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A superquadratic indefinite elliptic system and its Morse–Conley–Floer homology

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Abstract. We study critical points of the indefinite functional $f_H(u, v) = \int \{\nabla u \cdot \nabla v - H(x, u, v)\} \mathrm{d}x$ by applying Floer's homology construction to the ordinary gradient flow of the functional f on a suitable Sobolev space. One of our main observations is that even though this flow is well posed in both time directions and lacks any kind of smoothing property one can still obtain compactness of connecting orbit spaces and thus define the Floer homology for f_H.

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1 Introduction

In this paper we will use Floer's homology construction [10] to study the critical points of functionals of the type

$$f_H(u, v) = \int_\Omega \{\nabla u \cdot \nabla v - H(x, u, v)\} \, \mathrm{d}x \tag{1}$$

where $H : \Omega \times \mathbb{R}^2 \to \mathbb{R}$ is a C^2 function, *the Hamiltonian*, and $\Omega \subset \mathbb{R}^n$ is a bounded domain with smooth boundary.

The functional f_H is strongly indefinite,[1] but its Euler-Lagrange equations are elliptic:

$$\begin{cases} \Delta v + H_u(x,u,v) = 0, \\ \Delta u + H_v(x,u,v) = 0. \end{cases} \tag{2}$$

We will always assume that the functions u, v vanish on $\partial\Omega$, i.e. we will look for solutions of the system (2) with Dirichlet boundary conditions.

In Sect. 2 we give precise conditions on H for which our theory works. We consider two different classes of Hamiltonians. These two classes can best be illustrated by the following two special representatives

$$H(x,u,v) = a(x)\frac{|u|^{p+1}}{p+1} \pm b(x)\frac{|v|^{q+1}}{q+1}, \tag{3}$$

where $a, b \in C^0(\bar{\Omega})$ are strictly positive. For such H the Euler Lagrange equations are

$$\begin{cases} \Delta v + a(x)u^p = 0, \\ \Delta u \pm b(x)v^q = 0. \end{cases} \tag{4}$$

Here and elsewhere we shall write u^p for the odd extension of the power function, i.e. $u^p \stackrel{\text{def}}{=} u|u|^{p-1}$.

We will restrict p and q to the *subcritical* range

$$\frac{1}{p+1} + \frac{1}{q+1} > \frac{n-2}{n}. \tag{5}$$

This turns out to be the range of p, q for which boundedness of the integral

$$\int_\Omega \left(|u|^{p+1} + |v|^{q+1}\right) \mathrm{d}x$$

implies L^∞ and from there $C^{2,\alpha}$ bounds for the solutions u and v. Inequality (5) is sharp in the following sense. If p and q satisfy the reversed inequality it follows from results by Mitidieri [15] and by Van der Vorst [27] that (2) has no positive solutions on starshaped domains.

Problem (2) has previously been studied using the variational structure by Felmer and DeFigueiredo [11], and also by Hulshof and Van der Vorst [13]. In these papers a minimax theorem for strongly indefinite functionals due to Benci and Rabinowitz [5, 6] is employed to obtain existence of at least one non-trivial solution to (2). See also Szulkin's work [24, 14] for a Morse theory for strongly indefinite functionals based on different cohomology theories.

[1] At each critical point z of f_H there are infinite dimensional subspaces $E_\pm \subset (C^1)^2$ on which $d^2 f_H(z)$ is positive and negative definite respectively.

The functional f_H is very similar to the action functional

$$\mathcal{A}_H(p,q) = \int \{pq' - H(p,q,t)\}\mathrm{d}t$$

whose critical points are periodic orbits of the Hamiltonian system with Hamiltonian $H(p,q,t)$. Floer's study of the unregularized L^2 gradient flow of this functional led to a very succesfull Morse theory for $\mathcal{A}_H$. Motivated by this success we tried to develop an analogous Morse theory for the functional f_H. Our main conclusion is that this can indeed be done, but that one must not use the unregularized L^2 gradient flow, but rather a regular gradient flow in some fractional power space $E_\alpha = D((-\Delta)^\alpha) \times D((-\Delta)^{1-\alpha})$. This is perhaps surprising since the gradient flow in the space E_α is locally well posed in both time directions and has no smoothing properties, in contrast with the L^2 gradient flow which leads to an elliptic initial value problem all of whose solutions are smooth.

The best existence result which we have obtained by using Floer's method is as follows.

Theorem 1 *Let $H(x,u,v) = a(x)\frac{|u|^{p+1}}{p+1} + b(x)\frac{|v|^{q+1}}{q+1}$ with a and b strictly positive continuous functions. Then f_H has an unbounded sequence (in L^∞) of critical points $(u_k,v_k) \in C_0^1(\bar{\Omega})\times C_0^1(\bar{\Omega})$. If $\Omega \subset \mathbb{R}^n$ with $n \geq 3$ then the generalized Morse indices $\mu_-(z_k)$ of the critical points are also unbounded.*

As is to be expected from a Floer-homology approach, we define a generalized Morse index for critical points of f_H and obtain Morse inequalities for those H whose associated functional f_H is a Morse function. The analogue of Theorem 1 can also be obtained for a larger class of non-linearities H, see Sect. 9.

To prove that the generalized Morse indices of the critical points in Theorem 1 are unbounded we must prove an a priori estimate for solutions in terms of their Morse index analogous to a result of Bahri and Lions [4]. For the current setting we derive the estimate in [2], and we summarize it in Sect. 9.4.

In Sect. 2 we discuss the various gradient flows for f_H one can choose from. Section 3 establishes compactness properties of the set of bounded trajectories for the "E_α gradient flow," which motivates our preference for this flow. In Sect. 4 we discuss the generalized Morse index. The Sects. 5 upto 8 deal with the definition and continuation properties of the Floer homology groups for isolating neighborhoods for the E_α gradient flow of f_H. Finally in Sect. 9 we use this theory to derive various existence results.

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2 The gradient flow

To define Floer homology groups we must consider a gradient flow for f_H, i.e. we must specify an inner product with respect to which we take the gradient of f_H.

2.1 The L^2 gradient

The L^2 gradient of f_H is given by

$$\nabla^{L^2} f_H(z) = -\partial_\Delta z - H_z(x,z),$$

where

$$\partial_\Delta = \begin{pmatrix} 0 & \Delta \\ \Delta & 0 \end{pmatrix}. \tag{6}$$

The L^2 gradient flow is therefore equivalent to the following system of PDE's

$$\frac{\partial z}{\partial t} = \partial_\Delta z + H_z(x,z). \tag{7}$$

The initial value problem for this system is ill-posed in both time directions, as one sees by writing $z = \binom{u}{v}$ and noting that $w_\pm = u \pm v$ satisfy forward and backward semilinear heat equations, respectively. Parabolic regularity theory then also implies that bounded (i.e. L^∞) solutions to (7) which are defined for all $t \in \mathbb{R}$ actually have uniformly bounded derivatives. Thus to obtain the necessary compactness one would have to obtain L^∞ estimates for all connecting orbits between critical points of f_H.

Originally we found such estimates, but our proof required a stronger hypothesis on the exponents p, q than (5). We then found that our arguments also worked for the ordinary E_α gradient flow of f_H, which we now describe.

2.2 The E_α gradient flow

We will choose the innerproduct of the Hilbert space

$$E_\alpha \stackrel{\text{def}}{=} D((-\Delta)^\alpha) \times D((-\Delta)^{1-\alpha}) = \theta_\alpha \times \theta_{1-\alpha} \tag{8}$$

where $\theta_\alpha = D((-\Delta)^\alpha)$ is the domain of the fractional power of the L^2-realization of the Laplacian with Dirichlet boundary conditions (see [11] and [13] for a more detailed discussion). The number $\alpha \in (0,1)$ must satisfy

$$p < \frac{n+4\alpha}{n-4\alpha}, \qquad q < \frac{n+4(1-\alpha)}{n-4(1-\alpha)}. \tag{9}$$

Such an α exists if and only if p and q are subcritical (see (5)).

The Sobolev embedding theorems imply that E_α is continuously embedded in $L^{p+1}(\Omega) \times L^{q+1}(\Omega)$. Hence if the function H is smooth, and asymptotically grows like $a(x)|u|^{p+1} \pm b(x)|v|^{q+1}$, then f_H will be a C^2 functional on E_α, and its E_α gradient flow will be a local flow. To state this with more precision we observe that the $L^2(\Omega) \times L^2(\Omega)$ and E_α inner products are related by

$$(z, z')_{L^2} = (\mathcal{D}_\alpha z, z')_{E_\alpha},$$

where[2] $\mathcal{D}_\alpha : E_\alpha^* \to E_\alpha$ is given by

$$\mathcal{D}_\alpha = \begin{pmatrix} (-\Delta)^{-2\alpha} & 0 \\ 0 & (-\Delta)^{-2(1-\alpha)} \end{pmatrix}.$$

The L^2 and E_α gradients of f are related by $\nabla f = \mathcal{D}_\alpha \nabla^{L^2} f$, so the E_α gradient ∇f_H is given by

$$\nabla f_H(u,v) = \begin{pmatrix} (-\Delta)^{-2\alpha}\{-\Delta v - H_u(x,u,v)\} \\ (-\Delta)^{-2(1-\alpha)}\{-\Delta u - H_v(x,u,v)\} \end{pmatrix}. \tag{10}$$

Assume now, as we shall do throughout this paper, that

$$H \in C^2(\bar{\Omega} \times \mathbb{R}^2), \tag{11}$$

and that H satisfies the following growth conditions.

$$\begin{cases} |H(x,u,v)| \le C\left(1 + |u|^{p+1} + |v|^{q+1}\right) \\ |H_u(x,u,v)| \le C\left(1 + |u|^p + |v|^{\frac{p(q+1)}{p+1}}\right) \\ |H_v(x,u,v)| \le C\left(1 + |u|^{\frac{q(p+1)}{q+1}} + |v|^q\right) \end{cases} \tag{12}$$

Then the E_α gradient ∇f_H defines a continuous vector field on E_α. By imposing even more growth conditions on the second derivatives of H, we could ensure that the composition operator $z \mapsto \mathcal{D}_\alpha[H_z(\cdot, z(\cdot))]$ is locally

[2] The inclusion $E_\alpha \subset L^2 \times L^2$ induces a triple $E_\alpha \subset L^2 \times L^2 \subset E_\alpha^*$. The operator is in principle only defined on $L^2 \times L^2$, but it extends to a continuous operator from E_α^* to E_α.

Lipschitz on E_α, which makes ∇f_H locally Lipschitz on E_α, and which therefore implies that the gradient flow equation

$$\frac{\partial z}{\partial t} = -Q_\alpha z + \mathcal{D}_\alpha H_z(x,z) \tag{13}$$

generates a local flow on E_α. Here

$$Q_\alpha = -\mathcal{D}_\alpha \partial_\Delta = \begin{pmatrix} 0 & (-\Delta)^{1-2\alpha} \\ (-\Delta)^{2\alpha-1} & 0 \end{pmatrix} \tag{14}$$

Our main observation in this paper is that even though the E_α gradient flow has no smoothing properties at all (the initial value problem is well posed in both time directions), connecting orbits between critical points still form precompact sets. Moreover, this compactness can be obtained for a larger range of p, q than one would get for the L^2 gradient flow (7). See Sect. 3

2.3 The flow on X_α

As we just noted, the growth conditions (12) are not sufficient to guarantee that the gradient flow produces a local flow on E_α. Even if we impose more conditions to make ∇f_H locally Lipschitz, the construction of Floer homology requires genericity arguments which rely on Sard's theorem, which in turn requires certain differentiability of the functionals involved. Although $f_H : E_\alpha \to \mathbb{R}$ is C^1 it will in general not have many more derivatives, even if H is C^∞. Fortunately, as will become clear in a moment, the E_α gradient flow of f_H leaves the space X_α defined by

$$X_\alpha \stackrel{\text{def}}{=} h_0^{2\alpha}(\bar{\Omega}) \oplus h_0^{2(1-\alpha)}(\bar{\Omega})$$

invariant. Here $h_0^{2\alpha}(\bar{\Omega})$ is, by definition

$$\{\varphi \in h^{2\alpha}(\bar{\Omega}) \mid \varphi|_{\partial\Omega} \equiv 0\}$$

when $0 < \alpha < \frac{1}{2}$, and

$$\{\varphi \in h^{1,2\alpha-1}(\bar{\Omega}) \mid \varphi|_{\partial\Omega} \equiv 0\}$$

when $\frac{1}{2} < \alpha < 1$. For $\alpha = \frac{1}{2}$ we define

$$h_0^{2\alpha}(\bar{\Omega}) \stackrel{\text{def}}{=} \{\varphi \in C^1(\bar{\Omega}) \mid \varphi|_{\partial\Omega} \equiv 0\}.$$

The space $h^\beta(\Omega)$ is the so-called "little Hölder space", which is the closure in the usual Hölder space $C^\beta(\Omega)$ of the subspace of smooth functions. The little Hölder spaces have the advantage of being separable.

For any $\alpha \in (0,1)$ the operator $(-\Delta)^{1-2\alpha}$ is an isomorphism from $h_0^{2(1-\alpha)}(\bar{\Omega})$ to $h_0^{2\alpha}(\bar{\Omega})$. It follows that $Q_\alpha : X_\alpha \to X_\alpha$ is also an isomorphism (one even has $(Q_\alpha)^2 = \mathrm{I}_{X_\alpha}$. See for instance [7].) Furthermore, $\mathcal{D}_\alpha$ is a compact operator on X_α, and, if H is a C^{m+3} function, then the substitution operator $z \mapsto H_z(\cdot, z(\cdot))$ is C^m from X_α to X_α. Thus we arrive at

Lemma 2 *If $H \in C^{m+3}$ then ∇f_H defines a C^m vector field on X_α.*

For $H \in C^{m+3}$ the gradient flow equation (13) therefore generates a C^m local flow on X_α, which we denote by $\phi^t : \mathcal{D}_t \to X_\alpha$ ($\mathcal{D}_t \overset{\text{open}}{\subset} X_\alpha$ is the domain of ϕ^t).

2.4 The perturbed flows

To define the Floer homology we need all connecting orbits between critical points to be transverse, at least for a generic set of flows. We achieve this by slightly varying the Riemannian metric on E_α, thereby perturbing the gradient of f_H.

Let $K : E_\alpha \to \mathrm{NSym}(H^{-1}, C_0^2)$ be a smooth map, where

$$\mathrm{NSym}(H^{-1}, C_0^2) = \left\{ K \in L(H^{-1}, C_0^2) \;\middle|\; \begin{array}{c} K \text{ is nuclear and symmetric} \\ \text{w.r.t. the inner product of } E_\alpha \end{array} \right\}$$

and where $H^{-1} = H^{-1}(\Omega; \mathbb{R}^2) = H^{-1}(\Omega) \times H^{-1}(\Omega)$, $C_0^2 = C_0^2(\bar{\Omega}; \mathbb{R}^2)$.

Recall that a bounded operator $T : X \to Y$ between Banach spaces is *nuclear* if it can be written as an absolutely convergent sum, i.e.

$$T = \sum_{i=1}^{\infty} y_i \otimes x_i^* \tag{15}$$

with $x_i^* \in X^*$, $y_i \in Y$ and

$$\|T\|_{\mathrm{nucl}} \overset{\text{def}}{=} \inf \sum_{i=1}^{\infty} \|y_i\|_Y \|x_i^*\|_{X^*} \tag{16}$$

Clearly the space of finite rank operators from X to Y is dense in the space of nuclear operators, and since both H^{-1} and C_0^2 are separable, the space $\mathrm{NSym}(H^{-1}, C_0^2)$ is also separable.

The set $\mathrm{NSym}(H^{-1}, C_0^2)$ with the nuclear norm (16) is a Banach space. We define a Riemannian metric on E_α by

$$g_z^K(\zeta_1, \zeta_2) \overset{\text{def}}{=} (\zeta_1, (\mathrm{I} + K(z))\zeta_2)_{E_\alpha}$$

for all $z \in E_\alpha$ and $\zeta_i \in T_z E_\alpha \cong E_\alpha$. Our modified gradient flow will be

$$z'(t) + (\mathrm{I} + K(z(t)))\nabla f_H(z(t)) = 0\ , \tag{17}$$

where ∇f_H still denotes the E_α gradient of f_H. We shall assume that

$$\|K(z)\|_{E_\alpha \to E_\alpha} \le \rho, \quad \forall z \in E_\alpha, \tag{18}$$

for some $\rho < \frac{1}{2}$, and also that the map K can be written as

$$K(z) = e^{-\|z\|^2_{E_\alpha}} K_0(z), \tag{19}$$

where K_0 satisfies the following Gevrey type estimates:

$$\sup_{z \in E_\alpha} \|K_0^{(n)}(z)\|_{L_n(E_\alpha, \mathrm{NSym}(H^{-1}, C_0^2))} \le C(n!)^2 \tag{20}$$

for some $C < \infty$.

We define $\tilde{\mathsf{K}}$ to be the set of maps $K(\cdot)$ which satisfy (19, 20). By defining the norm of $K(\cdot)$ to be the infimum of all constants C for which (20) holds, $\tilde{\mathsf{K}}$ becomes a Banach space. The bounds (20) ensure that $\tilde{\mathsf{K}}$ is a Banach space which at least contains maps of the form

$$\phi(\lambda\|z - z_0\|)K_1, \tag{21}$$

where $K_1 \in \mathrm{NSym}(H^{-1}, C_0^2)$ is a constant, and $\phi(s) = e^{-1/(1-s^2)}$ for $s < 1$, $\phi(s) = 0$ for $s \ge 1$, is a common cutoff function. Since E_α is a separable Hilbert space, the set of maps of the form (21) is a separable subset of $\tilde{\mathsf{K}}$. We now define K to be the closed linear subspace of $\tilde{\mathsf{K}}$ spanned by the maps (21). This is the class of metrics which we shall allow, and from the construction it follows that this space is a separable Banach space. We will need this in Sect. 7.

We write K_ρ for the subset of K consisting of those $K(\cdot)$ which also satisfy (18).

The assumption (19) directly implies the following uniform bound

$$\|K(z)\nabla f_H(t,z)\|_{C_0^2} \le C, \tag{22}$$

where C does not depend on z or t. To verify this one considers the two terms in $\nabla f_H(t,z)$. The first term is linear, and bounded in E_α by $C\|z\|_{E_\alpha}$; the second involves the Hamiltonian function H, and using (12), (9) and the Sobolev inequalities, one shows that the second term is bounded in $H^{-1}(\Omega)$ by $C(1 + \|z\|^m_{E_\alpha})$, with $m = \max(p,q)$. The rapid decay we require in (19) now implies (22).

Lemma 3 *For $H \in C^\infty$ and $K : E_\alpha \to \mathrm{NSym}(H^{-1}, C_0^2)$ as above, (17) generates a C^∞ local flow on X_α.*

3 Compactness

Our compactness theorems will apply to maximal invariant sets, or to their analogues in the case of nonautonomous equations.

We recall that the *maximal invariant set* $S = S(N)$ of a closed subset $N \subset X_\alpha$ for the flow ϕ^t consists of all points $z \in N$ such that $\phi^t(z)$ is defined and lies in N for all $t \in \mathbb{R}$. If S is contained in the interior of N then S is called an *isolated invariant set*, and N is an *isolating neighborhood* for S (compare for instance Conley [8]).

For any pair of regular values a and b of f_H the set

$$N_{a,b} = N_{a,b}(H) \stackrel{\text{def}}{=} \{z \in X_\alpha \mid a \leq f_H(z) \leq b\}$$

is an isolating neighborhood for the perturbed gradient flow (17) of ∇f_H.

3.1 Hypotheses on H

The Hamiltonians which we have in mind are the homogeneous examples (3) from the introduction and lower order perturbations of these. For the proofs of the compactness theorems of this section it turns out that only the following properties of the examples (3) are of importance.

The $|u|^{p+1} + |v|^{q+1}$ type Hamiltonians satisfy

$$\begin{aligned} uH_u(x,u,v) + vH_v(x,u,v) - 2H(x,u,v) \\ \geq -C + \delta\left(|u|^{p+1} + |v|^{q+1}\right) \end{aligned} \tag{23}$$

and the $|u|^{p+1} - |v|^{q+1}$ type Hamiltonians satisfy

$$uH_u(x,u,v) - vH_v(x,u,v) \geq -C + \delta\left(|u|^{p+1} + |v|^{q+1}\right) \tag{24}$$

for certain constants $\delta, C > 0$.

3.2 Boundedness in E_α for the autonomous equation

In this subsection we will prove the following.

Lemma 4 *Suppose $H \in C^2(\bar{\Omega} \times \mathbb{R}^2)$ satisfies* (23) *or* (24). *Then the maximal invariant set of the flow generated by* (17) *in $N_{a,b}$ is bounded in E_α.*

The proof of this lemma proceeds in three steps.

Step 1 Let $z : \mathbb{R} \to E_\alpha$ be a solution of (17), with $a \le f_H(z) \le b$. Since (17) is a gradient flow we get the following estimate for free:

$$\int_{\mathbb{R}} \Big(\{\mathrm{I} + K(z(t))\}^{-1}\, z'(t), z'(t) \Big)_{E_\alpha} \mathrm{d}t \le b - a.$$

Since $\|K(z)\| \le 1/2$, we have

$$\big((I+K)^{-1}z, z\big)_{E_\alpha} \ge \frac{1}{2}\|z\|^2_{E_\alpha},$$

and hence

$$\int_{\mathbb{R}} \|z'(t)\|^2_{E_\alpha} \mathrm{d}t \le 2(b-a). \tag{25}$$

Step 2 If H satisfies (23) then we take the E_α inner product of (17) with $[I + K(z)]^{-1} z$ and find

$$\big([I+K(z)]^{-1} z'(t), z(t)\big)_{E_\alpha} + 2 f_H(z) = \int_{\Omega} \{uH_u + vH_v - 2H\} \mathrm{d}x$$

By (23) this implies

$$\int_{\Omega} |u|^{p+1} + \int_{\Omega} |v|^{q+1} \le C + C\|z(t)\|_{E_\alpha} \|z'(t)\|_{E_\alpha}. \tag{26}$$

If H satisfies (24) then we take the inner product with $\tilde{z} = [I + K(z)]^{-1} \times \binom{u}{-v}$ and find

$$\big([\mathrm{I} + K(z)]^{-1} z'(t), \tilde{z}(t)\big)_{E_\alpha} = \int_{\Omega} \{uH_u - vH_v\} \mathrm{d}x$$

From (24) we have $uH_u - vH_v \ge C\{|u|^{p+1} + |v|^{q+1}\} - C$ so that we again get (26). By integrating (26) in time and using (25) we get for any $T \in \mathbb{R}$

$$\int_T^{T+1} \int_{\Omega} |u|^{p+1} + \int_T^{T+1} \int_{\Omega} |v|^{q+1} \le C + C \sup_{\mathbb{R}} \|z(t)\|_{E_\alpha}. \tag{27}$$

Step 3 The gradient flow may be written as

$$z'(t) + Q_\alpha z = \mathcal{D}_\alpha H_z(z) + b(t) \tag{28}$$

where $b(t) = -K(z(t))\nabla f_H(z)$ is uniformly bounded in $C^2(\bar{\Omega})$ by (22). Since $(-\Delta)^{1-2\alpha} : D((-\Delta)^{1-\alpha}) \to D((-\Delta)^\alpha)$ is an isometry, $Q_\alpha : E_\alpha \to E_\alpha$ also is an isometry. Moreover, one has $Q_\alpha^2 = \mathrm{I}_{E_\alpha}$, so that one can split $E_\alpha = E_+ \oplus E_-$, with $Q_\alpha|_{E_\pm} = \pm \mathrm{I}_{E_\pm}$. If we denote the corresponding projections by $P_\pm$ then the operator

$$G(t) = \chi_{\{t>0\}} e^{-t} P_- - \chi_{\{t<0\}} e^t P_+ \tag{29}$$

is a fundamental solution for $\frac{\mathrm{d}}{\mathrm{d}t} + Q_\alpha$ and we can write (28) as

$$z(t) = \int_{\mathbb{R}} G(t-s)\left\{\mathcal{D}_\alpha H_z(z(s)) + b(s)\right\}\mathrm{d}s. \tag{30}$$

Using $\|G(t)\|_{E_\alpha \to E_\alpha} \le e^{-|t|}$ one then gets the estimate

$$\|z(t)\|_{E_\alpha} \le C \sup_{T\in\mathbb{R}} \int_T^{T+1} \|\mathcal{D}_\alpha H_z(z(s))\|_{E_\alpha} \mathrm{d}s + C. \tag{31}$$

By an appropriate version of the Sobolev embedding theorem one has

$$\begin{aligned}
&\|\mathcal{D}_\alpha H_z(z(s))\|_{E_\alpha} \\
&\le \|(-\Delta)^{-2\alpha} H_u\|_{\Theta_\alpha} + \|(-\Delta)^{-2(1-\alpha)} H_v\|_{\Theta_{1-\alpha}} \\
&= \|(-\Delta)^{-\alpha} H_u\|_{L^2} + \|(-\Delta)^{-(1-\alpha)} H_v\|_{L^2} \\
&\le C\|H_u\|_{L^{2n/(n+4\alpha)}} + C\|H_v\|_{L^{2n/(n+4(1-\alpha))}} \text{ (by Sobolev)} \\
&\le C + C\|u\|_{L^{p+1}}^p + C\|v\|_{L^{q+1}}^q
\end{aligned}$$

where we have used (12), and the assumptions (9) on p and q. Using Hölder's inequality we find

$$\begin{aligned}
\int_T^{T+1} \|\mathcal{D}_\alpha H_z\| \mathrm{d}s &\le C + C\left(\int_T^{T+1} \|u\|_{L^{p+1}}^{p+1} \mathrm{d}s\right)^{\frac{p}{p+1}} \\
&\quad + C\left(\int_T^{T+1} \|v\|_{L^{q+1}}^{q+1} \mathrm{d}s\right)^{\frac{q}{q+1}} \\
&\le C + C\left(\sup_{\mathbb{R}} \|u\|_{\theta_\alpha}^{\frac{p}{p+1}} + \sup_{\mathbb{R}} \|v\|_{\theta_{1-\alpha}}^{\frac{q}{q+1}}\right) \qquad \text{(by (27))} \\
&\le C_\varepsilon + \varepsilon \sup_{\mathbb{R}} \|z(t)\|_{E_\alpha}
\end{aligned}$$

for arbitrary $\varepsilon > 0$. Combined with (31) this shows that there is a constant C_0 such that any bounded orbit of the gradient flow which stays in $N_{a,b}$ is in fact bounded by $\|z(t)\|_{E_\alpha} \le C_0$. □

3.3 Boundedness in E_α for nonautonomous equations

We allow K and H to depend on time, and consider the resulting nonautonomous equation

$$z'(t) + (\mathrm{I} + K(z(t)))\nabla f_H(t, z(t)) = 0. \tag{32}$$

We again derive an E_α bound for solutions for which $f_H(t,z)$ is bounded.

3.3.1 Hypotheses for time dependent H, K.

(1) For some positive constant ℓ, $H(t,z)$ does not depend on t in the two intervals $t \geq \ell$ and $t \leq -\ell$.
(2) For some constant $M < \infty$ we have

$$\int_{-\infty}^{\infty} \sup_{x,u,v} \left| \frac{\partial H}{\partial t}(x,t,u,v) \right| \mathrm{d}t \leq M. \tag{33}$$

(3) Each $H(\cdot, t)$ satisfies (23) or (24) uniformly in $t \in \mathbb{R}$.
(4) $K \in C^0(\mathbb{R}, \mathsf{K}_{1/2})$.

Lemma 5 *Let $H \in C^2$ satisfy the above conditions. Then for any $a < b$ there is a constant $C < \infty$ such that any solution $z(t)$ of* (32) *with* $\lim_{t\to\pm\infty} f_H(t, z(t)) \in [a,b]$ *satisfies* $\|z(t)\|_{E_\alpha} \leq C$.

If $z(t)$ is a solution of (32) for which $f_H(t, z(t))$ remains bounded, then, since (32) is autonomous for $|t| \geq \ell$, both limits

$$c_\pm = \lim_{t\to\pm\infty} f_H(t, z(t))$$

exist. Moreover (32) implies

$$\begin{aligned}
&\Big(z'(t), \{\mathrm{I} + K(t, z(t))\}^{-1} z'(t)\Big)_{E_\alpha} \\
&= \big(z'(t), \nabla f_H(t, z(t))\big)_{E_\alpha} \\
&= \frac{\mathrm{d} f_H(t, z(t))}{\mathrm{d}t} - \frac{\partial f_H}{\partial t}(t, z(t)) \\
&= \frac{\mathrm{d} f_H(t, z(t))}{\mathrm{d}t} + \int_\Omega \frac{\partial H}{\partial t}(t, z(x,t))\mathrm{d}x \\
&\leq \frac{\mathrm{d} f_H(t, z(t))}{\mathrm{d}t} + |\Omega| \sup_{x,z} H_t(t,x,z).
\end{aligned}$$

Recall that (18) implies $(\zeta, \{\mathrm{I}+K\}^{-1}\zeta)_{E_\alpha} \geq \frac{1}{2}\|\zeta\|^2_{E_\alpha}$, and integrate over $\mathbb{R}$. One then finds

$$\int_{\mathbb{R}} \|z'(t)\|^2_{E_\alpha} \mathrm{d}t \leq 2(c_+ - c_-) + 2M|\Omega|. \tag{34}$$

We now observe that in the proof of Lemma 4, time independence of $H(t, x, z)$ was only used in step 1 to derive the bound (25). Our current hypotheses imply (34) which, for the arguments in Lemma 4, is just as good as (25). Hence we may conclude Lemma 5.

3.4 Boundedness in Hölder spaces

So far we have seen that entire orbits must be bounded in E_α. In this section we show that one can bootstrap from this to conclude compactness of invariant sets in X_α. More precisely, we will prove:

Lemma 6 (Regularization) *Assume $H \in C^2$ satisfies the conditions in Sect.* 3.3.1. *If $z(t)$ is a solution of* (32) *which is bounded in E_α, then $z(t)$ is actually bounded in $C_0^\beta(\bar{\Omega}) \oplus C_0^{\beta'}(\bar{\Omega})$, where $0 < \beta < \min(2, 4\alpha)$, $0 < \beta' < \min(2, 4(1-\alpha))$. Moreover $\sup_{t\in\mathbb{R}} \|z(t)\|_{C_0^\beta \oplus C_0^{\beta'}}$ is bounded by a constant which only depends on $\sup_{t\in\mathbb{R}} \|z(t)\|_{E_\alpha}$*

As before, for $0 < \beta < 1$, $C_0^\beta(\bar{\Omega})$ denotes the space of β-Hölder continuous functions on $\bar{\Omega}$ which vanish on $\partial\Omega$; if $1 < \beta < 2$ then $C_0^\beta(\bar{\Omega})$ stands for those C^1 functions on $\bar{\Omega}$ which vanish on $\partial\Omega$, and whose gradient is β-Hölder continuous.

Since one can choose $\beta > 2\alpha$ and $\beta' > 2\alpha'$ the compact embedding of $C_0^\beta(\bar{\Omega}) \oplus C_0^{\beta'}(\bar{\Omega})$ in X_α implies that solutions of (32) which remain bounded in E_α actually stay inside a compact subset of X_α.

For the proof we show that the operator $G(t)\mathcal{D}_\alpha$ has a regularizing effect, and we apply this to (30). A bootstrapping argument then shows that $z(t)$ is uniformly bounded in $C_0^\beta(\bar{\Omega}) \oplus C_0^{\beta'}(\bar{\Omega})$.

We begin with an explicit formula for $G(t)\mathcal{D}_\alpha$. By diagonalizing the Laplacian one finds that the eigenspace projections $P_\pm$ are given by

$$P_\pm = \tfrac{1}{2}\begin{pmatrix} \mathrm{I} & \pm(-\Delta)^{1-2\alpha} \\ \pm(-\Delta)^{2\alpha-1} & \mathrm{I} \end{pmatrix}. \tag{35}$$

Hence

$$P_\pm \mathcal{D}_\alpha = \tfrac{1}{2}\begin{pmatrix} (-\Delta)^{-2\alpha} & \pm(-\Delta)^{-1} \\ \pm(-\Delta)^{-1} & (-\Delta)^{-2(1-\alpha)} \end{pmatrix},$$

and

$$|G(t)\mathcal{D}_\alpha H_z(t,z)| \le \tfrac{1}{2}e^{-|t|}\begin{pmatrix} (-\Delta)^{-2\alpha}|H_u| + (-\Delta)^{-1}|H_v| \\ (-\Delta)^{-1}|H_u| + (-\Delta)^{-2(1-\alpha)}|H_v| \end{pmatrix},$$

where the inequality holds pointwise, and componentwise. We have used the fact that the powers $(-\Delta)^{-\alpha}$ with $0 \le \alpha \le 1$ are positive operators (they have positive integral kernels and satisfy $|(-\Delta)^{-\alpha}\varphi| \le (-\Delta)^{-\alpha}|\varphi|$.) We now use the following well known mapping properties[3] of $(-\Delta)^\alpha$ between L^p and Hölder spaces.

[3] The fractional powers $(-\Delta)^{-\alpha}$ satisfy $|(-\Delta)^{-\alpha}f| \le I_\alpha|f|$ where the I_α are the Riesz transforms [23, page 117]. This implies the L_p estimates in the Lemma.

Lemma 7 *Assume* $\varphi \in L^r(\Omega)$.

If $r < \frac{n}{2\alpha}$ *then* $(-\Delta)^{-\alpha}\varphi \in L^s(\Omega)$, *with* $\frac{1}{s} = \frac{1}{r} - \frac{2\alpha}{n}$.

If $r > \frac{n}{2\alpha}$ *then* $(-\Delta)^{-\alpha}\varphi \in C_0^{2\alpha - n/r}(\bar{\Omega})$.

Consider the norm

$$[z]_{r,s} \stackrel{\text{def}}{=} \sup_{t\in\mathbb{R}} \left\{ \|u(t,\cdot)\|_{L^r(\Omega)} + \|v(t,\cdot)\|_{L^s(\Omega)} \right\},$$

where $z(t) = (u(t,\cdot), v(t,\cdot))$. Combined with the growth conditions (12), the L^p and Hölder mapping properties of $(-\Delta)^{-\alpha}$ imply the following

Lemma 8 (bootstrap) *If* $z(t)$ *is a solution of* (30) *with* $[z]_{r,s} < \infty$, *then* $[z]_{r',s'} < \infty$, *where*

$$\begin{cases} \frac{1}{r'} = \max\left(\frac{p}{r} - \frac{2\alpha}{n}, \frac{q}{s} - \frac{2}{n}\right) \\ \frac{1}{s'} = \max\left(\frac{p}{r} - \frac{2}{n}, \frac{q}{s} - \frac{2(1-\alpha)}{n}\right) \end{cases} \tag{36}$$

provided $1/r'$ *and* $1/s'$ *defined by these formulas are positive.*

If $1/r' < 0$, *then* $u(t,\cdot)$ *is uniformly bounded in* $C_0^{-1/r'}(\bar{\Omega})$.

If $1/s' < 0$, *then* $v(t,\cdot)$ *is uniformly bounded in* $C_0^{-1/s'}(\bar{\Omega})$.

By the Sobolev embedding theorems we know that boundedness of $z(t)$ in E_α implies $[z]_{r^*,s^*} < \infty$, with

$$r^* = \frac{2n}{n-4\alpha}, \qquad s^* = \frac{2n}{n-4(1-\alpha)}.$$

Given any r, s for which $[z]_{r,s} < \infty$ we can compute the corresponding r', s' from (36). It now follows from (9) that $r' > r$ and $s' > s$ whenever $r \geq r^*$ and $s \geq s^*$. It is not hard to show that a finite number of iterations gives one $[z]_{r,s} < \infty$ with the corresponding $1/r'$ and $1/s'$ negative. Hence we get L^∞ bounds for u, v, and in particular we get $[z]_{r,s} < \infty$ for all finite r, s. Applying the bootstrap lemma one more time we find that z is bounded in $C_0^\beta \oplus C_0^{\beta'}$, as claimed in the regularization lemma. □

3.5 Action bounds and a Palais Smale condition

The compactness results we have derived so far are very similar to the Palais Smale condition. In the sequel we will need the Palais Smale condition, so we prove it here, first for "+" type Hamiltonians, then for the "−" type.

Lemma 9 *Assume that* H *satisfies* (23). *If* $z_n \in X_\alpha$ *is a sequence for which*

$$\lim_{n\to\infty} \|\nabla f_H(z_n)\|_{E_\alpha} = 0$$

$$f_H(z_n) = a,$$

for some $a \in \mathbb{R}$ *then* a *is a critical value of* f_H.

Proof. Let $\varepsilon_n = \|\nabla f_H(z_n)\|_{E_\alpha}$, so that

$$\begin{aligned} C\varepsilon_n\|z_n\|_{E_\alpha} &\geq (-\nabla f_H(z_n), z_n)_{E_\alpha} \\ &= (-\nabla^{L^2} f_H(z_n), z_n)_{L^2} \\ &\geq (z_n, \partial_\Delta z_n + H_z(x, z_n))_{L^2} \\ &= \int_\Omega \{-2\nabla u_n \cdot \nabla v_n + u_n H_u + v_n H_v\} \,\mathrm{d}x \\ &\geq -2f_H(z_n) - C|\Omega| + \delta \int_\Omega \left\{|u_n|^{p+1} + |v_n|^{q+1}\right\} \mathrm{d}x. \end{aligned}$$

On the other hand $Q_\alpha z_n = \mathcal{D}_\alpha H_z(x, z_n) + \nabla f_H(z_n)$, which gives

$$\begin{aligned} \|z_n\|_{E_\alpha} = \|Q_\alpha z_n\|_{E_\alpha} &\leq \|\mathcal{D}_\alpha H_z(x, z_n)\| + \varepsilon_n \\ &\leq C|\Omega| + \varepsilon_n + C\|u_n\|^p_{L^{p+1}} + C\|v_n\|^q_{L^{q+1}}. \end{aligned}$$

Combining this estimate with the previous one, we obtain

$$\|z_n\|_{E_\alpha} \leq C_\varepsilon + \varepsilon\|z_n\|_{E_\alpha},$$

for arbitrary small $\varepsilon > 0$, which proves the boundedness of $\{z_n\}$. After passing to a subsequence we may assume that z_n converges weakly in E_α to some z_*. Consequently, since $z_n \to z_*$ in $L^{p+1} \times L^{q+1}$ (Sobolev embeddings) and $\partial_\Delta^{-1} : E_\alpha^* \to E_\alpha$, the right hand side of

$$z_n = -\partial_\Delta^{-1} H_z(x, z_n) + Q_\alpha \nabla f_H(z_n). \tag{37}$$

converges in E_α-norm and $\|z_n - z_*\|_{E_\alpha} \to 0$. The limit z_* must then be a critical point in E_α of f_H with action $f_H(z_*) = a$. In view of our regularization lemma z_* actually belongs to X_α. □

Lemma 10 *Assume that H satifies* (24)*. If $z_n \in X_\alpha$ is a sequence for which*

$$\lim_{n\to\infty} \|\nabla f_H(z_n)\|_{E_\alpha} = 0$$

then z_n has a E_α convergent subsequence.

Proof. The proof is very similar to that of the previous lemma. In the present situation we take the inner product with $\tilde{z}_n = \binom{u_n}{-v_n}$. This leads to

$$\begin{aligned} C\varepsilon_n\|z_n\|_{E_\alpha} &\geq \int_\Omega \{u_n H_u - v_n H_v\} \,\mathrm{d}x \\ &\geq -C|\Omega| + \delta \int_\Omega \left\{|u|^{p+1} + |v|^{q+1}\right\} \mathrm{d}x. \end{aligned}$$

From here on the same arguments apply. □

3.5.1 Corollaries In the proof of Lemma 9 we only used $f_H(z_n) \le a$ to extract an E_α convergent subsequence. Hence it follows that the set of critical points z of f_H with $f_H(z) \le a$ is compact in E_α. Elliptic regularity then implies that the same set is also compact in X_α.

Lemma 10 does not refer to the function values of z_n at all, so for "$-$type" Hamiltonians we conclude that the set of critical points is compact.

4 The Morse index and Fredholm properties

If $z \in X_\alpha$ is a critical point of f_H, then the second derivative of f_H at z is given by the linear operator

$$\mathrm{d}\nabla f_H(z) \cdot w = Q_\alpha w - \mathcal{D}_\alpha[H_{zz}(x,z)w].$$

This operator is a compact perturbation of Q_α, which is invertible on X_α, so $\mathrm{d}\nabla f_H(z)$ is a Fredholm operator of index 0 on X_α. We will call z a *Morse critical point* if $\mathrm{d}\nabla f_H(z) : X_\alpha \to X_\alpha$ is invertible (i.e. injective). We will call f_H a Morse function if all critical points $z \in X_\alpha$ are Morse points.[4]

In this section we will consider the solutions to (32) for some possibly time dependent Hamiltonian which satisfies the hypotheses of Sect. 3.3.1

$$f_H(\pm\infty, \cdot) \text{ are Morse functions on } X_\alpha. \tag{38}$$

For such H we shall study the space of C^1 solutions $z : \mathbb{R} \to X_\alpha$ of (32) which satisfy

$$\lim_{t\to\pm\infty} z(t) = z_\pm$$

for certain critical points z_+ and $z_- \in X_\alpha$ of f_{H_+} and f_{H_-}, respectively, where $H_\pm(x,z) = H(\pm\infty, x, z)$. We denote the space of such solutions by $\mathcal{M}_{H,K}(z_-, z_+)$. By definition, $\mathcal{M}_{H,K}(z_-, z_+)$ is the zeroset of the map $\Phi_{H,K} : \mathcal{Q}^1 \to \mathcal{Q}^0$, where

$$\begin{aligned}
\Phi_{H,K}(z) &= \frac{\mathrm{d}z}{\mathrm{d}t} + (\mathrm{I} + K(z))\nabla f_H(t,z),\\
\mathcal{Q}^1(z_-, z_+) &= z_0(\cdot) + W^{1,2}(\mathbb{R}, X_\alpha),\\
\mathcal{Q}^0(z_-, z_+) &= L^2(\mathbb{R}, X_\alpha).
\end{aligned}$$

and where $z_0 : \mathbb{R} \to X_\alpha$ is an arbitrary smooth map which satisfies $z_0(t) \equiv z_-$ for $t \le -1$ and $z_0(t) \equiv z_+$ for $t \ge 1$. Thus $\mathcal{Q}^1$ is not a vector space but only an affine space, and z_0 is an arbitrarily chosen base point in this space.

[4] We repeat these seemingly standard definitions since X_α is not a Hilbert space, so that the second derivative $f''_H(z)$, in its incarnation as a bilinear form on X_α, can never be coercive.

Lemma 11 *If $H(x, t, z)$ is a smooth function on $\bar{\Omega} \times \mathbb{R} \times \mathbb{R}^2$ then the map $\Phi_H : \mathsf{K} \times \mathcal{Q}^1 \to \mathcal{Q}^0$ defined by*

$$\Phi_H(K, z) = \Phi_{H,K}(z) \tag{39}$$

is smooth.

We leave it to the reader to verify this.

4.1 The linearized equation

The Fréchèt derivative of $\Phi_{H,K}$ is given by

$$\begin{aligned} \mathrm{d}\Phi_{H,K}(z) \cdot w = \frac{\mathrm{d}w}{\mathrm{d}t} &+ \{\mathrm{I} + K(z(t))\}\, \mathrm{d}\nabla f_H(t, z(t)) \cdot w(t) \\ &+ \{\mathrm{d}K(z(t)) \cdot w(t)\}\, \nabla f_H(t, z(t)). \end{aligned} \tag{40}$$

Since we assume that $z_\pm$ are critical points, we have $\lim_{t\to\pm\infty} \nabla f_H(t, z(t)) = 0$; by (19) the linear map $w \mapsto \{\mathrm{d}K(z) \cdot w\}\, \nabla f_H(z)$ is bounded from X_α to $C^2(\bar{\Omega})$, and hence compact from X_α to X_α. The following Proposition now implies compactness of the third term in (40).

Proposition 12 *Let X be a Banach space. If $\{B(t) \mid t \in \mathbb{R}\}$ is a norm-continuous family of compact operators on X, with $\lim_{t\to\pm\infty} \|B(t)\|_{L(X)} = 0$, then the multiplication operator $w(t) \mapsto B(t)w(t)$ is a compact operator from $W^{1,2}(\mathbb{R}, X)$ to $L^2(\mathbb{R}, X)$.*

Thus $\mathrm{d}\Phi_{H,K}$ is a Fredholm operator if and only if the operator

$$T \cdot w \stackrel{\text{def}}{=} \frac{\mathrm{d}w}{\mathrm{d}t} + \{\mathrm{I} + K(z(t))\}\, \mathrm{d}\nabla f_H(t, z(t)) \cdot w(t)$$

is Fredholm. We conjugate this operator with $S(t) = \sqrt{\mathrm{I} + K(z(t))}$, with result

$$S(t)^{-1} T S(t) \cdot w = \frac{\mathrm{d}w}{\mathrm{d}t} + S(t)^{-1} S'(t) w(t) + S(t) \mathrm{d}\nabla f_H(t, z(t)) S(t) w(t).$$

Proposition 13 *$S(t)^{-1} S'(t)$ is a compact operator from $W^{1,2}(\mathbb{R}, X_\alpha)$ to $L^2(\mathbb{R}, X_\alpha)$.*

Proof. By assumption $K = K(z(t))$ is a compact operator on X_α whose E_α operator norm is not more than $\frac{1}{2}$. Hence the spectrum of K is contained in the disk with radius $\frac{1}{2}$ and we can define the square root via a Dunford integral of the resolvent $R(\lambda, t) = (\lambda - K(z(t)))^{-1}$ of $K(z(t))$:

$$\sqrt{\mathrm{I} + K(z(t))} = \frac{1}{2\pi i} \oint_{|\lambda|=\rho} R(\lambda, t) \sqrt{1 + \lambda}\, \mathrm{d}\lambda,$$

with $\frac{1}{2} < \rho < 1$, so that $\sqrt{1+\lambda}$ is analytic inside the contour, and so the contour encloses the spectrum of $K(z(t))$. The derivative $S'(t)$ is given by

$$\begin{aligned} S'(t) &= \frac{1}{2\pi i} \oint \frac{\partial R}{\partial t}(\lambda, t)\sqrt{1+\lambda}\mathrm{d}\lambda \\ &= \frac{1}{2\pi i} \oint R(\lambda, t)\left\{\mathrm{d}K \cdot z'(t)\right\} R(\lambda, t)\sqrt{1+\lambda}\mathrm{d}\lambda. \end{aligned}$$

Since $z'(t)$ vanishes at $t = \pm\infty$, the $L(X_\alpha, C_0^2(\bar{\Omega}))$ norm of $\mathrm{d}K(z(t)) \cdot z'(t)$ also vanishes at $t = \pm\infty$. By proposition 12 $S'(t)$ is therefore compact. □

Dropping the term $S(t)^{-1}S'(t)$ we see that $\mathrm{d}\Phi_{H,K}$ is Fredholm if and only if the operator

$$L_A = \frac{\mathrm{d}}{\mathrm{d}t} + A(t) \tag{41}$$

is Fredholm, where $A(t) = S(t)\mathrm{d}\nabla f_H(t, z(t))S(t)$. Furthermore, $\mathrm{d}\Phi_{H,K}(z)$ will have the same Fredholm index as L_A.

Lemma 14 *$A(t)$ is a bounded operator on X_α and symmetric with respect to the E_α inner product. Moreover, $A(t)$ can be written as*

$$A(t) = Q_\alpha + B(t) \tag{42}$$

where $B(t)$ is compact on X_α and symmetric with respect to the E_α inner product.

Proof. We already know that $A(t)$ is bounded on X_α.

By definition $\mathrm{d}\nabla f_H(t, z)$ is selfadjoint on E_α, since it is an operator representing a symmetric form. For any bounded operator S the operator $S^*\mathrm{d}\nabla f_H(t, z)S$ is therefore also selfadjoint on E_α. Since $S(t)$ is selfadjoint, we find that $S(t)\mathrm{d}\nabla f_H(t, z)S(t)$ is selfadjoint on E_α, which implies that it is symmetric. To obtain the representation (42) we recall that $\mathrm{d}\nabla f_H$ is a compact perturbation of Q_α, so, modulo compact operators $S(t)\mathrm{d}\nabla f_H S(t) = S(t)Q_\alpha S(t)$.

We now observe that

$$\sqrt{\mathrm{I}+K} = \mathrm{I} + KM, \qquad M = \left(\mathrm{I} + \sqrt{\mathrm{I}+K}\right)^{-1},$$

which shows that $S(t) = \mathrm{I}+$ a compact operator on X_α. Therefore $S(t)Q_\alpha \times S(t) = Q_\alpha$ modulo a compact operator on X_α. □

We now consider the following subsets of the space of bounded operators on X_α

$$\mathcal{A} = \left\{ Q_\alpha + B \;\middle|\; \begin{array}{c} B \text{ is a compact operator on } X_\alpha \\ \text{and } B \text{ is } E_\alpha\text{-symmetric.} \end{array} \right\}$$

and

$$\Sigma = \{A \in \mathcal{A} \mid \ker(A) \neq \{0\}\}.$$

We have just shown that $A(t) \in \mathcal{A}$ for all $t \in \mathbb{R}$. By (38) $\mathrm{d}\nabla f_H(\pm\infty, z_\pm)$ and hence $A(\pm\infty)$ are invertible operators on X_α.

Lemma 15 *Let $A : \mathbb{R} \to \mathcal{A}$ be a continuous family of operators, with $A_\pm = A(\pm\infty) \notin \Sigma$. Then the operator L_A defined in* (41) *is a Fredholm operator from $W^{1,2}(\mathbb{R}, X_\alpha)$ to $L^2(\mathbb{R}, X_\alpha)$. Its index $\mu(A_-, A_+)$ only depends on the limiting operators $A_\pm$.*

We split the proof in three parts.

Step 1 We first consider the case in which $A(t) \equiv A$ is constant. By assumption A must be invertible. Since A is a compact perturbation of Q_α, its spectrum consists of $\sigma(Q_\alpha) = \{-1, 1\}$ and pure point spectrum. Since A is symmetric with respect to the X_α inner product, the point spectrum must be real. Hence A is *hyperbolic*, i.e. its spectrum is disjoint from the imaginary axis. The spectrum of A then divides into a part on the right of the imaginary axis, and a part on the left of this axis. Denoting the projections on the corresponding A-invariant subspaces of X_α by $\pi_\pm$, we can write the inverse of L_A as

$$(L_A)^{-1}z(t) = \int_{\mathbb{R}} G(t-s)z(s)\mathrm{d}s,$$

where

$$G(t) = e^{-tA}\left\{\chi_{\{t>0\}}\pi_+ - \chi_{\{t<0\}}\pi_-\right\}$$

is the fundamental solution of L_A.

Step 2 We consider the case in which $A(t)$ is constant for $t \geq 1$ and $t \leq -1$ with both $A_\pm = A(\pm\infty)$ invertible. For all $u \in W^{1,2}(\mathbb{R}, X_\alpha)$ with support outside $[-1, 1]$ step 1 implies

$$\|z\|_{W^{1,2}} \leq C\|L_A[z]\|_{L^2}.$$

On the other hand, any $z \in W^{1,2}(\mathbb{R}, X_\alpha)$ with support in $[-2R, 2R]$ satisfies

$$\begin{aligned}\|z\|_{W^{1,2}} &\leq C\|z' + A_- z\|_{L^2}\\ &\leq C\left\{\|L_A z\|_{L^2} + \|K_R(t)z\|_{L^2}\right\}\end{aligned}$$

in which $K_R(t)z = \xi(t)(A(t) - A_-)z$ for some continuous function $\xi(t)$ with $\xi(t) \equiv 1$ for $|t| \leq R$, and $\mathrm{supp}\xi \subset [-2R, 2R]$.

Let $\zeta \in C^\infty(\mathbb{R})$ satisfy $\zeta(t) = 0$ for $|t| \leq 1$ and $\zeta(t) = 1$ for $|t| \geq 2$. Put $\eta_\infty(t) = \zeta(t/R)$ and $\eta_0(t) = 1 - \eta_\infty(t)$. Then any $z \in W^{1,2}(\mathbb{R}, X_\alpha)$ satisfies

$$\begin{aligned}\|z\|_{W^{1,2}} &\leq \|\eta_\infty z\|_{W^{1,2}} + \|\eta_0 z\|_{W^{1,2}} \\ &\leq C\left\{\|L_A(\eta_\infty z)\|_{L^2} + \|L_A(\eta_0 z)\|_{L^2} + \|K_R(t)\eta_0 z\|_{L^2}\right\} \\ &\leq C\left\{\|L_A z\|_{L^2} + R^{-1}\|\zeta'\|_\infty \|z\|_{W^{1,2}} + \|K_R(t)z\|_{L^2}\right\}\end{aligned}$$

Thus for large enough R we obtain

$$\|z\|_{W^{1,2}} \leq C\left\{\|L_A z\|_{L^2} + \|K_R(t)z\|_{L^2}\right\}.$$

Since the multiplication operator $z \mapsto K_R(t)z(t)$ is compact from $W^{1,2}$ to L^2, this estimate implies that the kernel of L_A is finite dimensional, and that the range of L_A is closed.

To see that the range of L_A has finite codimension we consider the adjoint equation. The dual of $L^2(\mathbb{R}, X_\alpha)$ is $L^2(\mathbb{R}, {X_\alpha}^*)$, and any functional $\Lambda \in L^2(\mathbb{R}, {X_\alpha}^*)$ which annihilates $\mathrm{Range}(L_A)$ is a distribution solution of $-\Lambda'(t) + A(t)^*\Lambda(t) = 0$ (see [26]). The equation implies that Λ actually belongs to $W^{1,2}(\mathbb{R}, {X_\alpha}^*)$, and hence that Λ belongs to the kernel of $L_{-A^*} : W^{1,2}(\mathbb{R}, {X_\alpha}^*) \to L^2(\mathbb{R}, X_\alpha)$. Our arguments for the finite dimensionality of $\mathrm{kern}(L_A)$ also apply to L_{-A^*}, and therefore the annihilator of $\mathrm{Range}(L_A)$ is finite dimensional, as claimed.

Step 3 We conclude with the general case by noting that any $A(t)$ satisfying our hypotheses can be written as $A(t) = A_0(t) + K(t)$, where $A_0(t)$ is constant for $|t| \geq 1$, and $K(t)$ is a continuous family of compact operators on X_α with $K(\pm\infty) = 0$. The multiplication operator $z \mapsto K(t)z(t)$ is compact from $W^{1,2}(\mathbb{R}, X_\alpha)$ to $L^2(\mathbb{R}, X_\alpha)$. We have shown in Step 2 that L_{A_0} is Fredholm, so that L_A is a compact perturbation of a Fredholm operator, and hence is also Fredholm. □

4.2 Cocycle Property and the renormalized index

Lemma 16 *For any A_1, A_2, $A_3 \in \mathcal{A} \setminus \Sigma$ one has*

$$\mu(A_1, A_2) + \mu(A_2, A_3) = \mu(A_1, A_3).$$

Hence, if we define $\mu_0(A) = \mu(A, Q_\alpha)$, we have

$$\mu(A_1, A_2) = \mu_0(A_1) - \mu_0(A_2).$$

Our proof follows Robbin and Salamon [17].

Proof. For $(i,j) = (1,2)$ or $(2,3)$ we let $A_{ij}(t) \in \mathcal{A}$ depend continuously on t with $A_{ij}(t) = A_i$ for $t \leq -1$, $A_{ij}(t) = A_j$ for $t \geq 1$. We also put

$$A_{13}(t) = \begin{cases} A_{12}(t+2R) & \text{for } t \leq 0, \\ A_{23}(t-2R) & \text{for } t \geq 0 \end{cases}$$

in which $R > 1$ is a large constant.

Consider the operators

$$L = \frac{d}{dt} + \begin{pmatrix} A_{12}(t) & 0 \\ 0 & A_{23}(t) \end{pmatrix},$$

$$M = \frac{d}{dt} + \begin{pmatrix} A_{13}(t) & 0 \\ 0 & A_2 \end{pmatrix}.$$

Both are Fredholm operators from $W^{1,2} \oplus W^{1,2}$ to $L^2 \oplus L^2$. The first has $\operatorname{index}(L) = \mu(A_1, A_2) + \mu(A_2, A_3)$, while the second has $\operatorname{index}(M) = \mu(A_1, A_3) + \mu(A_2, A_2) = \mu(A_1, A_3)$ (in step 1 of the proof of lemma 15 we saw that $\mu(A_2, A_2) = 0$ for any A_2.)

Choose a smooth function $\zeta(t)$ with $\zeta(t) \equiv 0$ for $\eta \leq -1$, and $\zeta(t) \equiv \pi/2$ for $t \geq 1$. We now conjugate M with $R(t) = \left(\begin{smallmatrix} \cos\zeta(t/R) & -\sin\zeta(t/R) \\ \sin\zeta(t/R) & \cos\zeta(t/R) \end{smallmatrix}\right)$ and find

$$\begin{aligned} \hat{M} &= R(t)^{-1} \cdot M \cdot R(t) \\ &= \frac{\mathrm{d}}{\mathrm{d}t} + \frac{\zeta'(t/R)}{R} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} + \begin{pmatrix} A_{12}(t+2R) & 0 \\ 0 & A_{23}(t-2R) \end{pmatrix}. \end{aligned}$$

Next we conjugate $\hat{M}$ with a shift operator $S(u,v)(t) = (u(t+2R), v(t-2R))$, with result

$$S^{-1}\hat{M}S = L + \begin{pmatrix} 0 & \zeta'(t/R-2)/R \\ -\zeta'(t/R-2)/R & 0 \end{pmatrix}.$$

Thus we see that M is conjugate with $S^{-1}\hat{M}S$, and that $S^{-1}\hat{M}S$ differs from L by an operator with norm bounded by $\|\zeta'\|_\infty/R$. If R is large enough, then $S^{-1}\hat{M}S$ and L must have the same Fredholm index, and consequently L and M have the same index. This proves the cocycle property for μ. □

We can now define the generalized Morse index of critical points of f_H.

Definition 17 *Let $z \in X_\alpha$ be a Morse critical point of f_H. Then the* **renormalized Morse index** *of z is $\mu_0(z) = \mu_0(\mathrm{d}\nabla f_H(z))$.*

4.3 The spectral flow formula

If $A(t) \in \mathcal{A}$ is a continuous family with $A(\pm\infty)$ invertible, then one can compute the index $\mu(A_1, A_2)$ from the way the spectrum of $A(t)$ changes as t increases from $-\infty$ to ∞. Since $A(t) = Q_\alpha + K(t)$ with $K(t)$ compact, the essential spectrum of $A(t)$ is $\{-1, +1\}$. Thus there is a neighborhood of $\lambda = 0$ which only contains point spectrum of $A(t)$. If $A(t)$ is a generic family in $\mathcal{A}$, then there will only be a finite number of times $t_j \in \mathbb{R}$ at which $A(t)$ fails to be invertible. At these times $\mathrm{kern}(A(t_j))$ will be one dimensional and, near t_j, the corresponding eigenvalue $\lambda_j(t)$ depends smoothly on t, with $\lambda_j'(t) \neq 0$. The *Spectral flow formula*[5] for $\mu(A_-, A_+)$ states that the index of $\frac{\mathrm{d}}{\mathrm{d}t} + A(t)$ is the number of t_j with $\lambda_j'(t_j) > 0$ minus the number of t_j with $\lambda_j'(t_j) < 0$, i.e. $\mu(A_-, A_+)$ is the number of eigenvalues of $A(t)$ that cross the imaginary axis as t increases from $-\infty$ to $+\infty$. In other words,

$$\mu(A^-, A^+) = \sum_j \mathrm{sign}\lambda_j'(t_j). \tag{43}$$

This was proved in [17] in the context of unbounded operators $A(t)$ with compact resolvent, in particular for elliptic operators on compact manifolds.

Our setting is slightly different, since our $A(t)$ are bounded (integral) operators. We now outline an argument that proves (43) still holds in our setting. The idea is simple and standard: first we note that, given $A_0 \in \mathcal{A} \setminus \Sigma$, $\mu(A_1, A_0)$ cannot change if A_1 varies within one connected component of $\mathcal{A} \setminus \Sigma$. Then we see how $\mu(A_1, A_0)$ changes if A_1 crosses Σ from one component of $\mathcal{A} \setminus \Sigma$ to another.

Lemma 18 *Let A_1 and A_2 lie in the same connected component of $\mathcal{A} \setminus \Sigma$. Then $\mu(A_1, A_2) = 0$.*

In particular, if $A = S(t)\mathrm{d}\nabla f_H(z)S(t)$, for some nondegenerate critical point z of f_H, then $\mu_0(A)$ is independent of $S(t)$, i.e. of the chosen metric $K(\cdot)$. Indeed, by deforming $K(\cdot)$ linearly to zero one obtains a family $A_\theta = S_\theta(t)\mathrm{d}\nabla f_H(z)S_\theta(t)$ in $\mathcal{A} \setminus \Sigma$ which connects $S(t)\mathrm{d}\nabla f_H(z)S(t)$ with $\mathrm{d}\nabla f_H(z)$.

Proof of Lemma 18. Choose $\zeta \in C^\infty(\mathbb{R})$ with $\zeta(t) = 0$ for $t \leq -1$, and $\zeta(t) = 1$ for $t \geq 1$. Consider for any pair A_1, $A_2 \in \mathcal{A}$ the operator $L_{A_1,A_2} = \frac{\mathrm{d}}{\mathrm{d}t} + A_1 + \zeta(t)(A_2 - A_1)$. The index of L_{A_1,A_2} is $\mu(A_1, A_2)$.

If $\|A_2 - A_1\|$ is sufficiently small then L_{A_1,A_2} is a small perturbation of L_{A_1}, so that L_{A_1,A_2} is invertible. Thus we find that $\mu(A_1, A_2) = 0$ for all A_2 in a neighborhood of A_1. Since this is true for all A_1 we find that the equivalence classes of the relation $A_1 \sim A_2 \Leftrightarrow \mu(A_1, A_2) = 0$ are open in

[5] The phrase was coined in [3].

$\mathcal{A} \setminus \Sigma$. Any connected component of $\mathcal{A} \setminus \Sigma$ must therefore be contained in one such equivalence class, which is what was claimed. $\square$

One can write Σ as $\Sigma = \cup_{n\geq 1}\Sigma_n$ with

$$\Sigma_n = \{A \in \mathcal{A} \mid \dim\ker(A) = n\}$$

(See [9].) Each Σ_n is a smooth submanifold of $\mathcal{A}$ of codimension $n(n+1)/2$. Indeed, let $A \in \Sigma_n$, and split $H = N \oplus N^\perp$, $N = \ker A$. Any operator $A' \in \mathcal{A}$ near A is of the form $A' = \left(\begin{smallmatrix} P & Q \\ Q^* & S \end{smallmatrix}\right)$ with $P \in \mathcal{L}(N, N)$ self-adjoint, $Q \in \mathcal{L}(N^\perp, N)$ arbitrary, and with $S : N^\perp \to N^\perp$ invertible. Such an operator A' then has the same corank as

$$\begin{aligned} A'' &= \begin{pmatrix} 1 & -QS^{-1} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} P & Q \\ Q^* & S \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -S^{-1}Q^* & 1 \end{pmatrix} \\ &= \begin{pmatrix} P - QS^{-1}Q^* & 0 \\ 0 & S \end{pmatrix}. \end{aligned}$$

Thus A' has corank n exactly when $P = QS^{-1}Q^*$, which shows that Σ_n is analytic with codimension $n(n+1)/2$. We also find that the $n(n+1)/2$ dimensional subpace

$$\Lambda_A = \left\{ A + \begin{pmatrix} P\, 0 \\ 0\, 0 \end{pmatrix} \mid P \in \mathcal{L}(N, N), P = P^* \right\}$$

is transverse to Σ_n at A.

Consider $A_0, A_1 \in \mathcal{A} \setminus \Sigma$, and connect them by a smooth curve $\{A(\theta) \mid 0 \leq \theta \leq 1\}$ in $\mathcal{A}$. After perturbing the curve slightly, if necessary, one may assume it intersects each Σ_n transversely; in particular it does not intersect any of the Σ_n with $n \geq 2$ since their codimension is too high.

Suppose $A(\theta_0) \in \Sigma_1$, and let $u \in X_\alpha$ span the kernel of $A(\theta_0)$. We may again modify the family $A(\theta)$ for θ close to θ_0 so that either

$$A(\theta_0 + \delta) = A(\theta_0) \pm \delta u \otimes u \tag{44}$$

holds for small $\delta \in \mathbb{R}$. Since $u \otimes u$ is positive definite both signs can occur. For small δ $A(\theta_0 + \delta)$ will have $\pm\delta$ as isolated simple eigenvalue. The plus sign will occur if an eigenvalue of $A(\theta)$ crosses the imaginary axis from the left halfplane to the right halfplane as θ increases beyond θ_0: if the eigenvalue goes the opposite way, we will get a minus sign in (44).

We compute $\mu(A(\theta_0 - \delta), A(\theta_0 + \delta))$ by looking at the Fredholm index of the operator $L = \frac{\mathrm{d}}{\mathrm{d}t} + A(\theta_0 + \delta \tanh t)$. If we split $X_\alpha = [u] \oplus [u]^\perp$ then this operator can be written as the direct sum of the operator $L_0 = \frac{\mathrm{d}}{\mathrm{d}t} + A_0$, where A_0 is the restriction of $A(\theta)$ to $[u]^\perp$, and the operator $L_1^\pm = \frac{\mathrm{d}}{\mathrm{d}t} \pm \delta \tanh t$ from $W^{1,2}(\mathbb{R}, \mathbb{R})$ to $L^2(\mathbb{R}, \mathbb{R})$. The Fredholm index of L is the sum of the indices of L_0 and $L_1^{\pm}$. Since A_0 is invertible L_0 has index zero; the index of $L_1^{\pm}$ can be computed directly: the kernel of L_1^{+} is spanned by $\operatorname{sech}(\delta t)$, while L_1^{-}, the adjoint of L_1^{+} is injective. Hence $L_1^{\pm}$ has index ± 1.

The same arguments apply each time $A(\theta)$ crosses Σ_1, and we therefore find that $\mu(A(0), A(1))$ is the number of eigenvalues of $A(\theta)$ that went from the left to the right halfplane, minus the number of eigenvalues that crossed the imaginary axis in the other direction. This concludes the proof of (43).

4.4 Upper and lower indices

If $A \in \mathcal{A}$ is degenerate (i.e. $A \in \Sigma$) then $\mu(A)$ is not defined. Instead we introduce the *upper* and *lower Morse indices* of A which are defined by

$$\mu_+(A) = \limsup_{\varepsilon \downarrow 0} \left\{ \mu(A') \;\middle|\; \begin{array}{c} A' \in \mathcal{A} \setminus \Sigma \\ \|A - A'\| < \varepsilon \end{array} \right\},$$

and

$$\mu_-(A) = \liminf_{\varepsilon \downarrow 0} \left\{ \mu(A') \;\middle|\; \begin{array}{c} A' \in \mathcal{A} \setminus \Sigma \\ \|A - A'\| < \varepsilon \end{array} \right\},$$

respectively.

Lemma 19 $\mu_+(A) - \mu_-(A) = \dim \operatorname{kern} A.$

Proof. Let m be the dimension of kern A. Then any $A' \in \mathcal{A}$ near A will have m eigenvalues close to zero, while the rest of the spectrum of will be bounded away from zero. Hence, if $A', A'' \in \mathcal{A}$ are two operators near A then the spectral flow formula applied to the linear arc $A_\theta = (1-\theta)A' + \theta A''$ implies that $\mu(A')$ and $\mu(A'')$ differ at most m. On the other hand if one chooses $A' = A + \varepsilon P$, $A'' = A - \varepsilon P$, where P is orthogonal projection on kern A, then for sufficiently small $\varepsilon > 0$ one has $\mu(A'') = \mu(A') - m$. □

5 The space $\mathcal{M}_{H,K}(z_-, z_+)$ and its closure

5.1 Broken orbits

Let $H : \mathbb{R} \times \bar{\Omega} \times \mathbb{R}^2 \to \mathbb{R}$ be a smooth Hamiltonian satisfying (12), one of (23) or (24), and the hypotheses in Sect. 3.3.1. For given $z_\pm \in X_\alpha$ we have defined $\mathcal{M}_{H,K}(z_-, z_+)$ to be the set of solutions of $z' = -(\mathrm{I} + K(z))\nabla f_H(t, z)$ with $z(t) \to z_\pm$ as $t \to \pm\infty$.

Although $\mathcal{M}_{H,K}(z_-, z_+)$ was given as a subset of the affine space $\mathcal{Q}^1(z_-, z_+)$, we will consider a different topology on $\mathcal{M}_{H,K}(z_-, z_+)$, namely, the topology of $C^0_{\mathrm{loc}}(\mathbb{R}, X_\alpha)$: uniform X_α norm convergence on bounded intervals. Evaluation of a $z \in \mathcal{M}_{H,K}(z_-, z_+)$ at $t = 0$ defines a map

$z \mapsto z(0)$ from $\mathcal{M}_{H,K}(z_-, z_+)$ to X_α, which is continuous. Since the $z \in \mathcal{M}_{H,K}(z_-, z_+)$ are solutions of an O.D.E. on a Banach space, the map $z \mapsto z(0)$ is one-to-one, and its inverse is continuous. We have therefore identified $\mathcal{M}_{H,K}(z_-, z_+)$ with a subset of $\mathcal{M}^*_{H,K} \subset X_\alpha$ (all initial values $z_0 \in X_\alpha$ whose corresponding solution $z(t)$ to the O.D.E. exists for all time and has the right limits at $t = \pm\infty$). In general this subset will not be closed, but our compactness theorems do imply that $\mathcal{M}^*_{H,K}(z_-, z_+)$ is *precompact in* X_α.

The closure of $\mathcal{M}^*_{H,K}(z_-, z_+)$ is easiest understood in the case where H is autonomous. Assume that H is time independent and let $z_0^i \in \mathcal{M}^*_{H,K}(z_-, z_+)$ be any given sequence. Consider the corresponding orbits

$$\gamma^i = \{z^i(t) \mid t \in \mathbb{R}\}$$

where $z^i(t)$ is the solution to the gradient flow with $z^i(0) = z_0^i$. The closures $\bar\gamma^i = \gamma^i \cup \{z_-, z_+\}$ form a sequence of compact sets in X_α, all of which lie in one larger compact set (they are uniformly bounded in $C_0^\beta \oplus C_0^{\beta'}$; see Lemma 6.) Along some subsequence they will therefore converge to some other compact set $\gamma^* \subset X_\alpha$ which must be invariant under the (K, H)-gradient flow. If the critical points of f_H are isolated (e.g. if f_H is a Morse function) then γ^* consists of a finite number of critical points $z_1^*, \ldots, z_m^*$ and heteroclinic orbits between each pair (z_i^*, z_{i+1}^*).

If H is time dependent, we make the system autonomous by introducing a new indepedent variable $\tau = \tanh(t)$, and considering a system of O.D.E.s for $\tau, z)$. The pair (τ, z) then satisfies

$$\left\{\begin{aligned} \frac{\mathrm{d}z}{\mathrm{d}t} &= -(\mathrm{I} + K(z))\nabla f_H(\operatorname{artanh}\tau, z) \\ \frac{\mathrm{d}\tau}{\mathrm{d}t} &= 1 - \tau^2 \end{aligned}\right. \tag{45}$$

Although $\operatorname{artanh}\tau = \ln\sqrt{(\tau+1)/(\tau-1)}$ is singular at $\tau = \pm 1$, this singularity is nullified by the hypothesis that $f_H(t, z)$ becomes constant for $|t| \geq \ell$. The system (45) therefore defines a smooth O.D.E. on $\mathbb{R} \times X_\alpha$.

By identifying a $z \in \mathcal{M}_{H,K}(z_-, z_+)$ with its trace $\gamma = \{(\tau(t), z(t)) \mid t \in \mathbb{R}\}$ in $[-1, 1] \times X_\alpha$ one can repeat the above arguments to see what happens to an arbitrary sequence in $\mathcal{M}_{H,K}(z_-, z_+)$. After passing to a subsequence one can again show that the γ converge to a broken trajectory. Of this broken trajectory one orbit crosses from $\tau = -1$ to $\tau = +1$ (i.e. corresponds to a solution of the original nonautonomous equation) while all others have $\tau \equiv 1$ or $\tau \equiv -1$, i.e. they correspond to orbits of the autonomous gradient flows at $t = \pm\infty$.

5.2 $\mathcal{M}_{H,K}(z_-, z_+)$ under the assumption of transversality

Since $\mathcal{M}_{H,K}(z_-, z_+)$ is by definition the zero set of $\Phi_{H,K} : \mathcal{Q}^1 \to \mathcal{Q}^0$, it follows from the implicit function theorem that if H is smooth and if 0 is a regular value of $\Phi_{H,K}$, then $\mathcal{M}_{H,K}(z_-, z_+)$ is a smooth manifold whose dimension is exactly the Fredholm index of $\mathrm{d}\Phi_{H,K}$, i.e.

$$\dim \mathcal{M}_{H,K}(z_-, z_+) = \mu_0(z_-) - \mu_0(z_+) \tag{46}$$

A number of special cases are of interest.

Since 0 is a regular value of $\Phi_{H,K} : \mathcal{Q}^1(z_-, z_+) \to \mathcal{Q}^0(z_-, z_+)$ for all possible pairs (z_-, z_+), the derivative $\mathrm{d}\Phi_{H,K}$ always has nonnegative Fredholm index. Hence $\mathcal{M}_{H,K}(z_-, z_+)$ is empty whenever $\mu_0(z_-) < \mu_0(z_+)$.

Index 0 orbits. If $z_\pm$ have the same renormalized index then $\mathcal{M}_{H,K}(z_-, z_+)$ is zero dimensional, so that it consists of isolated solutions $z \in \mathcal{Q}^1(z_-, z_+)$.

If, in addition, H and K are time independent, then all $z \in \mathcal{M}_{H,K}(z_-, z_+)$ must be constant (once $\mathcal{M}_{H,K}(z_-, z_+)$ contains z it also contains all time translates $z_\vartheta(t) = z(t+\vartheta)$; if z is isolated, then one must have $z_\vartheta \equiv z$ for all ϑ.)

Returning to the general non-autonomous case, one observes that any (geometric) limit of $z_i \in \mathcal{M}_{H,K}(z_-, z_+)$ is a broken trajectory $z_- \to z' \to z'' \to \cdots \to z_+$ where each segment $z^{(j)} \to z^{(j+1)}$ has index 0. Only one of these segments is a solution of the nonautonomous equation. The others are solutions of the autonomous equations at $t = \pm\infty$, which we have just pointed out are constant. Since one does not include constant orbits in broken orbits, it follows that the broken orbit actually only has one segment, and that the z_i converge in $\mathcal{Q}^1(z_-, z_+)$ to some other $z_* \in \mathcal{M}_{H,K}(z_-, z_+)$. Therefore $\mathcal{M}_{H,K}(z_-, z_+)$ is compact, and consists of a finite number of trajectories of the (H, K) flow.

Index 1 orbits. If H and K are autonomous and $\mu_0(z_-) = \mu_0(z_+) + 1$ then $\mathcal{M}_{H,K}(z_-, z_+)$ is one dimensional. Since $\mathcal{M}_{H,K}(z_-, z_+)$ contains all time translates z_ϑ of any solution $z \in \mathcal{M}_{H,K}(z_-, z_+)$, one sees that the connected components of $\mathcal{M}_{H,K}(z_-, z_+)$ consist of arcs formed by the time translates of heteroclinic orbits of the (H, K) flow. Any limit of such trajectories is again a broken orbit only one segment of which can have nonzero index. Since zero index orbits are constant, we again find that $\mathcal{M}_{H,K}(z_-, z_+)$ is compact, after one divides out the time translations.

6 Gluing

In general the solution spaces $\mathcal{M}_{H,K}(z_-, z_+)$ are not compact, and a sequence $z_i \in \mathcal{M}_{H,K}(z_-, z_+)$ may converge to a broken orbit. The so-called "gluing construction" provides a more detailed description of the way a sequence $z_i \in \mathcal{M}_{H,K}(z_-, z_+)$ converges to a broken orbit. We illustrate the construction for a broken orbit with only two segments

$$z_0 \xrightarrow{z_{01}} z_1 \xrightarrow{z_{12}} z_2 \tag{47}$$

of an autonomous (H, K) flow. Thus $z_{01} \in \mathcal{M}_{H,K}(z_0, z_1)$ and $z_{12} \in \mathcal{M}_{H,K}(z_1, z_2)$, and we assume $\mu_0(z_0) > \mu_0(z_1) > \mu_0(z_2)$.

Given these connecting orbits one defines for any large $T > 0$

$$z_{02,T}(t) = \left(1 - \eta\left(\frac{t}{T}\right)\right)\; z_{01}(t + 2T) + \eta\left(\frac{t}{T}\right) z_{12}(t - 2T),$$

where $0 \le \eta \in C^\infty(\mathbb{R})$ satisfies $\eta(t) \equiv 0$ for $t \le -1$, $\eta(t) \equiv 1$ for $t \ge 1$ and $\eta'(t) \ge 0$ for all t.

For large T the patched curve $z_{02,T} \in \mathcal{Q}^1(z_0, z_2)$ is almost a solution of $\Phi_{H,K}(z) = 0$. The crux of the gluing construction is that the nondegeneracy of $\mathrm{d}\Phi_{H,K}(z_{01})$ and $\mathrm{d}\Phi_{H,K}(z_{12})$ allows one to find an actual solution of $\Phi_{H,K}(z) = 0$ near $z_{02,T}$ in the $\mathcal{Q}^1$ norm.

The actual solution is constructed as follows. Under the assumption of transversality both $\mathrm{d}\Phi_{H,K}(z_{01})$ and $\mathrm{d}\Phi_{H,K}(z_{12})$ are surjective Fredholm operators, so one can choose bounded right inverses $M_{01}, M_{12} : L^2(\mathbb{R}, X_\alpha) \to W^{1,2}(\mathbb{R}, X_\alpha)$. One "patches" these operators together to form

$$M^T \overset{\text{def}}{=} \zeta_T^- \tau_{-2T} M_{0,1} \tau_{2T} \zeta_T^- + \zeta_T^+ \tau_{2T} M_{0,1} \tau_{-2T} \zeta_T^+,$$

in which $\tau_a \varphi(t) = \varphi(t-a)$, and $\zeta_T^\pm(t)$ is a pair of smooth functions satifying $\zeta_1^-(t)^2 + \zeta_1^+(t)^2 \equiv 1$, $\zeta_1^+(t) \equiv 0$ for $t \le -1$, $\zeta_1^-(t) = \zeta_1^+(-t)$, $\zeta_T^\pm(t) = \zeta_1^\pm(t/T)$.

The operator M^T is an approximate right-inverse for $\mathrm{d}\Phi_{H,K}(z_{02,T})$, in the sense that the operator norm of $\mathrm{d}\Phi_{H,K}(z_{02,T}) \circ M^T - \mathrm{I}_{\mathcal{Q}^0}$ tends to zero as $T \to \infty$.

One then substitutes the ansatz $z = z_{02,T} + M^T w$, for some $w \in \mathcal{Q}^0$, in the equation $\Phi_{H,K}(z) = 0$. Expansion of this equation in a Taylor series results in a fixed point problem for w on $\mathcal{Q}^0$ which has a unique small solution when T is large. We omit the details as they are very similar to those in the cited works on Floer homology[6] – indeed, our situation is a bit simpler since we only have to estimate the $\mathcal{Q}^0 = L^2(\mathbb{R}, X_\alpha)$ norm of w, whereas the usual gluing procedure is done in a Fréchèt space of C^∞ functions where one has to estimate a sequence of seminorms.

If one denotes the solution obtained this way by

$$z_{01} \#_T z_{12}(t) = z_{02,T}(t) + (M_{02} w)(t),$$

[6] In [1] we applied the same procedure in a different setting.

then the conclusion of the gluing procedure is that one can approximate the broken orbit (47) by smooth "glued" orbits $z_{01}\#_T z_{12} \in \mathcal{M}_{H,K}(z_0, z_2)$, *and that these glued orbits are the only ones which can approximate the broken orbit* (47). The glued orbits thus provide a description of the noncompact ends of the solution spaces $\mathcal{M}_{H,K}$.

The solution $z_{01}\#_T z_{12}$ whose construction we have just sketched actually depends on a number of parameters: namely, instead of the ansatz $z = z_{02,T} + M^T w$ one could choose any small $v \in \ker \mathrm{d}\Phi_{H,K}(z_{02,T})$ and put $z = z_{02,T} + v + M^T w$. Substitution in $\Phi_{H,K}(z) = 0$ again leads to a fixed point problem for w with $v \in \ker \mathrm{d}\Phi_{H,K}(z_{02,T})$ as a parameter, and one obtains for small v and large T unique small solutions $w_{T,v}$.

In our setting one can view the gluing procedure from a more dynamical point of view. The solutions $z \in \mathcal{M}_{H,K}(z_-, z_+)$ are actually orbits of a smooth ODE on a Banach space. The λ-lemma from dynamical systems then implies that, when all stable and unstable manifolds of critical points intersect transversally, any broken orbit (47) is "shadowed" by actual orbits of the flow. Since both stable and unstable manifolds of all our critical points are infinite dimensional it seems one can not compute the dimension of the intersection of such manifolds without considering the renormalized index $\mu_0(z)$ of the critical points.

7 Generic flows

In this section we show that nontransversality only occurs for exceptional choices of K and H, and can always be destroyed by arbitrarily small modifications of H and K. We summarize this as follows.

Lemma 20 *1. f_H is a Morse function for generic $H \in C^\infty$.*
2. *Given $H \in C^\infty$, the K-gradient flow of f_H has the Kupka-Smale property for generic $K \in \mathsf{K}_{1/2}$.*
3. *For generic time dependent $H \in C^\infty$ and $K \in C^\infty(\mathbb{R}, \mathsf{K}_{1/2})$ solutions of the nonautonomous equation* (32) *are nondegenerate (0 is a regular value of $\Phi_{H,K}$.)*

7.1 Generically a Morse function

Let H be the space of C^∞ functions h on $\bar{\Omega} \times \mathbb{R}^2$ for which

$$[h]_{\mathsf{H}} \stackrel{\mathrm{def}}{=} \sup_{\bar{\Omega}\times\mathbb{R}^2, n\geq 0} (n!)^{-2} \|\mathrm{d}^n h(x,u,v)\|$$

is finite. This space is a separable Banach space [21], and the map $\psi(h, z) = \nabla f_{H+h}(z)$ is smooth from $\mathsf{H} \times X_\alpha$ to X_α. The derivative of ψ with respect to

z is given by $\mathrm{d}_z\psi(h,z)w = Q_\alpha w - \mathcal{D}_\alpha\left(H_{zz} + h_{zz}\right)w$, which is a compact perturbation of Q_α, and thus Fredholm.

The derivative with respect to h is $\mathrm{d}_h\psi(h,z)k = \mathcal{D}_\alpha[k_z(x,z(x))]$, i.e. an evaluation operator followed by the linear operator $\mathcal{D}_\alpha$. The range of this composition is dense in X_α. Indeed, the range contains all functions of the form $\mathcal{D}_\alpha[\tilde{k}(x)]$, where we let $\tilde{k}$ be independent of z, i.e. $\tilde{k} \in C^\infty(\bar{\Omega})$, with $\sup_\Omega |\mathrm{d}^n\tilde{k}(x)| \leq C(n!)^2$ for some $C < \infty$, and $\tilde{k} \equiv 0$ on $\partial\Omega$. The set of such $\tilde{k}$ is dense in, say $C_0(\bar{\Omega}) \oplus C_0(\bar{\Omega})$, and $\mathcal{D}_\alpha$ maps this space densely into X_α.

We now claim that 0 is a regular value of ψ. Indeed, let $\psi(h,z) = 0$. Then the range of $\mathrm{d}_z\psi(h,k)$ has finite codimension, while the range of $\mathrm{d}_h\psi(h,z)$ is dense in X_α. The sum of these ranges must therefore be all of X_α.

Thus $\Gamma = \{(h,z) \in \mathsf{H} \times X_\alpha \mid \psi(h,z) = 0\}$ is a smooth manifold, and the projection $\pi_1(h,z) = h$ is a Fredholm map from Γ to H. The Sard-Smale theorem [22] says that almost every $h \in \mathsf{H}$ is a regular value (in the sense of Baire category). Any $h \in \mathsf{H}$ is a regular value of the projection π_1 if and only if f_{H+h} is a Morse function. Thus any H can be approximated by $H + h$ whose corresponding f_{H+h} is a Morse function. □

The third part of the lemma can be proved using the same strategy, and the same type of pertubations.

7.2 Generic Kupka-Smale property

We use the same strategy as above to prove part 2 of Lemma 20; the only difference is that one must choose different perturbations. For given $H \in C^\infty(\bar{\Omega} \times \mathbb{R}^2)$ we consider the map $\Phi_H : \mathsf{K}_{1/2} \times \mathcal{Q}^1 \to \mathcal{Q}^0$ given by (39).

Lemma 21 *$0 \in \mathcal{Q}^0$ is a regular value of Φ_H.*

In the proof of this lemma we will need the following observation which tells us that the class of perturbations we have chosen is large enough.

Proposition 22 *For any $\xi \in X_\alpha$, $\xi \neq 0$ and $\eta \in C_0^2$ there exists a $k_0 \in \mathrm{NSym}(H^{-1}, C_0^2)$ with $k_0(\xi) = \eta$.*

Proof. If $(\xi,\eta)_{E_\alpha} \neq 0$ then we define

$$k_0(x) \stackrel{\text{def}}{=} \frac{(x,\eta)_{E_\alpha}}{(\xi,\eta)_{E_\alpha}}\eta.$$

Otherwise we choose a $\zeta \in C_0^2$ with $(\xi,\zeta)_{E_\alpha} \neq 0$ and we put

$$k_0(x) \stackrel{\text{def}}{=} \frac{\zeta(\eta,x)_{E_\alpha} + \eta(\zeta,x)_{E_\alpha}}{(\zeta,\xi)_{E_\alpha}}.$$

In both cases k_0 is an E_α-symmetric finite rank operator which maps E_α into C_0^2. From

(48) $$(\eta, x)_{E_\alpha} = (\mathcal{D}_{-\alpha}\eta, x)_{L^2}$$

one sees that since $\eta \in C_0^2$ one also has $\mathcal{D}_{-\alpha}\eta \in C_0^2 \hookrightarrow H_0^1$ which implies that (48) extends to a continuous functional of $x \in H^{-1}$. Hence the operators k_0 we have defined belong to $\mathrm{NSym}(H^{-1}, C_0^2)$. □

Proof of Lemma 21. Let $\Phi_H(K, z) = 0$ and assume $\mathrm{d}\Phi_H(K, z)$ is not surjective. Then there is a $\Lambda \in L^2(\mathbb{R}, X_\alpha{}^*)$ which annihilates $\mathrm{Range}(\mathrm{d}\Phi_H(K, z))$. The derivative $\mathrm{d}_z\Phi_H(K, z)$ is the same as $\mathrm{d}\Phi_{H,K}(z)$, which we know to be a Fredholm operator. Hence the range of $\mathrm{d}\Phi_H(H, z)$ has finite codimension. By assumption Λ annihilates the range of $\mathrm{d}\Phi_{H,K}(z)$ and thus must be a weak solution of the adjoint equation $-\Lambda'(t) + (Q_\alpha + B(t)^*)\Lambda(t) = 0$, where we have written $\mathrm{d}\Phi_{H,K}(z) = \frac{\mathrm{d}}{\mathrm{d}t} + Q_\alpha + B(t)$. By regularity Λ belongs to $W^{1,2}(\mathbb{R}, X_\alpha{}^*)$; in particular, $\Lambda(t)$ is continuous.

The derivative of Φ_H with respect to K is given by

$$\mathrm{d}_K\Phi_H(K, z) \cdot k = k(z(t))\nabla f_H(z(t)).$$

By assumption Λ also annihilates the range of $\mathrm{d}_K\Phi_H(K, z)$. We now choose a fixed $k_0 \in \mathrm{NSym}(H^{-1}, C_0^2)$, some $t_0 \in \mathbb{R}$, and put

$$k^\varepsilon(z) = \frac{1}{\varepsilon}\phi\left(\frac{\|z - z(t_0)\|_{E_\alpha}}{\varepsilon}\right) k_0$$

with $\phi(s)$ as in (21). Then

$$\begin{aligned} 0 &= \lim_{\varepsilon\downarrow 0} \langle \Lambda, \mathrm{d}_K\Phi_H(K, z) \cdot k^\varepsilon \rangle \\ &= \lim_{\varepsilon\downarrow 0} \frac{1}{\varepsilon} \int_{\mathbb{R}} \langle \Lambda(t), k_0\nabla f_H(z(t))\rangle \phi\left(\frac{\|z(t) - z(t_0)\|_{E_\alpha}}{\varepsilon}\right) \mathrm{d}t \\ &= \frac{\int_{-1}^{1} e^{-1/(1-s^2)}\mathrm{d}s}{\|z'(t_0)\|_{E_\alpha}} \langle \Lambda(t_0), k_0\nabla f_H(z(t_0))\rangle . \end{aligned}$$

This holds for all $k_0 \in \mathrm{NSym}(H^{-1}, C_0^2)$. Since $\nabla f_H(z(t_0)) \neq 0$ Proposition 22 guarantees us that we can let $k_0\nabla f_H(z(t_0))$ be any vector in C_0^2 we like, by choosing the right k_0. Since C_0^2 is dense in X_α this implies $\Lambda(t_0) = 0$. This holds for any t_0, so we see that $\mathrm{d}\Phi_H(K, z)$ is surjective after all. □

The zero set of Φ_H is a smooth manifold. The canonical projection π_1 onto $\mathsf{K}_{1/2}$ must be Fredholm, so that Baire-almost every $K \in \mathsf{K}_{1/2}$ is a regular value of π_1. And such regular values correspond to K for which $0 \in \mathcal{Q}^0$ is a regular value of $\Phi_{H,K}$.

8 Floer homology

8.1 Definition in the transverse case

We now describe the definition of the Floer homology groups $FH_p(N, H, K)$ of a particular isolating neighborhood $N \subset X_\alpha$ for some (H, K) gradient flow.

We first treat with the case in which $H \in C^\infty$, for which f_H is a Morse function, and for which K is such that the (H, K) flow is a Kupka-Smale flow. Let $\mathrm{Cr}_p(N, H)$ denote the set of critical points of f_H which are contained in N and which have renormalized index $p \in \mathbb{Z}$. Then denote the $\mathbb{Z}_2$ module generated by the critical points $z \in \mathrm{Cr}_p(N, H)$ by $C_p(N, H)$. Thus the elements of $C_p(N, H)$ are finite formal linear combinations $a_1[z_1] + \cdots + a_m[z_m]$ of critical points in N of index p, and with coefficients $a_i \in \mathbb{Z}_2$.

One defines a boundary operator $\partial_p(N, H) = \partial : C_p(N, H) \to C_{p-1}(N, H)$ by setting

$$\partial[z] = \sum_{z' \in C_{p-1}(N,H)} n_{N,H,K}(z, z')[z'], \tag{49}$$

where $n_{N,H,K}(z, z')$ is the number (mod 2) of heteroclinic orbits of the gradient flow $z' = -(\mathrm{I} + K(z))\nabla f_H(z)$ from z to z' which stay entirely within the isolating neighborhood N. For every pair z, z' for which $\mu_0(z) - \mu_0(z') = 1$ the number $n_{N,H,K}(z, z')$ is well-defined since $\mathcal{M}_{H,K}(z, z')$ contains only finitely many (modulo translations) orbits (compactness, see Sect. 5).

Lemma 23 *The sum in* (49) *is finite.*

Proof. All critical points z' at the end of a heteroclinic orbit starting at z have $f_H(z') < f_H(z)$. The corollaries to Lemmas 9 and 10 imply that the set of critical points to which z can connect is compact. Since critical points of a Morse function are isolated the sum in (49) is indeed finite. □

The proof that the operator ∂ actually is a boundary operator, i.e. that $\partial_{p-1} \circ \partial_p : C_p \to C_{p-2}$ vanishes is completely standard. The argument begins with the observation that $\partial\partial[z] = \sum_{z''} \sum_{z'} n(z, z')n(z', z'')[z'']$, so that one must show that for any $z \in \mathrm{Cr}_p(N, H)$ and $z'' \in \mathrm{Cr}_{p-2}(N, H)$ the sum $\sum_{z'} n(z, z')n(z', z'')$ vanishes (mod 2). One proves this by considering connecting orbits from z to z'': these occur in one parameter families which are trapped in N, since N is an isolating neighborhood. The ends of such a one parameter family of connecting orbits from z to z'' can be used to pair up the possible broken paths $z \to z' \to z''$, thereby showing their number to be even. This uses the structure of the sets $\mathcal{M}_{H,K}(z,z')$ for $\mu_0(z)-\mu_0(z')=2$ obtained from the gluing-construction (Sect. 6).

In this argument one needs stable and unstable manifolds of the critical points for the (H,K) flow to be transverse, and one needs the connecting orbits to stay within compact sets of the phase space X_α of the flow. We have assumed transversality, and, since the action $f_H(z)$ along a connecting orbit $z \to z'$ is always bounded by $f_H(z)$, we have proved compactness in Sect. 3.

Thus

$$\cdots \to C_p(N,H,K) \xrightarrow{\partial_p} C_{p-1}(N,H,K) \to \cdots$$

is a chain complex. Its homology groups are by definition the Floer homology groups of (N,H,K):

$$FH_p(N,H,K) \stackrel{\text{def}}{=} \frac{\text{Kern}(\partial_p)}{\text{Range}(\partial_{p+1})}. \tag{50}$$

If the set N is not bounded, e.g. if $N = X_\alpha$, then there is no a priori reason why the groups $FH_p(N,H,K)$ should be finite.

If the Hamiltonian H is of "$-$" type, i.e. satisfies (24), then the set of critical points is compact (see Lemma 10), and hence finite. So in this case $FH_p(N,H,K)$ will be finite for any choice of N.

If the Hamiltonian is of "$+$" type (satisfies (23)) then we showed in [2] under additional assumptions on H (which are satisfied by our example Hamiltonian (3)) and under the assumption that the space dimension n is at least 3, that $\text{Cr}_p(X_\alpha,H)$ is compact for each p. In this situation each group $C_p(N,K)$ must therefore be finite, and hence the Floer homology groups are finite. It follows later on that, without using [2], the groups $FH_p(X_\alpha,H,K)$ are finite (Theorem 32).

8.2 Morse relations and Poincaré polynomials

We recall that the Poincaré series of any graded $\mathbb{Z}_2$ module $\{C_p\}_{p\in\mathbb{Z}}$ is defined by

$$P_{\{C_p\}}(t) = \sum_{p\in\mathbb{Z}} \text{rank}(C_p)t^p.$$

The Poincaré polynomial in general is a formal power series in t and t^{-1}, with nonnegative integer coefficients, some or all of which may be $+\infty$. One can add such series and multiply them with polynomials with nonnegative integer coefficients without introducing ambiguity.

If $0 \to C'_p \to C_p \to C''_p \to 0$ is an exact sequence, then one has

$$P_{\{C_p\}}(t) = P_{\{C'_p\}}(t) + P_{\{C''_p\}}(t).$$

If $\{C_p, \partial_p\}$ is a chain complex, then one puts $Z_p = \mathrm{kern}\partial_p \subset C_p$ and $B_p = \mathrm{range}(\partial_{p+1}) \subset C_p$, so that the homology of the chain complex is given by $H_p = Z_p/B_p$. From the exact sequence $0 \to B_p \to Z_p \to H_p$ one deduces

$$P_{\{Z_p\}}(t) = P_{\{H_p\}}(t) + P_{\{B_p\}}(t).$$

On the other hand $C_p \cong Z_p \oplus B_{p-1}$, so that we get

$$\begin{aligned} P_{\{C_p\}}(t) &= P_{\{H_p\}}(t) + P_{\{B_p\}}(t) + P_{\{B_{p-1}\}}(t) \\ &= P_{\{H_p\}}(t) + (1+t)Q(t), \end{aligned} \tag{51}$$

where $Q(t)$ is some formal series in t, t^{-1} with nonnegative, possibly infinite coefficients (here we have $Q(t) = P_{\{B_p\}}(t)$.)

When $C_p = C_p(N, H, K)$ is the chain complex defined by some (H, K) flow in an isolating neighborhood $N \subset X_\alpha$, the coefficient of t^p in $P_{\{C_p\}}(t)$ is the number of critical points of renormalized index p, and the identity (51) provides the usual lower bounds of the number of critical points (e.g. by substituting $t = 1$ one obtains that the total number of critical points must be at least the sum of the ranks of the Floer homology groups.) The usual Morse inequalities can be obtained formally from (51) by multiplying both sides with $\frac{1}{1+t} = 1 - t + t^2 - \cdots$.

8.3 Index pairs

Let N and N_0 be isolating neighborhoods for some (H, K) flow. Assume that $N_0 \subset N$, and let N_0 be positively invariant relative to N, i.e. any orbit $\{z(t) \mid 0 \le t < T\}$ which starts in N_0 and stays in N must actually stay in N_0.

Lemma 24 $N_1 \overset{\text{def}}{=} \overline{N \setminus N_0}$ *is also an isolating neighborhood.*

Indeed, let $z \in C^1(\mathbb{R}, X_\alpha)$ be an orbit which stays in N_1 for all t, but at some t_0 touches the boundary ∂N_1. Then $z(t_0)$ cannot belong to ∂N, so $z(t_0) \in \partial N_1 \cap N_0$. But then $z(t)$ must remain in N_0 for all $t \ge t_0$, without entering the interior of N_0. On the other hand, as $t \to \infty$ the orbit $z(t)$ must converge to a critical point of f_H, which must lie in the interior of N_0. □

Consider the following exact sequence

$$0 \to C_p(N_0) \xrightarrow{\alpha} C_p(N) \xrightarrow{\beta} C_p(N_1) \to 0 \tag{52}$$

where $\alpha([z]) = [z]$ is just inclusion, and $\beta([z]) = [z]$ when $z \in N_1$, $\beta([z]) = 0$ when $z \in N_0$. One easily verifies that the sequence (52) is exact, and that α and β are chain homomorphisms.

We can therefore form the long exact sequence on homology

$$\cdots \to FH_{p+1}(N_1) \xrightarrow{\partial^*_{p+1}} FH_p(N_0) \to FH_p(N) \to FH_p(N_1) \xrightarrow{\partial^*_p} \cdots$$

from which we extract a short exact sequence

$$0 \to \frac{FH_p(N_0)}{\mathrm{range}(\partial^*_{p+1})} \to FH_p(N) \to \mathrm{kern}\, \partial^*_p \to 0.$$

Writing q_p for the rank of range(∂^*_{p-1}) we then get

$$\mathrm{rank}\, FH_p(N) + q_p + q_{p-1} = \mathrm{rank}\, FH_p(N_0) + \mathrm{rank}\, FH_p(N_1),$$

i.e. for some nonnegative formal series $Q(t)$ one has

$$P_{N,H,K}(t) + (1+t)Q(t) = P_{N_0,H,K}(t) + P_{N_1,H,K}(t), \tag{53}$$

where we have written $P_{N,H,K}(t)$ for the Poincaré polynomial of the Floer homology groups $FH_p(N, H, K)$.

8.4 Continuation

We now consider the way the Floer homology groups change if we modify the Hamiltonian H and metric K. To this end we fix one particular Hamiltonian H_0 and only consider other Hamiltonians H for which $H - H_0 \in \mathsf{H}$ (H was defined in Sect. 7.1.) We would like to show that these groups do not change if one continuously deforms H and K, keeping the isolating neighborhood N fixed. Unfortunately we will need unbounded isolating neighborhoods in the next section, and this makes that we have to strengthen our definition of isolating neighborhood a little. It turns out that we can prove independence of the Floer homology groups for "stable isolating neighborhoods." After defining these we will show that all isolating neighborhoods we will encounter later are indeed "stable" and discuss the invariance under continuation of the Floer homology.

8.4.1 Stable isolating neighborhoods Let $H \in C^2(\bar{\Omega} \times \mathbb{R}^2)$ satisfy (12), and (23) or (24), and let $K \in \mathsf{K}$.

We will say that $(\hat{H}, \hat{K})$ with $\hat{H} - H \in C^\infty(\mathbb{R}, \mathsf{H})$ and $\hat{K} \in C^\infty(\mathbb{R}, \mathsf{K})$ is an ε **perturbation** of (H, K) if

1. $\hat{H}$ satisfies the conditions (1, 2, 3) of Sect. 3.3.1 with $M < \varepsilon$, and
2. for all $t \in \mathbb{R}$ one has

$$\|\hat{K}(t,\cdot) - K\|_{\mathsf{K}} + \|\hat{H}(t,\cdot) - H\|_{\mathsf{H}} \le \varepsilon$$

Definition 25 *An isolating neighborhood N for the (H, K) gradient flow is* **stable** *if there exists an $\varepsilon > 0$ such that for all ε perturbations $\hat{K} : \mathbb{R} \to \mathsf{K}$ and $\hat{H} : \mathbb{R} \to \mathsf{H}$ any solution $z \in C^1(\mathbb{R}, X_\alpha)$ of*

$$z' = -(\mathrm{I} + \hat{K}(t, z(t)))\nabla f_{\hat{H}}(t, z(t)) \tag{54}$$

with $z(t) \in N$ for all t is actually contained in the interior of N.

Let $N \subset X_\alpha$ be a closed set, and let $\Sigma(N)$ be the set of all pairs (H, K) for which N is a stable isolating neighborhood for the (H, K) gradient flow on X_α.

By construction $\Sigma(N)$ is an open subset of $\mathsf{K} \times (H_0 + \mathsf{H})$.

We write $\Sigma_{\text{reg}}(N)$ for the set of $(H, K) \in \Sigma(N)$ for which f_H is

a Morse function and for which the (H, K) flow is a Kupka Smale flow. In the previous section we showed that $\Sigma_{\text{reg}}(N)$ is dense in $\mathsf{K} \times (H_0 + \mathsf{H})$.

Lemma 26 *Let $H \in H_0 + \mathsf{H}$ and $K \in \mathsf{K}_{1/2}$.*

1. *Let N be an isolating neighborhood for (H, K). If f_H is bounded on N then N is stable for (H, K).*
2. *X_α is always a stable isolating neighborhood.*
3. *If $a \in \mathbb{R}$ is a regular value of f_H then both*

$$N_a = \{x \in X_\alpha \mid f_H(x) \le a\} \text{ and } N^a = \{x \in X_\alpha \mid f_H(x) \ge a\}$$

are stable isolating neighborhoods for (H, K).

Proof. 1. (cf. [10, Theorem 3].) Let $(\hat{H}, \hat{K})$ be an ε perturbation of (H, K). Then

$$|f_{\hat{H}}(t, z) - f_H(z)| \le |\Omega| \sup |\hat{H}(t, \cdot) - H(\cdot)| \le \varepsilon |\Omega|$$

for any $t \in \mathbb{R}$. In particular $f_{\hat{H}}$ is also bounded on N.

Arguing by contradiction there must exist a sequence of ε_n perturbations $(\hat{H}_n, \hat{K}_n)$ with $\varepsilon_n \to 0$, a sequence of solutions $z_n \in C^1(\mathbb{R}, X_\alpha)$ of (54) with $z(t) \in N$ for all $t \in \mathbb{R}$, and a sequence $t_n \in \mathbb{R}$ for which $z_n(t_n) \in \partial N$. Lemmas 5 and 6 apply to solutions of (54), uniformly in $(\hat{H}_n, \hat{K}_n)$. Since the $f_{\hat{H}_n}$ are uniformly bounded on the z_n we can therefore extract a convergent subsequence of $\hat{z}_n(t) = z_n(t_n + t)$ whose limit $\hat{z}$ will be a solution of the autonomous (H, K) flow; this solution $\hat{z}$ stays in N but has $z(0) \in \partial N$. This is not possible since N is an isolating neighborhood.

2. The second statement is trivial since ∂X_α is empty.

3. We use the Palais-Smale condition proved in Lemmas 9 and 10 to conclude that

$$\|\nabla f_H(z)\|_{E_\alpha} \ge \delta \qquad \forall z \in f_H^{-1}(a)$$

for some $\delta > 0$. Now suppose $(\hat{H}, \hat{K})$ are ε perturbations of (H, K), and that $z(t)$ satifies (54) with $z(t_0) \in \partial N_a = \partial N^a = f_H^{-1}(a)$. Then

$$\begin{aligned}
&\left.\frac{\mathrm{d} f_H(z(t))}{\mathrm{d}t}\right|_{t=t_0} \\
&= -\Big\langle \nabla f_H(z(t_0)), \{\mathrm{I} + \hat{K}(t_0, z(t_0))\} \nabla f_{\hat{H}}(t_0, z(t_0)) \Big\rangle \\
&= -\Big\langle \nabla f_H, (\mathrm{I} + \hat{K}) \nabla f_H(t_0, z) \Big\rangle + \Big\langle \nabla f_H, (\mathrm{I} + \hat{K})(\nabla f_H - \nabla f_{\hat{H}}) \Big\rangle \\
&\leq -\frac{1}{2} \|\nabla f_H\|_{E_\alpha}^2 + 2 \|\nabla f_H\|_{E_\alpha} \cdot \|\nabla f_H - \nabla f_{\hat{H}}\|_{E_\alpha} \\
&\leq -\frac{1}{4} \|\nabla f_H\|_{E_\alpha}^2 + 4 \|\nabla f_H - \nabla f_{\hat{H}}\|_{E_\alpha}^2 \\
&\leq -\frac{\delta^2}{4} + 4 \Big\| \mathcal{D}_\alpha \Big\{ H_z(x, z(x, t_0)) - \hat{H}_z(t_0, x, z(x, t_0)) \Big\} \Big\|_{E_\alpha}^2 .
\end{aligned}$$

Since $\mathcal{D}_\alpha : E_\alpha^* \to E_\alpha$ is bounded and $L^\infty \times L^\infty \subset E_\alpha^*$ we can estimate the last term by $4C \|H_z - \hat{H}_z\|_{L^\infty}^2 \leq 4C\varepsilon^2$. So, if we take $\varepsilon = \delta / 8\sqrt{C}$, then

$$\left.\frac{\mathrm{d} f_H(z(t))}{\mathrm{d}t}\right|_{t=t_0} < -\frac{\delta^2}{8} \tag{55}$$

whenever $z(t_0) \in f_H^{-1}(a)$. Hence no solution of (54) which remains in N_a or in N^a can touch the boundary of N_a or N^a, respectively, since (55) would force such a solution to cross the boundary. □

8.4.2 Local continuation Let $(H, K) \in \Sigma(N)$, and let ε be as in the Definition 25. We define $\mathcal{U} \subset \Sigma(N)$ to consist of all (H', K') which satify

$$\|H - H'\|_{\mathsf{H}} + \|K - K'\|_{\mathsf{K}} < \frac{\varepsilon}{10}.$$

For any pair $(H_i, K_i) \in \mathcal{U} \cap \Sigma_{\mathrm{reg}}(N)$ $(i = 0, 1)$ we will now define a homomorphism $\Psi_{1,0} : FH_*(N, H_0, K_0) \to FH_*(N, H_1, K_1)$. The construction, which again is standard [19, 18, 20, 12, 10], proceeds as follows.

Let $\eta \in C^\infty(\mathbb{R})$ satisfy $\eta(t) \equiv 0$ for $t \leq -1$, $\eta(t) \equiv 1$ for $t \geq 1$, and $\eta'(t) \geq 0$ for all t. Define

$$\begin{cases} \hat{K}(t, z) = (1 - \eta(t)) K_0(z) + \eta(t) K_1(z), \\ \hat{H}(t, x, z) = (1 - \eta(t)) H_0(x, z) + \eta(t) H_1(x, z) \end{cases} \tag{56}$$

Then $(\hat{H}, \hat{K})$ is an $\frac{\varepsilon}{5}$ perturbation of (H, K), which we may assume to be regular in the sense of Lemma 20, i.e. 0 is a regular value of the map $\Phi_{\hat{H}, \hat{K}}$. If $(\hat{H}, \hat{K})$ is not regular then we can replace it by an arbitrary small perturbation which is regular, by Lemma 20.

For any pair of critical points $z_0 \in N$ of f_{H_0} and $z_1 \in N$ of f_{H_1} with the same renormalized index, we now define $n_{1,0}(z_0, z_1) \in \mathbb{Z}_2$ to be the number (mod 2) of solutions $z \in C^1(\mathbb{R}, X_\alpha)$ of (54) which remain in N, and which have boundary values $z(-\infty) = z_0$, $z(+\infty) = z_1$. For each $p \in \mathbb{Z}$ we define a homomorphism $\psi_{1,0} : C_p(N, H_0, K_0) \to C_p(N, H_1, K_1)$ by

$$\psi_{1,0}\left([z_0]\right) = \sum_{z_1 \in \mathrm{Cr}_p(N, H_1)} n_{1,0}(z_0, z_1)[z_1]. \tag{57}$$

This sum is again finite, since we have $f_{H_1}(z_1) \le f_{H_0}(z_0) + C$ for any z_1 occuring in the sum, so that the set of z_1 occuring in the sum is compact and hence finite (f_{H_1} is Morse).

Lemma 27 $\psi_{1,0}$ *is a chain homomorphism.*

The proof is identical to that given in [19,18,20,12,10], so we will not reproduce it here. The essential ingredients are compactness of the set of broken orbits of the extended system (45), tranversality of stable and unstable manifolds (which enables one to "glue" broken orbits) and the fact that continua of entire solutions of (54) cannot escape N.

Let $\Psi_{1,0} : FH_*(N, H_0, K_0) \to FH_*(N, H_1, K_1)$ denote the map induced on homology by $\psi_{1,0}$.

Lemma 28 *The homomorphisms $\Psi_{1,0}$ are natural in the sense that*

1. *If $(H_1, K_1) = (H_0, K_0)$ then $\Psi_{1,0}$ is the identity,*
2. *For any three pairs $(H_i, K_i) \in \Sigma_{\mathrm{reg}}(N)$ one has $\Psi_{2,0} = \Psi_{2,1} \circ \Psi_{1,0}$.*

In particular, the $\Psi_{1,0}$ are isomorphisms.

The proof is again largely standard: First one observes that (57) defines a chain homomorphism $\psi_{\hat{H},\hat{K}}$ for any regular ε perturbation $(\hat{H}, \hat{K})$ of (H, K) (not necessarily the one we constructed above.) The chain homomorphisms $\psi_{\hat{H},\hat{K}}$ obtained in this way may depend on the chosen ε perturbation $(\hat{H}, \hat{K})$, but they all induce the same map on (Floer-) homology. This is proved by connecting any pair $(\hat{H}_i, \hat{K}_i)$ of regular ε perturbations of (H, K) by a homotopy

$$\hat{K}_s = (1 - s)\hat{K}_0 + s\hat{K}_1, \quad \hat{H}_s = (1 - s)\hat{H}_0 + s\hat{H}_1$$

and by studying how the solutions of (54) change as one varies the parameter s from 0 to 1. After a making a small perturbation, if necessary, one may again assume that the map

$$\Phi(s, z) = z' + \left\{ \mathrm{I} + \hat{K}_s(t, z) \right\} \nabla f_{\hat{H}_s}(t, z)$$

from $[0, 1] \times \mathcal{Q}^1 \to \mathcal{Q}^0$ has 0 as regular value.

For any particular regular ε perturbation $(\hat{H}, \hat{K})$ solutions of (54) with Fredholm index -1, i.e. solutions connecting critical points of H_0 with index p to critical points of H_1 with index $p+1$ cannot occur. But for one parameter families $\{(\hat{H}_s, \hat{K}_s) \mid 0 \le s \le 1\}$ of ε perturbations solutions with index -1 will occur at isolated s values. Thus there will be a finite number of $(s_j, z_j) \in \Phi^{-1}(0)$ where z_j connects a critical point of index p to one of index $p+1$. Using these connections one can write down a homomorphism $C_p(N, H_0) \to C_{p+1}(N, H_1)$ which is a chain homotopy from $\psi_{\hat{H}_0, \hat{K}_0}$ to $\psi_{\hat{H}_1, \hat{K}_1}$. We refer to the quoted works for more details on this construction. The conclusion is that different ε perturbations lead to the same homomorphism $\Psi_{1,0}$ of Floer homology.

The second part of the proof of Lemma 28 relies on a gluing argument. Given three pairs $(H_i, K_i) \in \Sigma_{\mathrm{reg}}(N) \cap \mathcal{U}$ one considers the canonical homotopies $(\hat{H}_{1,0}, \hat{K}_{1,0})$ and $(\hat{H}_{2,1}, \hat{K}_{2,1})$ defined by (56), and defines another homotopy

$$\begin{cases} \bar{H}_{2,0}(t,x,z) = (1-\eta(t))\hat{H}_{1,0}(t+T,x,z) + \eta(t)\hat{H}_{2,1}(t-T,x,z), \\ \quad \bar{K}_{2,0}(t,z) = (1-\eta(t))\hat{K}_{1,0}(t+T,z) + \eta(t)\hat{K}_{2,1}(t-T,z). \end{cases}$$

One easily shows that $(\bar{H}_{2,0}, \bar{K}_{2,0})$ is an ε perturbation of (H, K), at least if T is large.

Given any $z \in \mathrm{Cr}_p(N, H)$ one can show by means of a gluing argument [19] that for large enough $T \in \mathbb{R}$ the homomorphism $\psi^T_{\bar{H}_{2,0}, \bar{K}_{2,0}}(z)$ equals the composition $\psi^T_{2,1} \circ \psi^T_{1,0}(z)$. Since we know that the homomorphism induced on Floer homology does not depend on the chosen regular ε perturbation, we conclude that $\Psi_{2,0}([z]) = \Psi_{2,1} \circ \Psi_{1,0}([z])$, for any $[z]$. Since the $[z]$ generate $FH_p(N, H)$ this shows that $\Psi_{2,0} = \Psi_{2,1} \circ \Psi_{1,0}$, as claimed.

Finally, if $(H_0, K_0) = (H_1, K_1)$, then the constant homotopy is regular, and the only orbits connecting critical points of the same index are constant orbits. Thus in this case one immediately gets $\psi_{0,0} = \underline{\mathrm{id}}$.

8.4.3 Global continuation In the previous section we have seen that every $(H, K) \in \Sigma(N)$ has a neighborhood $\mathcal{U}$ such that all $(H', K') \in \mathcal{U} \cap \Sigma_{\mathrm{reg}}(N)$ have isomorphic Floer homologies, and that the isomorphisms are natural. Hence, if the given pair (H, K) is not regular, then we may define the Floer homology of this pair to be the Floer homology of any of the regular pairs $(H', K') \in \mathcal{U}$ without introducing any ambiguity.

We will say that two Hamiltonian and metric pairs (H_i, K_i) are related by continuation if they lie in the same path component of $\Sigma(N)$. Since $\Sigma(N)$ is an open subset of an affine space $(H_0 + \mathsf{H}) \times \mathsf{K}$ it is locally path connected, so that path components and connected components coincide.

Given a path $\{(H_s, K_s) \in \Sigma(N) \mid 0 \le s \le 1\}$, from (H_0, K_0) to (H_1, K_1) one can find a partition $0 = s_0 < s_1 < \cdots < s_m = 1$ such that the Floer homologies of (H_{s_i}, K_{s_i}) and $(H_{s_{i+1}}, K_{s_{i+1}})$ are related by the natural isomorphisms of Lemma 28 (i.e. $(H_{s_{i+1}}, K_{s_{i+1}}) \in \mathcal{U}(H_{s_i}, K_{s_i})$). By composing these isomorphisms one obtains an isomorphism of the Floer homologies of (H_0, K_0) and (H_1, K_1). This isomorphism depends on the path taken in $\Sigma(N)$ from (H_0, K_0) to (H_1, K_1); but since the Floer homology of a pair (H, K) is locally constant, the isomorphism from $FH_*(H_0, K_0)$ to $FH_*(H_1, K_1)$ actually only depends on the homotopy class of the path $\{(H_s, K_s) \in \Sigma(N) \mid 0 \leq s \leq 1\}$. Should $\Sigma(N)$ be connected and simply connected all $(H, K) \in \Sigma(N)$ have naturally isomorphic Floer homologies. If $\Sigma(N)$ is multiply connected, then all $(H, K) \in \Sigma(N)$ still have isomorphic Floer homologies, but the isomorphisms need not be unique. In particular one obtains a representation of the fundamental group of $\Sigma(N)$ in $FH_*(H, K)$ by mapping each closed path in $\Sigma(N)$ with base point (H, K) to the corresponding automorphism of $FH_*(H, K)$.

To conclude this section we remark that one can also vary the isolating neighbourhood N and look for continuation theorems for Floer homology under such variation. Clearly if two isolating neighborhoods N_1 and N_2 contain the same isolating invariant set $S = S(N_1) = S(N_2)$, then their Floer homologies must be equal, since the Floer homology groups are defined in terms of the critical points and connecting orbits between them which lie on S. Thus Floer homology will not change under small perturbations of the isolating neighborhood. To go beyond this local continuation observation one could imitate the construction given by Conley in [8, Chapter IV]. We will not do this here.

9 Existence results for critical points

9.1 The index of constant coefficient linearizations

In [2] we studied the index of critical points with the objective of finding a priori estimates for critical points in terms of their index. Along the way we found the following simple properties of the index [2, Sect. 3]. Before describing these properties we note that the index of a critical point $z \in X_\alpha$ of f_H only depends on the 2×2 matrix function

$$x \in \Omega \mapsto H_{zz}(x, z(x)) = \begin{pmatrix} H_{uu}(x, z(x)) & H_{uv}(x, z(x)) \\ H_{vu}(x, z(x)) & H_{vv}(x, z(x)) \end{pmatrix}.$$

Lemma 29 *If z_0, $z_1 \in X_\alpha$ are critical points of f_{H_0} and f_{H_1} respectively, and if $H_{0,zz}(x, z_0(x)) \geq H_{1,zz}(x, z_1(x))$ pointwise, then $\mu_0(z_0) \geq \mu_0(z_1)$.*

Lemma 30 *Let z be a critical point with*

$$H_{zz}(x, z(x)) = \begin{pmatrix} A & -C \\ -C & B \end{pmatrix},$$

where A, B and C are constants, then

1. *z is a Morse critical point if either $AB < 0$ or else if $C + \sqrt{(AB)}$ is not an eigen value of $-\Delta$;*
2. *if $AB \leq 0$ then z has index 0.*
3. *if $AB > 0$ then let k be the number of eigenvalues of $-\Delta$ in the interval $(C - \sqrt{(AB)}, C + \sqrt{(AB)})$; If $A, B > 0$ then the index of z is k, if $A, B < 0$ then z has index $-k$.*

9.2 "$-$" type Hamiltonians

All existence results follow from the following basic fact.

Theorem 31 *Let $H \in C^2(\bar{\Omega} \times \mathbb{R}^2)$ satisfy* (12), (24). *Then for any $K \in \mathsf{K}_{1/2}$*

$$FH_p(X_\alpha, H, K) = \begin{cases} \mathbb{Z}_2 & \textit{if } p = 0 \\ 0 & \textit{otherwise} \end{cases}$$

Proof. We may assume that H is smooth. Let $h \in C^\infty(\bar{\Omega} \times \mathbb{R}^2)$ have compact support and choose h so that $(H + h)_z(x, 0, 0) = \binom{0}{0}$ and $(H + h)_{zz}(x, 0, 0) \equiv \left(\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right)$.

Since h has compact support, $H + \theta h$ satisfies (12), (24) for all $\theta \in [0, 1]$, and H and $H + h$ are related by continuation. Next put

$$\tilde{h}(u, v) \stackrel{\text{def}}{=} \frac{u^2 - v^2}{1 + \varepsilon(u^2 + v^2)}$$

Then H and $\tilde{H} \stackrel{\text{def}}{=} H + h + \lambda \tilde{h}$ are also related by continuation. For large enough λ and small enough $\varepsilon > 0$ one has

$$u\tilde{H}_u - v\tilde{H}_v \geq \delta \left(|u|^{p+1} + |v|^{q+1} \right), \tag{58}$$

for some $\delta > 0$, while $\tilde{H}_z(x, 0, 0) = \binom{0}{0}$ and $\tilde{H}_{zz}(x, 0, 0) = \left(\begin{smallmatrix} \lambda & 0 \\ 0 & -\lambda \end{smallmatrix}\right)$. It follows from lemma 30 that 0 is a Morse critical point with index 0. Finally, the following computation:

$$\begin{aligned} 0 &= \int_\Omega \{v\Delta u - u\Delta v\} \, \mathrm{d}x \\ &= \int_\Omega \left\{ u\tilde{H}_u - v\tilde{H}_v \right\} \mathrm{d}x \\ &\geq \delta \int_\Omega \left\{ |u|^{p+1} + |v|^{q+1} \right\} \mathrm{d}x \end{aligned}$$

shows that $(0, 0)$ is the only critical point of $f_{\tilde{H}}$. Hence $f_{\tilde{H}}$ is a Morse function with only one critical point, which happens to have index 0. $\square$

9.2.1 Existence of a nontrivial solution Let H be a "$-$" type Hamiltonian, and assume that $(0, 0)$ is a Morse critical point with index $m \neq 0$. It then follows immediately from Theorem 31 that $(0, 0)$ cannot be the only critical point. If it were, then f_H would be a Morse function whose only critical point has index $m \neq 0$; in particular, $FH_m(X_\alpha, H)$ would be $\mathbb{Z}_2$, contradicting Theorem 31.

If f_H is a Morse function then we can even assert the existence of two non zero critical points. Indeed, let b_p denote the rank (or dimension) of $FH_p(X_\alpha)$, and let c_p be the number of critical points of f_H with index p, i.e. the rank of $C_p(X_\alpha, H)$. Then, since $FH_p(X_\alpha)$ are the homology groups associated to the $C_p(X_\alpha, H)$ we have the following relation between the b_p and c_p:

(59) $$\sum c_p t^p = \sum b_p t^p + (1 + t)Q(t) \text{ for some } Q(t).$$

In the present context we know that all coefficients are finite, since the number of critical points of a "$-$" type Hamiltonian is finite. One could therefore recover the usual Morse inequalities by dividing both sides by $1 + t$.

We know that all $b_p = 0$, except for $b_0 = 1$, so that we get

$$\sum c_p t^p = 1 + (1 + t)Q(t),$$

which, if we write $Q(t) = \sum q_p t^p$, leads to

(60) $$c_p = \delta_{p,0} + q_p + q_{p-1}.$$

This implies $c_0 \geq 1$, so there exists a critical point of index 0.

To obtain a second critical point we first consider the case $m > 0$. Since $c_m \geq 1$ we find that either $q_m > 0$ or $q_{m-1} > 0$. If $q_m > 0$ then $c_{m+1} \geq q_m > 0$, so there must exist a critical point with index $m + 1$. In this case we therefore have two nontrivial critical points, with indices 0 and m.

If $q_{m-1} > 0$ then either $m > 1$ and we again get two nontrivial critical points whose indices are 0 and $m-1$; if $m = 1$ then (60) implies that $c_0 \geq 2$, so we still have two nonzero critical points, both of which have index 0.

The case $m < 0$ is similar.

9.2.2 Even "$-$" type Hamiltonians If we assume that H is even, i.e.

$$H(x, -u, -v) \equiv H(x, u, v),$$

then the Morse relations (59) imply the existence of more critical points.

Suppose f_H is Morse, an assume that 0 is a critical point with index $m > 0$, then we will show that there must exist critical points of indices $0, 1, 2, \ldots, m - 1$. Indeed, since $-z$ is critical point for any critical point z, all c_p must be even, except c_m which must be odd. Thus (60) implies that $q_p + q_{p-1}$ must be even, except when $p = 0$ or $p = m$, when $q_p + q_{p-1}$ must be odd. Hence all q_p with $p < 0$ have the same parity. Since H is a "$-$" type Hamiltonian there are only finitely many critical points so that all c_p with $p \leq 0$ must be even. Then q_0 must be odd, to make $q_{-1} + q_0$ odd. After that all q_p with $p \leq m$ must be odd, and finally all q_p with $p > m$ must be even. We conclude for $0 \leq p < m$ from (60) that $c_p \geq 2$, as claimed.

It seems very likely that the nondegeneracy condition is not crucial for this result, but an existence proof of $2m$ critical points without the nondegeneracy hypotheses would probably require cup-length estimates and thus information about cup products on Floer homology. We will not pursue this here.

9.3 "+" type Hamiltonians

We again begin by computing the Floer homology of the whole space X_α.

Theorem 32 *Let $H \in C^2(\bar{\Omega} \times \mathbb{R}^2)$ satisfy* (12)*,* (23)*. Assume moreover that for any $\lambda \in \mathbb{R}$ there exists $R < \infty$ such that for all $x \in \Omega$ and $z \in \mathbb{R}^2$ with $|z| \geq R$ one has*

$$\left(\xi, H_{zz}(x, z)\xi\right) \geq \lambda |\xi|^2, \quad \forall \xi \in \mathbb{R}^2. \tag{61}$$

Then for any $K \in \mathsf{K}_{1/2}$, $FH_p(X_\alpha, H, K) = 0$ for all $p \in \mathbb{Z}$.

Proof. Our hypothesis (61) implies that H_{zz} is uniformly bounded from below, say by $-C\mathrm{I}$. Let λ be any (large) number; choose R as given in (61). Construct a bounded function $h \in C^\infty(\mathbb{R}^2)$ with $h_{zz}(z) \geq (\lambda + C)\mathrm{I}$ for $|z| \leq R$ and $h_{zz}(z) \geq -\frac{\lambda}{2}\mathrm{I}$ for all $z \in \mathbb{R}^2$. Then H and $H + h$ are related by continuation. But $H + h$ satisfies $(H + h)_{zz}(x, z) \geq \frac{\lambda}{2}\mathrm{I}$ for all x, z, so Lemmas 29 and 30 imply that any critical point of f_{H+h} has index at least $m_{\lambda/2}$ where m_λ is the number of eigenvalues of $-\Delta$ less than λ. This number can be made arbitrarily high, so we see that for any given $p \in \mathbb{Z}$ we can find a Hamiltonian $H + h$ which is related to H by continuation which does not have critical points of index p. Hence $C_p(X_\alpha, H + h, K)$ must be trivial, and thus $FH_p(X_\alpha, H, K)$ is also trivial. □

9.3.1 Existence of a nontrivial solution In general our computation of the Floer homology groups of X_α does not force a "+" type Hamiltonian to have critical points at all! However, if f_H has one nondegenerate critical point then there must exist another critical point, for otherwise we could conclude that one of the Floer homology groups would be nontrivial. This way we recover the result in [11, 13], in the cases where their assumptions and ours overlap. Indeed, assuming $p_{\rm FF} > 2$ and $q_{\rm FF} > 2$, and $\alpha = p$, $\beta = q$ (notation as in [11]. DeFigeuiredo and Flemer define the exponents p and q differently: our p is $p_{\rm FF} - 1$) one of their conditions requires the Hamiltonian to vanish super quadratically at $z = 0$, which implies that 0 is a nondegenerate critical point of index 0. Hulshof and van der Vorst [13] make a similar hypothesis. The presence of one nondegenerate critical point $z = \binom{0}{0}$ then forces the existence of another critical point, for reasons we have just stated.

9.3.2 Infinitely many solutions if H is even We again assume that H is even, and for the moment assume that f_H is a Morse function. Then $z = 0$ is a critical point; let m be its index.

Theorem 33 *Either f_H has a pair of critical points $\{z_k, -z_k\}$ with index k for each $k > m$, or else there is an m_* such that f_H has infinitely many critical points with index m_*.*

Proof. Assume that f_H has only finitely many critical points with any given index p. We have called this number c_p. The Morse relations (59) imply

$$c_p = q_p + q_{p-1} \tag{62}$$

for all p. Since f_H is even, all c_p are even, except c_m, which is odd.

Since $H_{zz}(x, z)$ is bounded from below the index of critical points is also bounded from below, so that for $p \to -\infty$ one has $c_p = 0$. For all $p < m$ (62) implies that q_p and q_{p-1} have the same parity; they must all be even since $c_p = 0$ for very negative p. We may conclude that all q_p with $p \geq m$ have the same parity; since $q_m = c_m - q_{m-1}$, and c_m is odd, q_m and q_{m-1} have opposite parity, so we find in the end that all q_p with $p \geq m$ are odd. Therefore $c_p \geq 2$ for all $p \geq m$, and we find that f_H has infinitely many critical points. □

We now show how to obtain infinitely many critical points without the hypothesis that f_H is a Morse function.

Theorem 34 *Let H be an even "+" type Hamiltonian as above. Then the set of critical points cannot be bounded in L^∞.*

Proof. Suppose that for some "+" type H the critical point set of the functional f_H is bounded in L^∞. By regularity it would then also be compact in E_α and there would be an $a \in \mathbb{R}$ such that any $a' \geq a$ is a regular value of f_H. Define

$$N_a = \{z \in X_\alpha \mid f_H(z) \leq a\}, \quad N^a = \{z \in X_\alpha \mid f_H(z) \geq a\}.$$

Both sets are stable isolating neighborhoods for (H, K) for any $K \in \mathsf{K}_{1/2}$. By assumption N^a contains no critical points, so its Floer homology groups are all trivial.

To compute the Floer homology of (N_a, H, K) we perturb H and K slightly so H' becomes a Morse function and the (H', K') flow is a Kupka Smale flow. We may assume that the perturbation H' is again even. Thus all critical points of $f_{H'}$ occur in pairs, except for 0. The total number of critical points in N_a (which is finite, since the action $f_{H'}$ of the critical points is bounded by $a + o(1)$) must therefore be odd. By substituting $t = 1$ in the Morse relations (51) for (N_a, H', K') we therefore find that $\sum \mathrm{rank} FH_p(N_a, H', K')$ is odd.

Now apply (53) to the triple $(N_a, N = X_\alpha, N^a)$. We get

$$P_{X_\alpha,H,K}(t) + (1+t)Q(t) = P_{N_a,H,K}(t) + P_{N^a,H,K}(t).$$

The first term on the left and the last term on the right both vanish. If we substitute $t = 1$ we get an even number on the left, and an odd number on the right. The contradiction shows that the critical point set of our original f_H cannot be compact. □

9.4 Estimating the index of critical points

Theorem 33 allows the possibility of an H which has infinitely nondegenerate critical points with the same index m_*, and no other critical points. We would not expect this to happen, but have not been able to prove our expectation in full generality. However, for Hamiltonian functions of the form

$$H(x, u, v) = a(x)\frac{|u|^{p+1}}{p+1} + b(x)\frac{|v|^{q+1}}{q+1} + h(x, u, v) \tag{63}$$

with $h(x, u, v)$ uniformly bounded we have obtained in [2] the following compactness theorem (compare [4]).

Proposition 35 *Let $\Omega \subset \mathbb{R}^n$ with $n \geq 3$. Assume H is as in* (63) *and assume also that*

$$H_{uu}, H_{vv} \geq 0, \qquad H_{uv} \leq 0. \tag{64}$$

Then any critical point $z \in E_\alpha$ of f_H with lower Morse index m is bounded by $\|z\|_{L_\infty} \leq C(m, M)$ for some constant $C(m, M) < \infty$ depending only on m and $M = \sup_{x,u,v} |h(x, u, v)|$.

In fact we prove this for a slightly larger class of perturbations h (see [2, Theorem 1B] for details.) We have not been able to remove the hypothesis that the domain Ω be at least 3 dimensional; neither have we been able to weaken the assumptions (64). It seems likely to us that these hypotheses are not necessary, and that they could perhaps be removed by following a different method of proof.

The proposition can be applied directly to improve the statements of Theorems 33 and 34.

If f_H is a Morse function, and H, Ω are as in the proposition, then f_H cannot have infinitely many critical points with any given particular index, so that for each $k \geq m$ there must be a pair of critical points $\{\pm z_k\}$ with index k.

If f_H is not a Morse function, then one can still assert that the lower Morse indices of the unbounded sequence of critical points z_k obtained in Theorem 34 is unbounded, i.e. $\mu_-(z_k) \to \infty$.

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