u, v)) dx. \end{aligned} \quad (6.10)$$

Suppose now Problem (II) has a non-trivial regular solution $z \neq 0$ in Ω . Clearly by 1) and 3) the left hand side of (6.10) is non-positive. However if $z \neq 0$ in Ω , there is a measurable subset $\Omega_c \subset \Omega$ of positive measure, where $z \neq 0$, and thus by 2) and 3) the right hand side of (6.10) is strictly positive. Because solutions of Problem (II) satisfy identity (6.10), the hypotheses of Theorem 6.1 imply that $z = 0$, which proves the theorem. $\square$

Example: Consider the case $H(u, v) = \frac{1}{p+1}|u|^{p+1} + \frac{1}{q+1}|v|^{q+1}$. Then from Theorem 6.1 we deduce a necessary condition on the exponents p and q for existence of solutions of Problem (II), in the case Ω is star-shaped. On p and q we therefore impose the condition

$$\frac{1}{p+1} + \frac{1}{q+1} > \frac{N-1}{N}, \quad N \geq 1, \quad p, q > 0. \quad (6.11)$$

In order to state an existence theorem we proceed as follows.

According to Section 5 we look for non-trivial solutions of Problem (II) as critical points of $\mathcal{L}$. We now introduce the spaces on which $\mathcal{L}$ is defined:

$$T = \nabla, \quad \text{with } D(T) = H_0^1(\Omega) \quad (6.12)$$

and

$$H_1 = L^2(\Omega), \quad H_2 = (L^2(\Omega))^N, \quad \text{and } H = (L^2(\Omega))^{N+1}. \quad (6.13)$$

In the next lemma we summarize the required properties about the operators T , T^* and their domains.

Lemma 6.2. *The operator T , defined by (6.12), satisfies assumption (H_T) and its range $R(T)$ is closed in H_2 . The domain of the adjoint T^* is given by*

$$\begin{aligned} D(T^*) &= H_d^1(\Omega) = \{v \in (L^2(\Omega))^N; \operatorname{div} v \in L^2(\Omega)\} \\ &= \{v \in (L^2(\Omega))^N; v = v_1 + v_2, \operatorname{div} v_1 = 0 \text{ and} \\ &\quad v_2 = \nabla u, \text{ with } u \in H^2 \cap H_0^1(\Omega)\}. \end{aligned}$$

Moreover $T^*v = -\operatorname{div} v$ for $v \in H_d^1(\Omega)$.

Proof of Lemma 6.2. The operator T is closed since $H_0^1(\Omega)$ is a closed subspace of $H^1(\Omega)$ and the graph-norm of T is an equivalent norm on the Sobolev space $H^1(\Omega)$. The closedness of the range follows directly from the Poincaré-inequality. From Green's formulas it follows that $H_d^1(\Omega) \subset D(T^*)$ and $T^*v = -\operatorname{div} v$ for all $v \in D(T^*)$. Finally, we prove that $D(T^*) \subset H_d^1(\Omega)$.

Let $v \in D(T^*)$, then there is $M > 0$, such that

$$\left| \int_{\Omega} v \nabla u dx \right| \leq M \left(\int_{\Omega} u^2 dx \right)^{1/2}, \quad \forall u \in H_0^1(\Omega).$$

Since $H_0^1(\Omega)$ is dense in $L^2(\Omega)$, there exists a function $w \in L^2(\Omega)$ by the Riesz representation theorem, such that

$$\int_{\Omega} w u dx = \int_{\Omega} v \nabla u dx, \quad \forall u \in H_0^1(\Omega).$$

Because $\partial\Omega \in C^2$, there exists a function $\phi \in H^2 \cap H_0^1(\Omega)$, such that $-\Delta\phi = w$. Hence

$$\int_{\Omega} (v - \nabla\phi) \nabla u dx = 0, \quad \forall u \in H_0^1(\Omega),$$

or equivalently $v - \nabla\phi \in (L^2(\Omega))^N$ and $\operatorname{div}(v - \nabla\phi) = 0$ in the sense of distributions. $\square$

We have now the operator ∂_T

$$\partial_T = \begin{pmatrix} 0 & -\operatorname{div} \\ \nabla & 0 \end{pmatrix} : H_0^1(\Omega) \times H_d^1(\Omega) \subset (L^2(\Omega))^{N+1} \rightarrow (L^2(\Omega))^{N+1}, \quad (6.14)$$

which is of course a selfadjoint operator by Proposition 2.2, with domain $H_0^1(\Omega) \times H_d^1(\Omega)$. Our next step is now to describe the interpolation spaces E_α , as defined in (4.22), on which the quadratic form $Q(z) = \int_\Omega v \cdot \nabla u dx$ can be continuously extended (see Section 4). Let $E_0 = L^2(\Omega) \times H_d^1(\Omega)$, $E_1 = H_0^1(\Omega) \times (L^2(\Omega))^N$, then by (4.39) and Definition 4.2 we have

$$\begin{aligned} E_\alpha &= [L^2(\Omega), H_0^1(\Omega)]_{\alpha,2} \times [H_d^1(\Omega), (L^2(\Omega))^N]_{\alpha,2} = [E_0, E_1]_{\alpha,2} \\ &= \Theta_1^\alpha(\Omega) \times \Theta_2^{1-\alpha}(\Omega), \quad \alpha \in [0, 1]. \end{aligned} \quad (6.15)$$

In the next lemma we characterize these interpolation spaces.

Lemma 6.3. *The interpolation spaces E_α are*

$$E_\alpha = \Theta_1^\alpha(\Omega) \times \Theta_2^{1-\alpha}(\Omega), \quad \alpha \in [0, 1],$$

where

$$\Theta_1^\alpha(\Omega) = \begin{cases} H^\alpha(\Omega), & 0 \leq \alpha < 1/2, \\ H_{00}^\alpha(\Omega), & \alpha = 1/2, \\ H_0^\alpha(\Omega), & 1/2 < \alpha \leq 1, \end{cases} \quad (6.16)$$

and

$$\begin{aligned} \Theta_2^{1-\alpha}(\Omega) &= \left\{ v \in (L^2(\Omega))^N; v = v_1 + v_2, \text{ such that } \operatorname{div} v_1 = 0 \right. \\ &\quad \left. \text{and } v_2 = \nabla u \text{ with } u \in H^{2-\alpha} \cap H_0^1(\Omega) \right\}. \end{aligned} \quad (6.17)$$

Proof of Lemma 6.3. We have $E_0 = L^2(\Omega) \times H_d^1(\Omega)$, $E_1 = H_0^1(\Omega) \times (L^2(\Omega))^N$ and by (4.39) $E_\alpha = [E_0, E_1]_\alpha$. From [LM] we obtain (6.16) and from [Gr,HV1] we obtain (6.17). $\square$

The interpolation spaces E_α can also be expressed in terms of convergent series using the fact that $(-\Delta)^{-1} : L^2(\Omega) \rightarrow L^2(\Omega)$ is compact. Thus let $\{\phi_k\}$, $\{\psi_k\}$ be the eigenfunctions of respectively T^*T and TT^* and $\{\mu_k\}$ the non-negative eigenvalues. We have:

$$\Theta_1^\alpha(\Omega) = \left\{ u = \sum_{k=1}^{\infty} u_k \phi_k; \sum_{k=1}^{\infty} \mu_k^\alpha u_k^2 < \infty \right\} \quad (6.18)$$

$$\begin{aligned} \Theta_2^{1-\alpha}(\Omega) &= \left\{ v = \sum_{k=1}^{\infty} v_k \frac{\nabla \phi_k}{\mu_k^{1/2}}; \sum_{k=1}^{\infty} \mu_k^{1-\alpha} v_k^2 < \infty \right\} \oplus N(T^*) \\ &= \left\{ v = \sum_{k=1}^{\infty} v'_k \nabla \phi_k; \sum_{k=1}^{\infty} \mu_k^{2-\alpha} v_k'^2 < \infty \right\} \oplus N(T^*). \end{aligned} \quad (6.19)$$

From Lemma 6.3 and the Sobolev embeddings (using the interpolation property) it follows that the spaces $E_\alpha \ominus N(\partial_T)$ are embedded in $L^{\frac{2N}{N-2\alpha}}(\Omega) \times (L^{\frac{2N}{N-2+2\alpha}})^N$, where $N(\partial_T) = \{0\} \times N(T^*)$. If $N > 1$, then $\dim N(T^*) = \infty$. One deduces directly from (6.18) and (6.19) that the embedding

$$E_\alpha \ominus N(\partial_T) \hookrightarrow E_{\alpha-\epsilon} \ominus N(\partial_\nabla), \quad \forall \alpha \in (\epsilon, 1), \quad \epsilon > 0, \quad (6.20)$$

is compact. Summarizing we have

Lemma 6.4. *Let $\alpha \in (0, 1)$. Then the embedding*

$$E_\alpha \ominus E^0 = E^+ \oplus E^- \hookrightarrow L^{p_\alpha+1} \times (L^{q_\alpha+1})^N, \quad (6.21)$$

is compact, if

$$p_\alpha + 1 < \frac{2N}{N-2\alpha}, \quad q_\alpha + 1 < \frac{2N}{N-2+2\alpha}.$$

In order to apply Theorem 4.5.1 we take

$$X = L^{p_\alpha+1} \times (L^{q_\alpha+1})^N, \quad \alpha \in (0, 1). \quad (6.22)$$

Now we are in a position to consider Problem (II) as a ∂_T -system in $((L^2)^{N+1}, X)$ with Lagrangian (6.5). On the Hamiltonian density $H(z)$ we impose the following hypotheses:

$$(g1) \quad c_1|u|^p \leq |H_u(u, v)| \leq c_2|u|^p + c_3 \sum_k |u|^{\alpha_k-1} |v|^{\beta_k}, \quad H(0, 0) = 0,$$

$$c_4|v_i|^q \leq |H_{v_i}(u, v)| \leq c_5 \sum_k |u|^{\alpha_k} |v|^{\beta_k-1} + c_6|v|^q, \quad i = 1, \dots, N,$$

$$\text{with } \frac{1}{p+1} + \frac{1}{q+1} > \frac{N-1}{N}, \quad p, q > 1, \quad \frac{\alpha_k}{p+1} + \frac{\beta_k}{q+1} = 1, \quad \alpha_k, \beta_k > 1,$$

$$(g2) \quad H_z(z) \text{ is strictly monotone,}$$

$$(g3) \quad \frac{1}{2}z \cdot H_z(z) - H(z) \geq c_7|u|^{p+1} + c_8|v|^{q+1} - c_9.$$

Remark 6.5. Clearly in the case of the pure powers as stated above the hypotheses (g1)-(g3) on H are met. Even if we consider a more general form, i.e.

$$H(z) = \frac{1}{p+1}|u|^{p+1} + c \sum_k |u|^{\alpha_k} |v|^{\beta_k} + \frac{1}{q+1}|v|^{q+1}, \quad (6.23)$$

the conditions (g1)-(g3) are satisfied whenever

$$\frac{1}{p+1} + \frac{1}{q+1} > \frac{N-1}{N} \quad \text{and} \quad \frac{\alpha_k}{p+1} + \frac{\beta_k}{q+1} = 1, \quad \alpha_k, \beta_k > 1. \quad (6.24)$$

In the complementary case $\frac{1}{p+1} + \frac{1}{q+1} < \frac{N-1}{N}$, a non-existence result follows from Theorem 6.1, which reveals the 'sharpness' of these conditions especially in the pure power case.

Theorem 6.6. *Consider Problem (II) and let $H(\cdot)$ satisfy (g1)-(g3). Then Problem (II) has at least one non-trivial solution $z \in E_\alpha \cap X$ for some appropriate $\alpha \in (0, 1)$ depending on p and q .*

Proof of Theorem 6.6. We start off verifying conditions (s1)-(s4) of Propostion 5.1 in the case of Problem (II). Condition (s1) follows directly from Lemma 6.4. Conditions (s2) and (s3) are easily checked. In order to verify (s4) we consider $v \in N(T^*)$ and $\tilde{v}_n \in (C_0^\infty(\Omega))^N$ converging to v in $(L^2(\Omega))^N$. Set $\phi_n = -\operatorname{div} \tilde{v}_n$ and let $w_n \in H^{1, q_\alpha+1} \cap H_0^1(\Omega)$ satisfying

$$-\Delta w_n = \phi_n.$$

Finally we define $v_n = \tilde{v}_n - \nabla w_n$. Clearly we have $v_n \in (L^{q_\alpha+1}(\Omega))^N$, $\operatorname{div} v_n = 0$ in the sense of distributions and since $\phi_n = -\operatorname{div} \tilde{v}_n$ converges to $-\operatorname{div} v = 0$ in $H^{-1}(\Omega)$, we also have that $w_n \rightarrow 0$ in $H_0^1(\Omega)$. Therefore the sequence $\{v_n\}$ converges to v in $(L^2(\Omega))^N$, which completes the verification of condition (s4).

We recall that $\mathcal{H}(z) = \int_\Omega H(z) dx$. If we take a suitable $\alpha \in (0, 1)$ it follows from (g1) that $\mathcal{H} \in C^1(X_\alpha, \mathbf{R})$. From (g1)-(g2) we derive that

$$d_1|u|^{p+1} + d_2|v|^{q+1} \leq H(z) \leq d_3|u|^{p+1} + d_4|v|^{q+1}, \quad p, q > 1, \quad (6.25)$$

To see this one argues as follows. For any $z \in \mathbf{R}^{N+1}$ we have the identity

$$H(z) = \int_0^1 \frac{d}{ds} H(sz) ds = \int_0^1 (\nabla H(sz), sz) \frac{ds}{s} = \int_0^1 H_z(sz) sz \frac{ds}{s}.$$

Using the strict monotonicity of H_z (condition (g2)) one can easily derive that $H_u(z)u > 0$ and $H_{v_i}(z)v_i > 0$ for all $z \in \mathbf{R}^{N+1}$. Using this we derive from the previous identity

$$\begin{aligned} H(z) &= \int_0^1 H_u(sz) su \frac{ds}{s} + \sum_{i=1}^N \int_0^1 H_{v_i}(sz) sv_i \frac{ds}{s} \\ &= \int_0^1 |H_u(sz)| |su| \frac{ds}{s} + \sum_{i=1}^N \int_0^1 |H_{v_i}(sz)| |sv_i| \frac{ds}{s}. \end{aligned}$$

Using (g1) now we obtain

$$\begin{aligned} c_1 \int_0^1 |u|^{p+1} s^p ds + c_2 \sum_{i=1}^N \int_0^1 |v_i|^{q+1} s^q ds &\leq H(z) \leq c_3 \int_0^1 |u|^{p+1} s^p ds + \\ c_4 \sum_k \int_0^1 |su|^{\alpha_k} |sv|^{\beta_k} \frac{ds}{s} + c_5 \sum_{i=1}^N \sum_k \int_0^1 |su|^{\alpha_k} |sv|^{\beta_k} \frac{ds}{s} &+ c_6 \sum_{i=1}^N \int_0^1 |v_i|^{q+1} s^q ds. \end{aligned}$$

Working out the integral over $[0, 1]$ and using Young's inequality on the mixed terms we get

$$a_1|u|^{p+1} + a_2 \sum_{i=1}^N |v_i|^{q+1} \leq H(z) \leq a_3|u|^{p+1} + a_4 \sum_{i=1}^N |v_i|^{q+1}.$$

Using the equivalence of norms in $\mathbf{R}^N$, (6.25) immediately follows. By taking $\phi_1(t) = d_1|t|^{p+1}$, $\phi_2(t) = d_2|t|^{q+1}$, $\phi_3(t) = d_3|t|^{p+1}$, $\phi_4(t) = d_4|t|^{q+1}$, (k3) follows by setting $a = p+1 > 2$, $b = q+1 > 2$. Hypotheses (k1) and (k4) follow directly from (g1), (g2) and (g3).

Suppose that there exists a $z_0 \in X$, such that

$$\int_{\Omega} (H_z(z_n) - H_z(z_0)) \cdot (z_n - z_0) dx \longrightarrow 0,$$

along a subsequence. Then, since $h(z) = H_z(z)$ is strictly monotone, we have

$$(H_z(z_{n_k}) - H_z(z_0)) \cdot (z_{n_k} - z_0) \longrightarrow 0, \quad \text{a.e. in } \Omega \quad (6.26)$$

and

$$(H_z(z_{n_k}) - H_z(z_0)) \cdot (z_{n_k} - z_0) \leq g_1, \quad \forall n_k, \quad (6.27)$$

where g_1 is a fixed non-negative function in $L^1(\Omega)$. From (6.26) we deduce that $z_{n_k} \rightarrow z_0$ a.e. in Ω . Finally, by using (6.27) and (g1), one can prove that

$$|u_{n_k}|^{p+1} \leq g_2 \in L^1, \quad |v_{n_k}|^{q+1} \leq g_3 \in L^1 \quad (\text{fixed functions}). \quad (6.28)$$

Combining the pointwise limit $z_{n_k} \rightarrow z_0$ and (6.28) and using Lebesgue's dominated convergence theorem we obtain

$$z_{n_k} \longrightarrow z_0, \quad \text{in } X. \quad (6.29)$$

So it remains to derive (6.28) from (6.27), (g1) and (g2). We have

$$\begin{aligned} c_1|u_n|^{p+1} + c_4 \sum_{i=1}^N |v_{n,i}|^{q+1} &\leq |H_u(z_n)||u_n| + \sum_{i=1}^N |H_{v_i}(z_n)||v_{n,i}| = H_u(z_n) \cdot z_n \\ &\leq g_1 + H_z(z_0) \cdot z_n + H_z(z_n) \cdot z_0 - H_z(z_0) \cdot z_0 \\ &\leq g_1 + H_z(z_0) \cdot z_n + H_z(z_n) \cdot z_0 \\ &= g_1 + |H_u(z_0)||u_n| + \sum_{i=1}^N |H_{v_i}(z_0)||v_{n,i}| \\ &\quad + |H_u(z_n)||u_0| + \sum_{i=1}^N |H_{v_i}(z_n)||v_{0,i}| \\ &\leq g_1 + c_2|u_0|^p|u_n| + c_3 \sum_k |u_0|^{\alpha_k-1}|u_n||v_0|^{\beta_k} \\ &\quad + c_5N \sum_k |u_0|^{\alpha_k}|v_0|^{\beta_k-1}|v_{n,i}| + c_6 \sum_{i=1}^N |v_0|^q|v_{n,i}| \\ &\quad + c_2|u_n|^p|u_0| + c_3 \sum_k |u_n|^{\alpha_k-1}|u_0||v_n|^{\beta_k} \\ &\quad + c_5N \sum_k |u_n|^{\alpha_k}|v_n|^{\beta_k-1}|v_{0,i}| + c_6 \sum_{i=1}^N |v_n|^q|v_{0,i}| \\ &\leq g_1 + \epsilon|u_n|^{p+1} + \epsilon|v_n|^{q+1} + C_\epsilon|u_0|^{p+1} + C_\epsilon|v_0|^{q+1}. \end{aligned} \quad (6.30)$$

Taking ϵ small enough in (6.30) easily yields, after some rearrangements,

$$|u_n|^{p+1} + |v_n|^{q+1} \leq Cg_1 + C'|u_0|^{p+1} + C''|v_0|^{q+1}.$$

The latter inequality now proves (6.28). This shows that also (k2) is satisfied.

Applying Theorem 5.1 finally gives a solution $z \in E_\alpha \cap X$, for some fixed $\alpha \in (0, 1)$, satisfying the equation

$$\partial_T z = H_z(z), \quad \text{in } (E_\alpha \cap X)^*,$$

which proves the theorem. $\square$

If the Hamiltonian is $H(u, v) = \frac{r-1}{r}|v|^{\frac{r}{r-1}} + F(u)$, with $F(u) \sim |u|^{p+1}$ and $\frac{r}{r-1} < p+1 < \frac{Nr}{N-r}$, Problem (II) is equivalent to the quasilinear r -Laplace equation $-\Delta_r u = f(u)$. The above example applies when $\frac{2N}{N+2} < r \leq 2$.

2. A reaction-diffusion system.

Consider the system

$$(III) \begin{cases} -\Delta v = H_u(u, v), & \text{in } \Omega, \\ -\Delta u = H_v(u, v), & \text{in } \Omega, \\ u = v = 0, & \text{on } \partial\Omega, \end{cases} \quad \begin{matrix} (6.31) \\ (6.32) \\ (6.33) \end{matrix}$$

where $\Omega \subset \mathbf{R}^N$ is a bounded domain with a smooth boundary $\partial\Omega$ and $(u, v) \in (C^2(\Omega) \cap C^1(\overline{\Omega}))^2$. This system was studied in [FF,HV1] using variational techniques. The Lagrangian associated with Problem (III) is

$$\mathcal{L}(z) = \int_{\Omega} \nabla v \cdot \nabla u dx - \int_{\Omega} H(u, v) dx. \quad (6.34)$$

In [FF,HV1] system (III) was considered in $L^2(\Omega)^2$, with $T = T^* = -\Delta$, $D(-\Delta) = H^2 \cap H_0^1(\Omega)$ and $E_\alpha = D((-\Delta)^\alpha) \times D((-\Delta)^{1-\alpha})$. For these spaces one has the continuous injections

$$i : E_\alpha \longrightarrow L^{p_\alpha+1} \times L^{q_\alpha+1}, \quad (6.35)$$

with

$$\frac{1}{p_\alpha+1} + \frac{1}{q_\alpha+1} = \frac{N-2}{N}, \quad p_\alpha, q_\alpha \geq 1. \quad (6.36)$$

The embeddings (6.35) will not be valid if we extend the curve of critical exponents (6.36) up to its asymptots at $p = \frac{2}{N-2}$ and $q = \frac{2}{N-2}$. However for applications one would also like to consider Hamiltonians $H(z)$ which are not necessarily superquadratic in both terms u and v . The goal of this example is therefore to show that our general theory developed in Section 4 and 5 enables us to formulate $\mathcal{L}$ on a different family of spaces E_α , which has embeddings in $L^{p_\alpha+1} \times L^{q_\alpha+1}$ for all positive elements of the hyperbola (6.36) (i.e. up to its asymptots).

We assume that $\Omega = B_R \subset \mathbf{R}^N$, $R < \infty$. We consider the quadratic form $Q(z) = \int_{\Omega} \nabla u \cdot \nabla v dx$, which is properly defined on $C_0^\infty(\Omega) \times C_0^\infty(\Omega)$ and we restrict ourselves to radially symmetric functions, denoted by $C_{0,r}^\infty(B_R)$. The estimate

$$|Q(z)| \leq \left(\int_{\Omega} |\nabla u|^2 |x|^{N-2} dx \right)^{1/2} \left(\int_{\Omega} |\nabla v|^2 |x|^{2-N} dx \right)^{1/2}, \quad (6.37)$$

indicates that Q can be extended by continuity to Hilbert space $G = H_0^1(d\mu_{N-2}) \times H_0^1(d\mu_{2-N})$, where $H_0^1(d\mu_{N-2})$ and $H_0^1(d\mu_{2-N})$ are the closures of $C_{0,r}^\infty(\Omega)$ in respectively $H^1(d\mu_{N-2})$ and $H^1(d\mu_{2-N})$ ⁵. In $H_0^1(d\mu_{N-2})$ and $H_0^1(d\mu_{2-N})$ we use the equivalent norms $\left(\int_{\Omega} |\nabla u|^2 |x|^{N-2} dx \right)^{1/2}$ and $\left(\int_{\Omega} |\nabla u|^2 |x|^{2-N} dx \right)^{1/2}$. Notice that if one interchanges u and v the quadratic form Q can be extended to the space $F = H_0^1(d\mu_{2-N}) \times H_0^1(d\mu_{N-2})$. In order to apply our theory we would like to find an operator T , with closed range and spaces H_1, H_2 such that $2|Q|(z)$ can serve as an equivalent norm. Because the quadratic form Q is symmetric on $C_{0,r}^\infty(\Omega) \times C_{0,r}^\infty(\Omega)$ we can give a simple procedure to find T . We set $H_1 = H_2 = H_0^1(d\mu_{N-2})$ and $H = H_1 \times H_2$.

Let us denote by $[\cdot, \cdot]_\theta$, $\theta \in [0, 1]$ the real ($p = 2$) and complex interpolation between Hilbert spaces. We have $[G, F]_\theta = [H_0^1(d\mu_{N-2}), H_0^1(d\mu_{2-N})]_\theta \times [H_0^1(d\mu_{2-N}), H_0^1(d\mu_{N-2})]_\theta$. For these interpolation spaces one has the following lemma:

Lemma 6.7. *Let $N > 2$ and $\Omega \subset \mathbf{R}^N$ a smooth bounded domain. Consider the interpolation between $H_0^1(d\mu_{2-N})$ and $H_0^1(d\mu_{N-2})$, i.e. $[H_0^1(d\mu_{2-N}), H_0^1(d\mu_{N-2})]_\theta$. We have that the interpolation norms $\|\cdot\|_\theta$ are equivalent to the norms*

$$\left(\int_{\Omega} |\nabla \cdot|^2 |x|^{(2\theta-1)(N-2)} dx \right)^{1/2}. \quad (6.38)$$

Furthermore

$$[H_0^1(d\mu_{2-N}), H_0^1(d\mu_{N-2})]_\theta = H_0^1(d\mu_{(2\theta-1)(N-2)}), \quad (6.39)$$

where $H_0^1(d\mu_{(2\theta-1)(N-2)})$ is the closure of $C_{0,r}^\infty(\Omega)$ in $H^1(d\mu_{(2\theta-1)(N-2)})$, using the equivalent norms given in (6.38). $\square$

We notice now that the space $H_0^1(d\mu_{2-N})$ is densely and continuously embedded in $H_0^1(d\mu_{N-2})$ if $N > 2$. From the procedure described in for example [LM] we can find an unbounded positive selfadjoint operator T , with closed range, acting from $H_0^1(d\mu_{2-N}) \subset H_1$ to H_1 , with $D(T) = H_0^1(d\mu_{2-N})$ and $\|u\|_{H_0^1(d\mu_{2-N})} = \|Tu\|_{H_0^1(d\mu_{N-2})}$. Using Lemma 6.7 we have that $[H_0^1(d\mu_{2-N}), H_0^1(d\mu_{N-2})]_{1/2} = H_0^1(\Omega) = D(T^{1/2})$ and

$$\int_{\Omega} \nabla u \cdot \nabla v dx = (u, v)_{H_0^1} = (T^{1/2}u, T^{1/2}v)_{H_1}. \quad (6.40)$$

⁵ For $\beta < N$, $L^2(d\mu_\beta) \subset L_{\text{loc}}^1$ and $H^1(d\mu_\beta) = \{u \in L^2(d\mu_\beta); Du \in L^2(d\mu_\beta), \text{ (in the distributional sense) }\}$.

We see from (6.40) that

$$Q(z) = (Tu, v)_{H_1} = (u, Tv)_{H_1}, \quad (6.41)$$

for $z = (u, v)^t \in H_0^1(d\mu_{2-N}) \times H_0^1(d\mu_{2-N}) \subset H$.

From (4.22) and (4.39) we have that $E_\alpha = D(|T|^\alpha) \times D(|T|^{1-\alpha}) = [E_0, E_1]_\alpha$, where $E_0 = G$ and $E_1 = F$. One also derives from Section 4 that $2|Q|(z)$ can serve as an equivalent norm on E_α . From Lemma 6.7, Theorem 4.6, (4.24) and (4.36) one obtains that

$$\begin{aligned} \|z\|_{E_\alpha}^2 &= \|T^\alpha u\|_{H_1}^2 + \|T^{1-\alpha} v\|_{H_1}^2 \\ &= \int_\Omega |\nabla u|^2 |x|^{(1-2\alpha)(N-2)} dx + \int_\Omega |\nabla v|^2 |x|^{(2\alpha-1)(N-2)} dx. \end{aligned} \quad (6.42)$$

We set

$$\Theta^\alpha = \Theta_1^\alpha = \Theta_2^\alpha = [H_0^1(d\mu_{2-N}), H_0^1(d\mu_{N-2})]_{1-\alpha}. \quad (6.43)$$

For the spaces $H_0^1(d\mu_{N-2})$ and $H_0^1(d\mu_{2-N})$ one has the following embeddings in L^q -spaces (see eg. [MS], [M], [HL]).

Lemma 6.8. *Let $N \geq 3$ and $0 < R < \infty$. Then, for $\gamma \in (2-N, N-2)$ the embeddings*

$$i : H_0^1(d\mu_\gamma) \longrightarrow L^q(B_R), \quad 1 < q < \frac{2N}{N + \gamma - 2},$$

are compact.

For the sake of completeness we shall give a proof of Lemma 6.8. Before going to the proof we need a general Hardy-Littlewood inequality:

Lemma 6.9. *(Hardy-Littlewood inequality [HL]) Let $u \in C^\infty(\overline{\mathbf{R}^+})$, such that $u(r) \equiv 0$ for $r \geq r_0$, for some $r_0 > 0$. Then*

$$\left(\int_0^\infty |u|^q r^{aq} dr \right)^{1/q} \leq C \left(\int_0^\infty |u'|^p r^{bp} dr \right)^{1/p}, \quad (6.44)$$

provided p, q, a and b satisfy the relations

$$\frac{1}{p} - \frac{1}{q} + b - a = 1, \quad a + \frac{1}{q} > 0, \quad p > 1, \quad 0 \leq b - a \leq 1.$$

Proof of Lemma 6.9: We set $s = e^y$, $r = e^x$ and $v(x) = e^{(1+a+1/q)x} u'(e^x)$.

$$\begin{aligned} \left(\int_0^\infty |u(r)|^q r^{aq} dr \right)^{1/q} &= \left(\int_0^\infty \left| \int_r^\infty u'(s) ds \right|^q r^{aq} dr \right)^{1/q} \\ &= \left(\int_{-\infty}^\infty \left| \int_x^\infty u'(e^y) e^y dy \right|^q e^{aqx} e^x dx \right)^{1/q} \\ &= \left(\int_{-\infty}^\infty \left| \int_x^\infty e^{(a+1/q)x} e^y u'(e^y) dy \right|^q dx \right)^{1/q} \\ &= \left(\int_{-\infty}^\infty \left| \int_x^\infty e^{(a+1/q)x} e^y e^{-(1+a+1/q)y} v(y) dy \right|^q dx \right)^{1/q} \\ &= \left(\int_{-\infty}^\infty \left| \int_x^\infty e^{(a+1/q)(x-y)} v(y) dy \right|^q dx \right)^{1/q}. \end{aligned}$$

We now set

$$E(x) = \begin{cases} e^{(a+1/q)x}, & x \leq 0, \\ 0, & x > 0. \end{cases}$$

This yields $\int_x^\infty e^{(a+1/q)(x-y)} v(y) dy = (E * v)(x) = \int_{-\infty}^\infty E(x-y) v(y) dy$. Using the hypotheses on a, b, p and q one has that $1/d = 1/q - 1/p + 1 = b - a$ and thus $1 \leq d \leq \infty$. Therefore

$$\begin{aligned} \left(\int_0^\infty |u(r)|^q r^{aq} dr \right)^{1/q} &= \left(\int_{-\infty}^\infty |(E * v)(y)|^q dx \right)^{1/q} \\ &\leq \left(\int_{-\infty}^\infty |E(x)|^d dx \right)^{1/d} \left(\int_{-\infty}^\infty |v(x)|^p dx \right)^{1/p}, \quad 1 < p \leq \infty \\ &= \left(\frac{1}{(a+1/q)d} \right)^{1/d} \left(\int_{-\infty}^\infty |v(x)|^p dx \right)^{1/p}, \quad a + \frac{1}{q} > 0 \\ &= \left(\frac{1}{(a+1/q)d} \right)^{1/d} \left(\int_0^\infty |u'(r)|^p r^{p(1+a+1/q)-1} dr \right)^{1/p} \\ &= \left(\frac{1}{(a+1/q)d} \right)^{1/d} \left(\int_0^\infty |u'(r)|^p r^{bp} dr \right)^{1/p}, \end{aligned}$$

with $bp = p(1 + a + \frac{1}{q}) - 1$, where we used Young's inequality for convolutions and the hypotheses on a, b, p and q . This concludes the proof. $\square$

Now we are in a position to prove Lemma 6.8.

Proof of Lemma 6.8: In order to prove this lemma it suffices to prove the inequality

$$\left(\int_0^R |u|^q r^{N-1} dr \right)^{1/q} \leq C_R \left(\int_0^R |u'|^2 r^{N-1+\gamma} dr \right)^{1/p}, \quad (6.45)$$

for all $u \in C^\infty(\overline{\mathbf{R}^+})$ with $u(r) \equiv 0$ for $r \geq r_0$, for some $0 < r_0 < R$. To prove the latter inequality we distinguish two cases;

I. $0 < \gamma < N - 2$: Let $1 < p < 2$, $aq = bp = N - 1$, then we have from Lemma 6.9 that

$$\left(\int_0^R |u|^q r^{N-1} dr \right)^{1/q} \leq C \left(\int_0^R |u'|^p r^{N-1} dr \right)^{1/p},$$

with $q = \frac{pN}{N-p}$. We proceed as follows

$$\begin{aligned} \left(\int_0^R |u|^q r^{N-1} dr \right)^{1/q} &\leq C \left(\int_0^R |u'|^p r^{\delta} r^{-\delta} r^{N-1} dr \right)^{1/p}, \quad \delta > 0 \\ &\leq C \left(\int_0^R |u'|^2 r^{\delta \frac{2}{p}} r^{N-1} dr \right)^{1/2} \left(\int_0^R r^{-\delta \frac{2}{2-p} + N-1} dr \right)^{\frac{2-p}{2}} \\ &\leq C_R \left(\int_0^R |u'|^2 r^{N-1+\gamma} dr \right)^{1/2}, \end{aligned}$$

where we chose $\delta \frac{2}{p} = \gamma$ and $1 < p < \frac{2N}{\gamma+N}$. For fixed $\gamma \in (0, N-2)$ this yields that one can choose p such that $q \in (\frac{N}{N-1}, \frac{2N}{N+\gamma-2})$. Using Hölder's inequality once more we obtain (6.45) for all $q \in (1, \frac{2N}{N+\gamma-2})$.

II. $2 - N < \gamma \leq 0$: It is enough to prove (6.45) for $q = \frac{2N}{N+\gamma-2}$. By choosing $a = (N-1)/q$, $b = (N+\gamma-1)/2$ and $p = 2$ in Lemma 6.9, we obtain (6.45). This choice of p, q, a and b is possible since $\frac{1}{p} - \frac{1}{q} + b - a = 1$, $a + \frac{1}{q} = \frac{N}{q} > 0$, $p = 2 > 1$ and $\frac{1}{2} < b - a = \frac{2N-2+\gamma}{2N} \leq \frac{2N-2}{2N} < 1$. Using Hölder's inequality proves then that also (6.45) is satisfied in case II.

The compactness of the embedding (6.45) easily follows from the compactness of standard Sobolev embeddings. $\square$

For the spaces E_α we now have the embeddings

Lemma 6.10. *Let $\alpha \in (0, 1)$, then*

$$i : E_\alpha \longrightarrow L^{p_\alpha+1} \times L^{q_\alpha+1}, \quad \text{compactly,}$$

with

$$1 < p_\alpha + 1 < \frac{N}{(1-\alpha)(N-2)}, \quad 1 < q_\alpha + 1 < \frac{N}{\alpha(N-2)}. \quad (6.46)$$

Remark 6.11. Another way of constructing a family of interpolation spaces E_α in the radially symmetric case could be achieved by considering the non-selfadjoint operator $T = -\sigma^\lambda(x)\Delta$ acting from $H_1 = L^2(\sigma^\lambda)$ to $H_2 = L^2(\sigma^{-\lambda})$, where $\sigma(x)$ is a proper radially symmetric weightfunction on B_R . In this particular case λ is fixed and the interpolation spaces E_α vary the order of differentiation, while in the above construction the interpolation spaces E_α have fixed order of differentiation (i.e. 1) and the power of the weightfunction varies as α goes from 0 to 1.

Moreover our choice of the weightfunctions in the above treatment is not unique. If for example $\Omega = \mathbf{R}^N$ our choice of the weightfunction would require a more careful analysis, since the choice would depend on the Hardy-inequalities as given in [M].

Remark 6.12. From the above considerations one can easily derive that the space $H_0^1(\Omega)$ can be seen as 'espace pivot' and then $H_0^1(d\mu_{N-2})$ is the dual space of $H_0^1(d\mu_{2-N})$ with respect to the duality in $H_0^1(\Omega)$. We have the scheme

$$H_0^1(d\mu_{2-N}) \subset H_0^1(\Omega) = (H_0^1(\Omega))^* \subset (H_0^1(d\mu_{2-N}))^* = H_0^1(d\mu_{N-2}).$$

This now clearly indicates the subtle difference between the spaces E_α in [FF,HV1], where $H_0^1(\Omega)$ was employed as 'espace pivot', with $T = -\Delta$.

If $H_1 = H_2$ and one has a $\mathcal{J}$ -quadratic form which is also an innerproduct on a linear subspace of $D(Q) \subset H_1$, one can always perform the above procedure. When the weaker norm $2|Q|(z)$ is not equivalent one can pass to completion in this norm as described in section 4.

At this stage we can give an existence result for critical points of the functional (6.34), which are also weak solutions of the equations (6.31)-(6.33). The spaces introduced in this

example enable us to extend results of [FF] and [HV1] in the radially symmetric case ($\Omega = B_R$). In order to find solutions of Problem (III) one can apply an abstract critical point theorem due to Felmer, which is an extension of the Indefinite Functional Theorem of Benci & Rabinowitz [F,FF]. In that case one can treat general Hamiltonians H . In this example we shall restrict ourselves to convex Hamiltonians and we apply Theorem 5.1.

Theorem 6.13. *Let $\Omega = B_R \subset \mathbf{R}^N$ and $N \geq 5$. Consider Problem (III) with $H(z) = \frac{1}{p+1}|u|^{p+1} + \frac{1}{q+1}|v|^{q+1}$ and*

$$1 > \frac{1}{p+1} + \frac{1}{q+1} > \frac{N-2}{N}, \quad p, q > 0. \quad (6.47)$$

Then $\mathcal{L}(z)$, given by (6.34), has at least one non-trivial radially symmetric critical point $z = (u, v)^t \in E_\alpha$.

Proof of Theorem 6.13. We use the interpolation spaces E_α , $\alpha \in (0, 1)$, as introduced above. In order to prove this theorem we shall apply Theorem 5.1 with the Hamiltonian

$$\mathcal{H}(z) = \int_{\Omega} H(z) dx. \quad (6.48)$$

Because of (6.47) it is automatically clear that (k1) is satisfied for $\mathcal{H}$. As in Example 1 (k2) is also satisfied here. Choosing $\phi_1(t) = \frac{1}{p+1}|t|^{p+1} = \phi_3(t)$ and $\phi_2(t) = \frac{1}{q+1}|t|^{q+1} = \phi_4(t)$, with $a = p+1$, $b = q+1$ and $\frac{1}{p+1} + \frac{1}{q+1} < 1$, we have that the conditions of (k3) are met. Finally, if we set $\theta_1 = \frac{1}{p+1} + \epsilon_1$ and $\theta_2 = \frac{1}{q+1} + \epsilon_2$, such that $\theta_1 + \theta_2 = 1$,

$$\theta_1 < \mathcal{H}_u(z), u > + \theta_2 < \mathcal{H}_v(z), v > - \mathcal{H}(z) =$$

$$\epsilon_1 \int_{\Omega} |u|^{p+1} dx + \epsilon_2 \int_{\Omega} |v|^{q+1} dx \rightarrow \infty, \quad \text{as } \|z\|_X \rightarrow \infty,$$

where $X = L^{p_\alpha+1} \times L^{q_\alpha+1}$, for some appropriate $\alpha \in (0, 1)$ depending on p and q . Finally, because $E^0 = \{0\}$, (s1)-(s4) are satisfied. Notice that $E_\alpha \cap X = E_\alpha$. This completes the verification of the hypotheses of Theorem 5.1, which then yields the existence of a critical point of $\mathcal{L}$ and the equations (6.31) and (6.32) are satisfied in $E'_\alpha = (\Theta^\alpha)' \times (\Theta^{1-\alpha})'$, where E'_α is the dual of E_α with respect to $L^2 \times L^2$ duality (identifying $L^2 \times L^2$ with its dual instead of the identification of H with its dual). $\square$

Remark 6.14. It remains an open question how one should deal with the case of an arbitrary domain Ω . As in the radially symmetric case one would like to introduce weightfunctions associated with the support of the functions on Ω , such that the proper embeddings hold.

7. Refined structures and multiparameter interpolation spaces

In Section 2 we introduced $\mathcal{J}$ -selfadjoint operators A in a Hilbert space H and they are of the form $A = \partial_T$ for some $T : H_1 \rightarrow H_2$, with $H = H_1 \times H_2$. For applications it is also interesting to consider the operator ∂_T , where T has the following specific structure

$$T = \begin{pmatrix} 0 & T_2 \\ T_1 & 0 \end{pmatrix}, \quad (7.1)$$

with $T_1 : H_1 \rightarrow H_4$, $T_2 : H_2 \rightarrow H_3$, $T_1^* : H_4 \rightarrow H_1$ and $T_2^* : H_3 \rightarrow H_2$. We have

$$\partial_T = \begin{pmatrix} 0 & 0 & 0 & T_1^* \\ 0 & 0 & T_2^* & 0 \\ 0 & T_2 & 0 & 0 \\ T_1 & 0 & 0 & 0 \end{pmatrix} : D(T) \times D(T^*) \subset H \rightarrow H, \quad (7.2)$$

where $H = H_1 \times H_2 \times H_3 \times H_4$, $D(T) = D(T_1) \times D(T_2) \subset H_1 \times H_2$ and $D(T^*) = D(T_2^*) \times D(T_1^*) \subset H_3 \times H_4$. As before we require that T satisfies (H_T) , this is true if and only if T_i satisfy (H_{T_i}) , $i = 1, 2$. Furthermore, because ∂_T has a refined structure, we consider the mappings

$$\mathcal{J}_0 = I_H, \quad \mathcal{J}_2 = \mathcal{J} = \begin{pmatrix} I_{H_1} & 0 & 0 & 0 \\ 0 & I_{H_2} & 0 & 0 \\ 0 & 0 & -I_{H_3} & 0 \\ 0 & 0 & 0 & -I_{H_4} \end{pmatrix},$$

$$\mathcal{J}_1 = \begin{pmatrix} I_{H_1} & 0 & 0 & 0 \\ 0 & I_{H_2} & 0 & 0 \\ 0 & 0 & I_{H_3} & 0 \\ 0 & 0 & 0 & -I_{H_4} \end{pmatrix}, \quad \mathcal{J}_3 = \begin{pmatrix} I_{H_1} & 0 & 0 & 0 \\ 0 & -I_{H_2} & 0 & 0 \\ 0 & 0 & -I_{H_3} & 0 \\ 0 & 0 & 0 & -I_{H_4} \end{pmatrix}.$$

Then

$$\mathcal{J}_i^* = \mathcal{J}_i^{-1} = \mathcal{J}_i, \quad \mathcal{J}_i^2 = I_H \quad \text{and} \quad [\mathcal{J}_i, \mathcal{J}_j] = 0 \quad \forall i, j. \quad (7.3)$$

and

$$\mathcal{J}_i \partial_T = -\partial_T \mathcal{J}_{4-i}, \quad i = 1, 2, 3. \quad (7.4)$$

Notice that $\mathcal{J}_1, \mathcal{J}_2, \mathcal{J}_3$ are generators of a group of reflections on $H = H_1 \times H_2 \times H_3 \times H_4$, isomorphic to $\mathbf{Z}_2^3$. As before we reverse the question now.

Let $(H, (\cdot, \cdot)_H)$ be a real Hilbert space and let $G_3 \simeq \mathbf{Z}_2^3$ be an abelian group of isomorphisms on H , with generators $\mathcal{J}_1, \mathcal{J}_2, \mathcal{J}_3$. Clearly (7.3) is satisfied then. Suppose now that A is a selfadjoint operator in H such that

$$\mathcal{J}_i A = -A \mathcal{J}_{4-i}, \quad i = 1, 2, 3. \quad (7.5)$$

Definition 7.1. Let $(H, (\cdot, \cdot)_H)$ be a real Hilbert space and $G_3 \simeq \mathbf{Z}_2^3$ is an abelian group of isomorphisms on H , with the generators $\mathcal{J}_1, \mathcal{J}_2, \mathcal{J}_3$. Then a selfadjoint operator A in H is called G_3 -selfadjoint if (7.5) holds.

We introduce the projections

$$Q_1 = \frac{1}{2}(I + \mathcal{J}_2), \quad Q_2 = \frac{1}{2}(I - \mathcal{J}_2), \quad (7.6)$$

which coincide with the projections as defined in (2.9) and (2.10). Using the operators $\mathcal{J}_1$ and $\mathcal{J}_3$ one can introduce four additional orthogonal projections

$$P_1 = \frac{1}{2}(Q_1 + \mathcal{J}_3 Q_1), \quad P_2 = \frac{1}{2}(Q_1 - \mathcal{J}_3 Q_1),$$

$$P_3 = \frac{1}{2}(Q_2 + \mathcal{J}_1 Q_2), \quad P_4 = \frac{1}{2}(Q_2 - \mathcal{J}_1 Q_2)$$

and H_i are the ranges of P_i , $i = 1, \dots, 4$. We identify H with $H_1 \times H_2 \times H_3 \times H_4$ by setting $(u_1, u_2, u_3, u_4)^t = u_1 + u_2 + u_3 + u_4$, $u_i \in H_i$, $i = 1, \dots, 4$.

As in Section 2 one proves the following proposition, which gives a necessary and sufficient condition for A to be of the form (7.2).

Proposition 7.2. Let A be a selfadjoint operator in a Hilbert space H . Then $A = \partial_T$, defined in (7.2) for some T_i , satisfying (H_{T_i}) if and only if A is G_3 -selfadjoint. In this case we have

$$H_1 = R(Q_1 + \mathcal{J}_3 Q_1), \quad H_2 = R(Q_1 - \mathcal{J}_3 Q_1),$$

$$H_3 = R(Q_2 + \mathcal{J}_1 Q_2), \quad H_4 = R(Q_2 - \mathcal{J}_1 Q_2),$$

and $D(T_1) = H_1 \cap D(A)$, $D(T_2) = H_2 \cap D(A)$, $D(T_2^*) = H_3 \cap D(A)$, $D(T_1^*) = H_4 \cap D(A)$; $T_1 u_1 = A u_1$, $T_2 u_2 = A u_2$, $T_2^* u_3 = A u_3$, $T_1^* u_4 = A u_4$, for $u_1 \in D(T_1)$, $u_2 \in D(T_2)$, $u_3 \in D(T_2^*)$, $u_4 \in D(T_1^*)$.

As before we call a quadratic form G_3 -quadratic if

$$B(\mathcal{J}_1 z, \mathcal{J}_3 w) = -B(z, w), \quad B(\mathcal{J}_3 z, \mathcal{J}_1 w) = -B(z, w), \quad \forall z, w \in D(Q),$$

$$Q(\mathcal{J}_2 z) = -Q(z), \quad \forall z \in D(Q),$$

where $B(z, w) = Q'(z)w$, the derivative of Q . Summarizing the above conditions gives

$$B(\mathcal{J}_i z, \mathcal{J}_{4-i} w) = -B(z, w), \quad \forall z, w \in D(Q), \quad i = 1, 2, 3.$$

Because of the G_3 -symmetry of these quadratic forms we can formulate a theorem concerning normalized extensions of G_3 -quadratic forms, similar to Theorem 4.6.

As Section 4 we introduce a family of spaces on which $Q(z)$ can be extended canonically. As in Lemma 4.1 $Q(z)$ can be extended by continuity to the domains

$$D(|T_1|^\alpha) \times D(|T_2|^\beta) \times D(|T_2^*|^{1-\beta}) \times D(|T_1^*|^{1-\alpha}), \quad (\alpha, \beta) \in [0, 1] \times [0, 1],$$

and

$$\begin{aligned} Q_{\alpha,\beta}(z) &= \left(|T_1|^\alpha u_1, |T_1|^{1-\alpha} U_1^* u_4 \right)_{H_1} + \left(|T_2|^\beta u_2, |T_2|^{1-\beta} U_2^* u_3 \right)_{H_2} \\ &= \left(|T_1^*|^{1-\alpha} u_4, |T_1^*|^\alpha U_1 u_1 \right)_{H_4} + \left(|T_2^*|^{1-\beta} u_3, |T_2^*|^\beta U_2 u_2 \right)_{H_3} \end{aligned} \quad (7.7)$$

Thus we set

$$\Theta_1^\alpha = D(|T_1|^\alpha), \quad (7.8)$$

$$\Theta_2^\beta = D(|T_2|^\beta), \quad (7.9)$$

$$\Theta_3^{1-\beta} = D(|T_2^*|^{1-\beta}), \quad (7.10)$$

$$\Theta_4^{1-\alpha} = D(|T_1^*|^{1-\alpha}), \quad (7.11)$$

and so we introduce the product spaces (assuming $R(T)$ is closed)

Definition 7.3. For $(\alpha, \beta) \in [0, 1] \times [0, 1]$

$$E_{\alpha,\beta} = \Theta_1^\alpha \times \Theta_2^\beta \times \Theta_3^{1-\beta} \times \Theta_4^{1-\alpha}, \quad (7.12)$$

with the norms

$$\begin{aligned} \|z\|_{E_{\alpha,\beta}}^2 &= \| |T_1|^\alpha u_1 \|_{H_1}^2 + \| |T_2|^\beta u_2 \|_{H_2}^2 + \\ &\quad \| |T_2^*|^{1-\beta} u_3 \|_{H_3}^2 + \| |T_1^*|^{1-\alpha} u_4 \|_{H_4}^2 + \| z^0 \|_H^2 \end{aligned} \quad (7.13)$$

We have

Theorem 7.4. Let Q be a G_3 -quadratic form on a Hilbert space H , corresponding with an unbounded G_3 -selfadjoint operator $A = \partial_T$, with $R(T)$ closed. Then there exists a 2-parameter family of spaces $E_{\alpha,\beta}$, $(\alpha, \beta) \in [0, 1] \times [0, 1]$, given in (7.12), satisfying:

(i) $\mathcal{J}_i^{\alpha,\beta}$, the restrictions of $\mathcal{J}_i$ in $E_{\alpha,\beta}$ (i.e. $D(\mathcal{J}_i^{\alpha,\beta}) = \{z \in E_{\alpha,\beta}; \mathcal{J}_i z \in E_{\alpha,\beta}\}$), are the generators of a group of isomorphisms in $E_{\alpha,\beta}$ isomorphic to $\mathbf{Z}_2^3$,

(ii) Q has a continuous extension $Q_{\alpha,\beta}$ to $E_{\alpha,\beta}$, which is a normalized G_3 -quadratic form on $E_{\alpha,\beta}$ given by $Q_{\alpha,\beta}(z) = \frac{1}{2}(L_{\alpha,\beta} z, z)_{E_{\alpha,\beta}}$, where

$$L_{\alpha,\beta} = \begin{pmatrix} 0 & 0 & 0 & |T_1|^{1-2\alpha} U_1^* \\ 0 & 0 & |T_2|^{1-2\beta} U_2^* & 0 \\ 0 & |T_2^*|^{2\beta-1} U_2 & 0 & 0 \\ |T_1^*|^{2\alpha-1} U_1 & 0 & 0 & 0 \end{pmatrix}, \quad (7.14)$$

(iii) $\partial_T : E_{\alpha,\beta} \rightarrow E_{\alpha,\beta}^*$ is a partial isometry,

(iv) $L_{\alpha,\beta} : E_{\alpha,\beta} \rightarrow E_{\alpha,\beta}$ is a selfadjoint partial isometry, with closed range, inducing the decomposition

$$E_{\alpha,\beta} = E^+ \oplus E^0 \oplus E^-, \quad (7.15)$$

on $E_{\alpha,\beta}$, where

$$E^+ = \left\{ z \in E_{\alpha,\beta} \ominus N(\partial_T); |T_1|^{1-\alpha} U_1^* u_4 = |T_1|^\alpha u_1, \quad |T_2|^{1-\beta} U_2^* u_3 = |T_2|^\beta u_2 \right\},$$

$$E^- = \mathcal{J}_2^{\alpha,\beta} E^+, \quad (7.16)$$

(v) the spectrum $\sigma(L_{\alpha,\beta}) = \{-1, 1\}$ if $0 \in \rho(\partial_T)$ and $\sigma(L_{\alpha,\beta}) = \{-1, 0, 1\}$ if $0 \in \sigma(\partial_T)$. Furthermore, E^+ and E^- are the eigenspaces of $L_{\alpha,\beta}$ corresponding to the eigenvalues 1 and -1 .

If one takes a closer look at the definition of the spaces $E_{\alpha,\beta}$ one can give another intrinsic definition of $E_{\alpha,\beta}$, using a 'multiparameter' interpolation functor. We set

$$E_0 = E_{1,1}, \quad E_1 = E_{0,1}, \quad E_2 = E_{1,0}, \quad E_3 = E_{0,0},$$

the 'endpoints' of the interpolation. Let $\Sigma E_i = E_0 + E_1 + E_2 + E_3$ and define the function

$$K : \mathbf{R}^3 \times \Sigma E_i \longrightarrow \mathbf{R}, \quad (7.17)$$

given by

$$K(s, t, x, z, \Sigma E_i)^2 = \inf_{z \in \Sigma E_i} \left(\|z_0\|_{E_0}^2 + s^2 \|z_1\|_{E_1}^2 + t^2 \|z_2\|_{E_2}^2 + x^2 \|z_3\|_{E_3}^2 \right)$$

and

$$\|z\|_{(\theta_1, \theta_2, \theta_3)}^2 = c^{-1} \int_0^\infty \int_0^\infty \int_0^\infty s^{2(\theta_1-1)} t^{2(\theta_2-1)} x^{2(\theta_3-1)} K(s, t, x, z, \Sigma E_i)^2 \frac{ds}{s} \frac{dt}{t} \frac{dx}{x}, \quad (7.18)$$

where

$$c = c_{\theta_1, \theta_2, \theta_3} = \int_0^\infty \int_0^\infty \int_0^\infty s^{1-2\theta_1} t^{1-2\theta_2} x^{1-2\theta_3} \frac{1}{1+s^2+t^2+x^2} ds dt dx,$$

with $(\theta_1, \theta_2, \theta_3) \in T^3 = \{(\theta_1, \theta_2, \theta_3) \in [0, 1]^3; 2 \leq \theta_1 + \theta_2 + \theta_3 \leq 3\}$. We define the tetrahedral interpolation spaces by

$$\begin{aligned} [E_0, E_1, E_2, E_3]_{\theta_1, \theta_2, \theta_3} = & \left\{ z \in \sigma E_i; s^{\theta_1-1} t^{\theta_2-1} x^{\theta_3-1} K(s, t, x, z, \Sigma E_i) \right. \\ & \left. \in L^2((0, \infty)^3; \frac{ds}{s} \frac{dt}{t} \frac{dx}{x}) \right\}. \end{aligned} \quad (7.19)$$

It appears that with the particular choice of the spaces $E_0, \dots, E_3$ the number parameters automatically reduce to two parameters and we obtain:

Lemma 7.5. *Let $1 - \theta_1 = (1 - \alpha)\beta$, $1 - \theta_2 = (1 - \beta)\alpha$, $1 - \theta_3 = (1 - \alpha)(1 - \beta)$. Then the norms $\|\cdot\|_{(\theta_1, \theta_2, \theta_3)}$ and $\|\cdot\|_{E_{\alpha, \beta}}$ coincide on $E_{\alpha, \beta}$.*

Proof of Lemma 7.5. As in the proof of Lemma 4.3 and the definition of the spaces E_0, E_1, E_2, E_3 we have

$$K(s, t, x, z, \Sigma E_i)^2 = \inf_{z^0 = \Sigma E_i^0} \left(\|z_0^0\|_H^2 + s^2 \|z_1^0\|_H^2 + t^2 \|z_2^0\|_H^2 + x^2 \|z_3^0\|_H^2 \right) + \\ \inf_{z^\pm = \Sigma E_i \ominus N(\partial_T)} \left(\|z_0^\pm\|_H^2 + s^2 \|z_1^\pm\|_H^2 + t^2 \|z_2^\pm\|_H^2 + x^2 \|z_3^\pm\|_H^2 \right).$$

Evaluating the first minimum we obtain

$$z_0^0 = \frac{x^2}{1 + x^2 s^{-2} + x^2 t^{-2} + x^2} z^0,$$

which yields

$$K(s, t, x, z, \Sigma E_i)^2 = \inf_{z^\pm = \Sigma \tilde{E}_i \ominus N(\partial_T)} \left(\|z_0^\pm\|_{E_0}^2 + s^2 \|z_1^\pm\|_{E_1}^2 + t^2 \|z_2^\pm\|_{E_2}^2 + x^2 \|z_3^\pm\|_{E_3}^2 \right) \\ + \frac{x^2}{1 + x^2 s^{-2} + x^2 t^{-2} + x^2} \|z^0\|_H^2.$$

We denote the latter function by $K^\perp(s, t, x, z, \Sigma \tilde{E}_i)^2$ and $z_i = (u_{(1)i}, u_{(2)i}, u_{(3)i}, u_{(4)i})^t$, $\tilde{E}_i = E_i \ominus N(\partial_T)$, $i = 0, 1, 2, 3$. Then

$$K^\perp(s, t, x, z, \Sigma \tilde{E}_i)^2 = \\ \inf_{z \in \Sigma \tilde{E}_i} \left(\| |T| u_{(1)0} \|_{H_1}^2 + s^2 \| |u_{(1)1}| \|_{H_1}^2 + t^2 \| |T| u_{(1)2} \|_{H_1}^2 + x^2 \| |u_{(1)3}| \|_{H_1}^2 \right. \\ + \| |S| u_{(2)0} \|_{H_2}^2 + s^2 \| |S| u_{(2)1} \|_{H_2}^2 + t^2 \| |u_{(2)2}| \|_{H_2}^2 + x^2 \| |u_{(2)3}| \|_{H_2}^2 \\ + \| |u_{(3)0}| \|_{H_3}^2 + s^2 \| |u_{(3)1}| \|_{H_3}^2 + t^2 \| |S^*| u_{(3)2} \|_{H_3}^2 + x^2 \| |S^*| u_{(3)3} \|_{H_3}^2 \\ \left. + \| |u_{(4)0}| \|_{H_4}^2 + s^2 \| |T^*| u_{(4)1} \|_{H_4}^2 + t^2 \| |u_{(4)2}| \|_{H_4}^2 + x^2 \| |T^*| u_{(4)3} \|_{H_4}^2 \right).$$

By setting $z_3 = z - z_0 - z_1 - z_2$ and taking the infimum over all $z_i \in \tilde{E}_i$, $i = 0, 1, 2$ in the

above expansion of $K^\perp(s, t, x, z, \Sigma E_i)^2$ one is led to twelve Euler equations:

$$\begin{aligned}
(|T|^2 + x^2)u_{(1)0} &= x^2(u_{(1)} - u_{(1)1} - u_{(1)2}) \\
(s^2 + x^2)u_{(1)1} &= x^2(u_{(1)} - u_{(1)0} - u_{(1)2}) \\
(t^2|T|^2 + x^2)u_{(1)2} &= x^2(u_{(1)} - u_{(1)0} - u_{(1)1}) \\
(|S|^2 + x^2)u_{(2)0} &= x^2(u_{(2)} - u_{(2)1} - u_{(2)2}) \\
(s^2|S|^2 + x^2)u_{(2)1} &= x^2(u_{(2)} - u_{(2)0} - u_{(2)2}) \\
(t^2 + x^2)u_{(2)2} &= x^2(u_{(2)} - u_{(2)0} - u_{(2)1}) \\
(1 + x^2|S^*|^2)u_{(3)0} &= x^2|S^*|^2(u_{(3)} - u_{(3)1} - u_{(3)2}) \\
(s^2 + x^2|S^*|^2)u_{(3)1} &= x^2|S^*|^2(u_{(3)} - u_{(3)0} - u_{(3)2}) \\
(t^2|S^*|^2 + x^2|S^*|^2)u_{(3)2} &= x^2|S^*|^2(u_{(3)} - u_{(3)0} - u_{(3)1}) \\
(1 + x^2|T^*|^2)u_{(4)0} &= x^2|T^*|^2(u_{(4)} - u_{(4)1} - u_{(4)2}) \\
(s^2|T^*|^2 + x^2|T^*|^2)u_{(4)1} &= x^2|T^*|^2(u_{(4)} - u_{(4)0} - u_{(4)2}) \\
(t^2 + x^2|T^*|^2)u_{(4)2} &= x^2|T^*|^2(u_{(4)} - u_{(4)0} - u_{(4)1})
\end{aligned}$$

Solving these equations and substitution into $K^\perp(s, t, x, z, \Sigma E_i)^2$ gives

$$\begin{aligned}
K^\perp(s, t, x, z, \Sigma E_i)^2 &= \left(u_1, x^2|T|^2((1 + s^{-2}x^2)|T|^2 + (1 + t^{-2})x^2)^{-1}u_1 \right)_{H_1} \\
&+ \left(u_2, x^2|S|^2((1 + t^{-2}x^2)|S|^2 + (1 + s^{-2})x^2)^{-1}u_2 \right)_{H_2} \\
&+ \left(u_3, x^2(x^2(1 + s^{-2})|S^*|^2 + (1 + t^{-2}x^2))^{-1}|S^*|^2u_3 \right)_{H_3} \\
&+ \left(u_4, x^2(x^2(1 + t^{-2})|T^*|^2 + (1 + s^{-2}x^2))^{-1}|T^*|^2u_4 \right)_{H_4}
\end{aligned}$$

We recall the constant

$$c_{\theta_1, \theta_2, \theta_3} = \int_0^\infty \int_0^\infty \int_0^\infty s^{1-2\theta_1} t^{1-2\theta_2} x^{1-2\theta_3} \frac{1}{1 + s^2 + t^2 + x^2} ds dt dx, \quad (7.22)$$

which can be seen to be finite by using polar coordinates in $\mathbf{R}^3$. Indeed

$$c_{\theta_1, \theta_2, \theta_3} \leq C \int_0^\infty r^{5-2\theta_1-2\theta_2-2\theta_3} \frac{1}{1 + r^2} dr < \infty,$$

if $2 < \theta_1 + \theta_2 + \theta_3 < 3$, which is exactly the case if $(\theta_1, \theta_2, \theta_3) \in \text{int}(T^3)$. For the norm

(7.18) we now obtain

$$\begin{aligned}
\|z\|_{(\theta_1, \theta_2, \theta_3)}^2 &= c^{-1} \int_0^\infty \int_0^\infty \int_0^\infty s^{2(\theta_1-1)} t^{2(\theta_2-1)} x^{2(\theta_3-1)} \times \\
&\quad \frac{x^2}{1 + x^2 s^{-2} + x^2 t^{-2} + x^2} \|z^0\|_H^2 \frac{ds}{s} \frac{dt}{t} \frac{dx}{x} \\
&+ c^{-1} \int_0^\infty \int_0^\infty \int_0^\infty s^{2(\theta_1-1)} t^{2(\theta_2-1)} x^{2(\theta_3-1)} \times \\
&\quad \left(u_1, x^2 |T|^2 ((1 + s^{-2} x^2) |T|^2 + (1 + t^{-2} x^2)^{-1} u_1) \right)_{H_1} \frac{ds}{s} \frac{dt}{t} \frac{dx}{x} \\
&+ c^{-1} \int_0^\infty \int_0^\infty \int_0^\infty s^{2(\theta_1-1)} t^{2(\theta_2-1)} x^{2(\theta_3-1)} \times \\
&\quad \left(u_2, x^2 |S|^2 ((1 + t^{-2} x^2) |S|^2 + (1 + s^{-2} x^2)^{-1} u_2) \right)_{H_2} \frac{ds}{s} \frac{dt}{t} \frac{dx}{x} \\
&+ c^{-1} \int_0^\infty \int_0^\infty \int_0^\infty s^{2(\theta_1-1)} t^{2(\theta_2-1)} x^{2(\theta_3-1)} \times \\
&\quad \left(u_3, x^2 |S^*|^2 (x^2 (1 + s^{-2}) |S^*|^2 + (1 + t^{-2} x^2)^{-1} u_3) \right)_{H_3} \frac{ds}{s} \frac{dt}{t} \frac{dx}{x} \\
&+ c^{-1} \int_0^\infty \int_0^\infty \int_0^\infty s^{2(\theta_1-1)} t^{2(\theta_2-1)} x^{2(\theta_3-1)} \times \\
&\quad \left(u_4, x^2 |T^*|^2 (x^2 (1 + t^{-2}) |T^*|^2 + (1 + s^{-2} x^2)^{-1} u_4) \right)_{H_4} \frac{ds}{s} \frac{dt}{t} \frac{dx}{x}
\end{aligned}$$

As before using the spectral representation of the operators $|T|, |S|, |S^*|, |T^*|$ we obtain using (7.22)

$$\begin{aligned}
\|z\|_{(\theta_1, \theta_2, \theta_3)}^2 &= \|z^0\|_H^2 + (|T|^{2(\theta_1+\theta_3-1)} u_1, u_1)_{H_1} + (|S|^{2(\theta_2+\theta_3-1)} u_2, u_2)_{H_2} \\
&\quad + (|S^*|^{2(2-\theta_2-\theta_3)} u_3, u_3)_{H_3} + (|T^*|^{2(2-\theta_1-\theta_3)} u_4, u_4)_{H_4} \\
&= \|z^0\|_H^2 + \| |T|^{\theta_1+\theta_3-1} u_1 \|_{H_1} + \| |S|^{\theta_2+\theta_3-1} u_2 \|_{H_2} \\
&\quad + \| |S^*|^{2-\theta_2-\theta_3} u_3 \|_{H_3} + \| |T^*|^{2-\theta_1-\theta_3} u_4 \|_{H_4}
\end{aligned} \tag{7.23}$$

Now we take $1 - \theta_1 = (1 - \alpha)\beta$, $1 - \theta_2 = (1 - \beta)\alpha$, $1 - \theta_3 = (1 - \alpha)(1 - \beta)$, with $(\alpha, \beta) \in [0, 1] \times [0, 1]$. Then $\theta_1 + \theta_3 - 1 = \alpha$, $\theta_2 + \theta_3 - 1 = \beta$ and we deduce from (7.23) that

$$\|z\|_{(\theta_1, \theta_2, \theta_3)} = \|z\|_{E_{\alpha, \beta}},$$

which proves the lemma. $\square$

Theorem 7.6. *Let $1 - \theta_1 = (1 - \alpha)\beta$, $1 - \theta_2 = (1 - \beta)\alpha$, $1 - \theta_3 = (1 - \alpha)(1 - \beta)$. Then*

$$E_{\alpha, \beta} = [E_0, E_1, E_2, E_3]_{\theta_1, \theta_2, \theta_3}, \quad (\alpha, \beta) \in [0, 1] \times [0, 1], \tag{7.24}$$

and if B is a bounded operator from E_i to F_i , with $\|B\| = M_i$, $i = 1, \dots, 4$, then

$$\|Bz\|_{\alpha, \beta} \leq M_0^{\alpha\beta} M_1^{(1-\alpha)\beta} M_2^{(1-\beta)\alpha} M_3^{(1-\alpha)(1-\beta)} \|z\|_{E_{\alpha, \beta}}, \tag{7.25}$$

with $\|\cdot\|_{\alpha,\beta} = \|\cdot\|_{[F_0,F_1,F_2,F_3]_{\theta_1,\theta_2,\theta_3}}$.

Proof of Theorem 7.6. The first part of the theorem follows from Lemma 7.5. As for the second part we proceed as follows:

$$\begin{aligned} & K(s, t, x, Bz, \Sigma F_i) \\ & \leq \inf_{z \in \Sigma E_i} \left(\|Bz_0\|_{F_0}^2 + s^2 \|Bz_1\|_{F_1}^2 + t^2 \|Bz_2\|_{F_2}^2 + x^2 \|Bz_3\|_{F_3}^2 \right)^{1/2} \\ & \leq \inf_{z \in \Sigma E_i} \left(M_0^2 \|z_0\|_{E_0}^2 + M_1^2 s^2 \|z_1\|_{E_1}^2 + M_2^2 t^2 \|z_2\|_{E_2}^2 + M_3^2 x^2 \|z_3\|_{E_3}^2 \right)^{1/2} \\ & \leq M_0 K(M_1 s/M_0, M_2 t/M_0, M_3 x/M_0, z, \Sigma E_i). \end{aligned}$$

Using $s' = M_1 s/M_0$, $t' = M_2 t/M_0$, and $x' = M_3 x/M_0$, we obtain

$$\|Bz\|_{\alpha,\beta} \leq M_0^{\theta_1+\theta_2+\theta_3-2} M_1^{1-\theta_1} M_2^{1-\theta_2} M_3^{1-\theta_3} \|z\|_{(\theta_1,\theta_2,\theta_3)}.$$

Substitution of θ_1 , θ_2 and θ_3 then proves the theorem. $\square$

Remark 7.7. Setting $\sigma_i = 1 - \theta_i$, $i = 1, 2, 3$, we deduce that the parameters σ_i are in the tetrahedron given by $\sigma_i \geq 0$ and $0 \leq \sigma_1 + \sigma_2 + \sigma_3 \leq 1$.

Example: Consider the Lagrangian

$$\mathcal{L}(z) = \int_{B_R} \nabla u_1 \nabla u_4 + \int_{B_R} \nabla u_2 \nabla u_3 - \int_{\Omega} H(u_1, u_2, u_3, u_4). \quad (7.26)$$

Clearly the quadratic part of (7.26) is a G_3 -quadratic form. As in Example 2 of Section 6 we consider the G_3 -quadratic form on the Hilbert space $G = H_0^1(d\mu_{N-2})^2 \times H_0^1(d\mu_{2-N})^2 \subset H_0^1(d\mu_{N-2})^4$. Without going into the details (see Section 6) we obtain that the G_3 -quadratic form $Q(z) = \int_{B_R} \nabla u_1 \cdot \nabla u_4 + \int_{B_R} \nabla u_2 \cdot \nabla u_3$ can be continuously extended to the interpolation spaces

$$E_{\alpha,\beta} = \Theta^\alpha \times \Theta^\beta \times \Theta^{1-\beta} \times \Theta^{1-\alpha},$$

where

$$\Theta^\alpha = [H_0^1(d\mu_{2-N}), H_0^1(d\mu_{N-2})]_{1-\alpha}. \quad (6.43)$$

We consider now the Hamiltonian

$$H(u_1, u_2, u_3, u_4) = \frac{1}{q+1} |u_3|^{q+1} + u_2 u_4 + \frac{1}{p+1} |u_1|^{p+1}. \quad (7.27)$$

Clearly the Euler equations of (7.26) with the Hamiltonian H as given by (7.27), give rise to a system of four second order partial differential equations, which are equivalent with the equation $\Delta^2((\Delta^2 u_1)^{1/q}) = u_1^p$ and the boundary conditions $u_1 = \Delta u_1 = \Delta^2 u_1 = \Delta^3 u_1 = 0$. From the embeddings of the spaces $E_{\alpha,\beta}$ we deduce that for the proper choices of the parameters $\alpha, \beta \in (0, 1)$ we obtain the relation

$$\frac{1}{p+1} + \frac{1}{q+1} > \frac{N-4}{N}, \quad N > 8. \quad (7.28)$$

For the case $q = 1$ we refer to [vdV1].

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