

The Poincaré-Hopf Theorem for relative braid classes

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Abstract Braid Floer homology is an invariant of proper relative braid classes, cf. [12]. Closed integral curves of 1-periodic Hamiltonian vector fields on the standard 2-disc may be regarded as braids, and a such determine relative braid classes. If the Braid Floer homology of a proper relative braid class is non-trivial, then additional closed integral curves of the Hamilton equations are forced via a Morse type theory. In this article we show that the Euler-Floer characteristic of Braid Floer homology can be used to force closed integral curves of arbitrary vector fields and yields a Poincaré-Hopf type Theorem. The Euler-Floer characteristic for any proper relative braid class can be computed via a finite cube complex that serves as a model for the given braid class.

Keywords Floer homology · Braids · Parity and spectral flow · Closed integral curves · Euler-Floer characteristic · The Poincaré-Hopf Formula

1 Introduction

Let $\mathbb{D}^2 \subset \mathbb{R}^2$ denote the standard (closed) 2-disc in the plane with coordinates $x = (p, q)$ and let $X(x, t)$ be a smooth 1-periodic vector field on $\mathbb{D}^2$. To be more precise,

- (i) $X \in C^\infty(\mathbb{D}^2 \times \mathbb{R}/\mathbb{Z}; \mathbb{R}^2)$, i.e. $X(x, t + 1) = X(x, t)$ for all $x \in \mathbb{D}^2$ and $t \in \mathbb{R}$;
- (ii) $X(x, t) \cdot \mathbf{v} = 0$ for all $x \in \partial\mathbb{D}^2$, where $\mathbf{v}$ the outward unit normal on $\partial\mathbb{D}^2$.

The set of vector fields satisfying these hypotheses is denoted by $\mathcal{X}(\mathbb{D}^2 \times \mathbb{R}/\mathbb{Z})$. *Closed integral curves* $x(t)$ of X are integral curves¹ of X for which $x(t + \ell) = x(t)$ for some $\ell \in \mathbb{N}$. Every integral curve of X with minimal period ℓ can be regarded as geometric braid with ℓ *strands* by considering all translates $x^i(t) = x(t + i)$, for $t \in [0, 1]$, see Figure 1[left]. They indeed form a geometric braid since the strands

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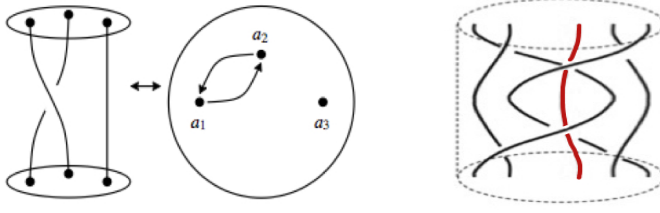


Fig. 1 A 2-periodic and 1-periodic closed integral curve represented as a 3-strand braids [left]. A closed integral curve go minimal period ℓ yields a ℓ -periodic point of the associated time-1 map [middle]. A relative braid with the skeleton (black) and one free strand (red) [right].

x^i are integral curves of X and cannot intersect therefore! Multiple closed integral curves of various periods yield a multi-strand braid.

Let y be a geometric braid consisting of closed integral curves of X , which will be referred to as a *skeleton*. The strands $y^i(t)$, $i = 1, \dots, m$ satisfy the periodicity condition $y(0) = y(1)$ as point sets, i.e. $y^i(0) = y^{\sigma(i)}(1)$ for some permutation $\sigma \in S_m$. Let $x = \{x^1(t), \dots, x^n(t)\}$ be a geometric braid such that the ‘union’ $x \text{ rel } y := \{x^1(t), \dots, x^n(t), y^1(t), \dots, y^m(t)\}$ is again a geometric braid, i.e. the strands in x do not intersect the strands in y . The pair $x \text{ rel } y$ is called a *relative braid*. Two relative braids $x \text{ rel } y$ and $x' \text{ rel } y'$ are equivalent if there exists a homotopy of relative braids connecting $x \text{ rel } y$ to $x' \text{ rel } y'$. The equivalence class is denoted by $[x \text{ rel } y]$ and is called a *relative braid class*. The set of relative braids $x' \text{ rel } y' \in [x \text{ rel } y]$, keeping y' fixed, is denoted by $[x'] \text{ rel } y'$ and is called a *braid class fiber*. A relative braid class $[x \text{ rel } y]$ is *proper* if components $x_c \subset x$ cannot be deformed onto (i) the boundary $\partial \mathbb{D}^2$, (ii) itself,² or other components $x'_c \subset x$, or (iii) components in $y_c \subset y$, see [12] for details. In this paper we are concerned with relative braids $x \text{ rel } y$ for which x has only one strand, i.e. $x(t+1) = x(t)$. The central question is: *given a skeleton y of integral curves, does there exist an integral curve x in the relative braid class $[x \text{ rel } y]$* . The theory also applies if x has more than one strand.

1.1 A brief summary of Braid Floer homology

Fix a Hamiltonian vector field X_H in $\mathcal{X}(\mathbb{D}^2 \times \mathbb{R}/\mathbb{Z})$ of the form $X_H(x, t) = J\nabla H(x, t)$, where

$$J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

and H is a Hamiltonian function with the properties:

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¹ Integral curves of X are smooth functions $x : \mathbb{R} \rightarrow \mathbb{D}^2 \subset \mathbb{R}^2$ that satisfy the differential equation $x_t = X(x, t)$.

² This condition is separated into two cases: (i) a component in x cannot be not deformed into a single strand, or (ii) if a component in x can be deformed into a single strand, then the latter necessarily intersects y or a different component in x .

- (i) $H \in C^\infty(\mathbb{D}^2 \times \mathbb{R}/\mathbb{Z}; \mathbb{R})$, i.e. $H(x, t+1) = H(x, t)$ for all $x \in \mathbb{D}^2$ and $t \in \mathbb{R}$;
- (ii) $H(x, t)|_{x \in \partial \mathbb{D}^2} = 0$, for all $t \in \mathbb{R}/\mathbb{Z}$.

For closed integral curves of X_H of period 1 we define the Hamilton action

$$\mathcal{A}_H(x) = \int_0^1 \frac{1}{2} Jx \cdot x_t - H(x, t) dt,$$

Critical points of the action functional $\mathcal{A}_H$ are in one-to-one correspondence with closed integral curves of period 1. Assume that $y = \{y^j(t)\}$ is a skeleton of closed integral curves of the Hamilton vector field X_H . Consider a proper relative braid class $[x \text{ rel } y]$ and seek 1-periodic solutions x of X_H (i.e. closed integral curves of X_H of period 1) in the fiber $[x] \text{ rel } y$. The set of critical points of $\mathcal{A}_H$ in $[x] \text{ rel } y$ is denoted by $\text{Crit}_{\mathcal{A}_H}([x] \text{ rel } y)$. In order to understand the set $\text{Crit}_{\mathcal{A}_H}([x] \text{ rel } y)$ we consider the negative L^2 -gradient flow of $\mathcal{A}_H$, which yields the Cauchy-Riemann equations

$$u_s(s, t) - Ju_t(s, t) - \nabla H(u(s, t), t) = 0,$$

for which the stationary solutions $u(s, t) = x(t)$ are the critical points of $\mathcal{A}_H$.

To a braid y one can assign an integer $\text{Cross}(y)$ which counts the number of crossings (with sign) of strands in the standard planar projection. In the case of a relative braid $x \text{ rel } y$ the number $\text{Cross}(x \text{ rel } y)$ is an invariant of the relative braid class $[x \text{ rel } y]$. In [12] a monotonicity lemma is proven, which states that along solutions $u(s, t)$ of the nonlinear Cauchy-Riemann equations, the number $\text{Cross}(u(s, \cdot) \text{ rel } y)$ is non-increasing (the jumps correspond to ‘singular braids’, i.e. ‘braids’ for which intersections occur). As a consequence an isolation property for proper relative braid classes exists: the set bounded solutions of the Cauchy-Riemann equations in a proper braid class fiber $[x] \text{ rel } y$, denoted by $\mathcal{M}([x] \text{ rel } y; H)$, is compact and isolated with respect to the topology of uniform convergence on compact subsets of $\mathbb{R}^2$. These facts provide all the ingredients to use Floer’s approach towards Morse Theory for the Hamiltonian action [7]. For generic Hamiltonians which satisfy (i) and (ii) above and for which y is a skeleton, the critical points in $[x] \text{ rel } y$ of the action $\mathcal{A}_H$ are non-degenerate and the set of connecting orbits $\mathcal{M}_{x_-, x_+}([x] \text{ rel } y; H)$ are smooth finite dimensional manifolds. To critical points in $\text{Crit}_{\mathcal{A}_H}([x] \text{ rel } y)$ we assign a relative index $\mu^{CZ}(x)$ (the Conley-Zehnder index, see Section 5) and

$$\dim \mathcal{M}_{x_-, x_+}([x] \text{ rel } y; H) = \mu^{CZ}(x_-) - \mu^{CZ}(x_+).$$

Define the free abelian groups C_k over the critical points of index k , with coefficients in $\mathbb{Z}_2$, i.e.

$$C_k([x] \text{ rel } y; H) := \bigoplus_{\substack{x \in \text{Crit}_{\mathcal{A}_H}([x] \text{ rel } y), \\ \mu(x)=k}} \mathbb{Z}_2 \langle x \rangle,$$

and the boundary operator

$$\partial_k = \partial_k([x] \text{ rel } y; H) : C_k \rightarrow C_{k-1},$$

which counts the number of orbits (modulo 2) between critical points of index k and $k-1$ respectively. Analysis of the spaces $\mathcal{M}_{x_-, x_+}([x] \text{ rel } y; H)$ reveals that (C_*, ∂_*) is

a chain complex, and its (Floer) homology is denoted by $\mathrm{HB}_*([x]\mathrm{rel}y; H)$. Different choices of H and different fibers yield isomorphic Floer homologies and

$$\mathrm{HB}_*([x]\mathrm{rel}y) = \varprojlim \mathrm{HB}_*([x]\mathrm{rel}y; H),$$

where the inverse limit is defined with respect to the canonical isomorphisms $f_k : \mathrm{HB}_k([x]\mathrm{rel}y, H) \rightarrow \mathrm{HB}_k([x']\mathrm{rel}y', H')$. Some properties are:

- (a) the groups $\mathrm{HB}_k([x]\mathrm{rel}y)$ are defined for all $k \in \mathbb{Z}$ and are finite, i.e. $\mathbb{Z}_2^d$ for some $d \geq 0$;
- (b) the groups $\mathrm{HB}_k([x]\mathrm{rel}y) = \varprojlim \mathrm{HB}_k([x]\mathrm{rel}y; H)$ are invariants for the fibers in the same relative braid class $[x]\mathrm{rel}y$, i.e. if $x\mathrm{rel}y \sim x'\mathrm{rel}y'$, then $\mathrm{HB}_k([x]\mathrm{rel}y) \cong \mathrm{HB}_k([x']\mathrm{rel}y')$;
- (c) if $(x\mathrm{rel}y) \cdot \Delta^{2\ell}$ denotes composition with ℓ full twists, then $\mathrm{HB}_k([(x\mathrm{rel}y) \cdot \Delta^{2\ell}]) \cong \mathrm{HB}_{k-2\ell}([x]\mathrm{rel}y)$.

In [12] Braid Floer homology is used as Morse type theory for closed integral curves forced by a skeleton y . The Floer theory applies to the Hamilton action and therefore to Hamiltonian vector fields X_H . It also provides a forcing theory for periodic points for area-preserving diffeomorphisms. Braid Floer homology cannot be applied to arbitrary vector fields X . The objective of this paper is twofold: (i) we extract an invariant from $\mathrm{HB}_*([x]\mathrm{rel}y)$ — the Euler-Floer characteristic — that applies to arbitrary vector fields X and non-triviality of the Euler-Floer characteristic yields forcing of closed integral curves, and (ii) we develop an algorithm to compute the Euler-Floer characteristic.

The remainder of the introduction deals with formulating the main theorems and gives an extensive outline of the steps required to prove the main results.

1.2 The Euler-Floer characteristic and the Poincaré-Hopf Formula

The *Euler-Floer characteristic* of $\mathrm{HB}_*([x]\mathrm{rel}y)$ is defined as follows:

$$\chi(x\mathrm{rel}y) = \sum_{k \in \mathbb{Z}} (-1)^k \dim \mathrm{HB}_k([x]\mathrm{rel}y). \quad (1)$$

A 1-periodic function $x \in C^1(\mathbb{R}/\mathbb{Z})$ is an *isolated* closed integral curve of X if there exists an $\varepsilon > 0$ such that x is the only solution of the differential equation

$$\mathcal{E}(x(t)) = \frac{dx}{dt}(t) - X(x(t), t) = 0, \quad (2)$$

in $B_\varepsilon(x) \subset C^1(\mathbb{R}/\mathbb{Z})$. For isolated, and in particular non-degenerate closed integral curves we can define an *index* as follows:

- (i) let $\Theta \in \mathrm{M}_{2 \times 2}(\mathbb{R})$ be a matrix satisfying $\sigma(\Theta) \cap 2\pi k i \mathbb{R} = \emptyset$, for all $k \in \mathbb{Z}$;
- (ii) let $\eta \mapsto \mathcal{R}(t; \eta)$ be a smooth path in $C^\infty(\mathbb{R}/\mathbb{Z}; \mathrm{M}_{2 \times 2}(\mathbb{R}))$, with $\mathcal{R}(t; 0) = \Theta$ and $\mathcal{R}(t; 1) = D_x X(x(t), t)$ — the linearization of X at $x(t)$.

Then $\eta \mapsto \mathcal{F}_\Theta(\eta) = \frac{d}{dt} - \mathcal{R}(t; \eta)$ defines a path in $\text{Fred}_0(C^1, C^0)$. Denote by $\Sigma \subset \text{Fred}_0(C^1, C^0)$ the set of non-invertible operators and by $\Sigma_1 \subset \Sigma$ the non-invertible operators with a 1-dimensional kernel. If the end points of F are invertible one can choose the path $\eta \mapsto \mathcal{R}(t; \eta)$ such that $\mathcal{F}_\Theta(\eta)$ intersects Σ in Σ_1 and all intersections are transverse. Define $\gamma = \#$ intersections of $\mathcal{F}_\Theta(\eta)$ with Σ_1 , then

$$\mathfrak{t}(x) = \text{sgn}(\det(\Theta))(-1)^{\gamma+1}. \quad (3)$$

This definition is independent of the choice of Θ , see Section 7. In Section 7 we also extend the definition of index to isolated closed integral curves, see Equation (32).

Theorem 1 (Poincaré-Hopf Formula) *Let y be a skeleton of closed integral curves of a vector field $X \in \mathcal{X}(\mathbb{D}^2 \times \mathbb{R}/\mathbb{Z})$ and let $[x \text{ rel } y]$ be a proper relative braid class. Suppose that all 1-periodic closed integral curves of X are isolated, then for all closed integral curves $x_0 \text{ rel } y$ in the fiber $[x_0] \text{ rel } y$ it holds that*

$$\sum_{x_0} \mathfrak{t}(x_0) = \chi(x \text{ rel } y). \quad (4)$$

The index formula can be used to obtain existence results for closed integral curves in proper relative braid classes.

Theorem 2 *Let y be a skeleton of closed integral curves of a vector field $X \in \mathcal{X}(\mathbb{D}^2 \times \mathbb{R}/\mathbb{Z})$ and let $[x \text{ rel } y]$ be a proper relative braid class. If $\chi(x \text{ rel } y) \neq 0$, then there exist closed integral curves $x_0 \text{ rel } y$ in the fiber $[x] \text{ rel } y$.*

Remark 1 In this paper we do not address the question whether the closed integral curves $x \text{ rel } y$ are non-constant, i.e. are not equilibrium points. However, closed integral curves in different relative braid classes correspond to different periodic points! By considering relative braid classes where x consists of more than one strand one can study non-constant closed integral curves. Braid Floer homology for relative braids with x consisting of n strands is defined in [12]. The ideas in this paper extend to relative braid classes with multi-strand braids x . In Section 11 we give an example of a multi-strand x in $x \text{ rel } y$ and explain how this yields the existence of non-trivial closed integral curves, which also provides detailed information about the linking of solutions.

1.3 Discretization and computability

The second part of the paper deals with the computability of the Euler-Floer characteristic. This is obtained through a finite dimensional model. The latter is constructed in three steps:

- (a) compose $x \text{ rel } y$ with $\ell \geq 0$ full twists Δ^2 , such that $(x \text{ rel } y) \cdot \Delta^{2\ell}$ is isotopic to a positive braid $x^+ \text{ rel } y^+$;
- (b) relative braids $x^+ \text{ rel } y^+$ are isotopic to Legendrian braids $x_L \text{ rel } y_L$ on $\mathbb{R}^2$, i.e. braids which have the form $x_L = (q_t, q)$ and $y_L = (Q_t, Q)$, where $q = \pi_2 x$ and $Q = \pi_2 y$, and π_2 is the projection onto the q -coordinate;

- (c) discretize q and $Q = \{Q^j\}$ to $q_d = \{q_i\}$, with $q_i = q(i/d), i = 0, \dots, d$ and $Q_D = \{Q_D^j\}$, with $Q_D^j = \{Q_i^j\}$ and $Q_i^j = Q^j(i/d)$ respectively, and consider the piecewise linear interpolations connecting the *anchor* points q_i and Q_i^j for $i = 0, \dots, d$, see Figure 2. A discretization $q_D \text{ rel } Q_D$ is *admissible* if the linear interpolation is isotopic to $q \text{ rel } Q$. All such discretizations form the discrete relative braid class $[q_D \text{ rel } Q_D]$, for which each fiber is a finite cube complex, cf. [8].

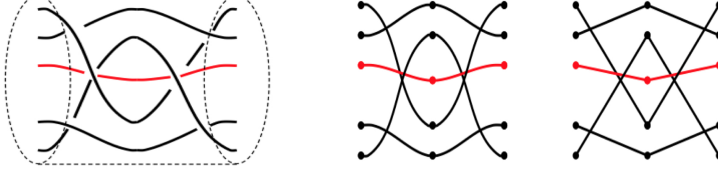


Fig. 2 A positive relative braid [left], anchor points [middle], and piecewise linear discretization [right].

Remark 2 If the number of discretization points is not large enough, then the discretizations may not be admissible and therefore not capture the topology of the braid class. See [8] and Section 10.1 for more details.

For $d > 0$ large enough there exists an admissible discretization $q_D \text{ rel } Q_D$ for any Legendrian representative $x_L \text{ rel } y_L \in [x \text{ rel } y]$ and thus an associated discrete relative braid class $[q_D \text{ rel } Q_D]$. In [8] an invariant for discrete braid classes was introduced. Let $[q_D] \text{ rel } Q_D$ denote a fiber in $[q_D \text{ rel } Q_D]$, which is a cube complex with a finite number of connected components and their closures are denoted by N_j . The faces of the hypercubes N_j can be co-oriented in direction of decreasing the number of crossings in $q_D \text{ rel } Q_D$, and we define N_j^- as the closure of the set of faces with outward pointing co-orientation. Figure 3 below explains the sets N_j and N_j^- for the example in Figure 2. The sets N_j^- are called *exit sets*.

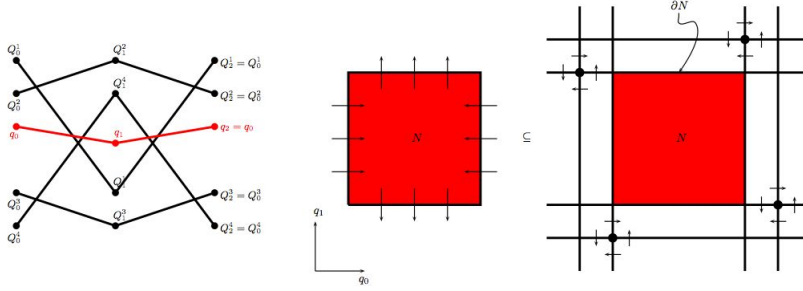


Fig. 3 The relative braid fiber $[q_D] \text{ rel } Q_D$ and $N = \text{cl}([q_D] \text{ rel } Q_D)$.

The invariant for a fiber is given by the Conley index

$$\text{HC}_*([q_D] \text{ rel } Q_D) = \bigoplus_j H_*(N_j, N_j^-).$$

This discrete braid invariant is well-defined for any $d > 0$ for which there exist admissible discretizations and is independent of both the particular fiber and the discretization size d . For the associated Euler characteristic we therefore write $\chi(q_D \text{rel } Q_D)$. The Euler characteristic of the Braid Floer homology $\chi(x \text{rel } y)$ can be related to the Euler characteristic of the associated discrete braid class.

Theorem 3 *Let $[x \text{rel } y]$ a proper relative braid class and $\ell \geq 0$ is an integer such that $(x \text{rel } y) \cdot \Delta^{2\ell}$ is isotopic to a positive braid $x^+ \text{rel } y^+$. Let $q_D \text{rel } Q_D$ be an admissible discretization, for some $d > 0$, of a Legendrian representative $x_L \text{rel } y_L \in [x^+ \text{rel } y^+]$. Then*

$$\chi(x \text{rel } y) = \chi(q_D \text{rel } Q_D^*),$$

where Q_D^* is an augmentation of Q_D by adding the constant strands ± 1 to Q_D .

1.4 Outline of the paper

The first part of the paper, Sections 2 through 7, is concerned with proving Theorems 1 and 2. The second part of the paper, Sections 8 through 11, is concerned with proving Theorem 3, which deals with computing the Euler-Floer characteristic and applying it to specific braid classes. We will now give an outline of the main steps to prove Theorems 1, 2 and 3 and in which sections these steps are proved.

Degree. Reformulate the equation $x_t = X(x, t)$ to a Leray-Schauder equation $\Phi(x) = 0$. Proper relative braid class fibers yield admissible domains $\Omega = [x] \text{rel } y$ for the Leray-Schauder degree and $\deg_{LS}(\Phi, \Omega, 0)$ is well-defined, see Lemma 2.

Homotopy. For a skeleton y for X we can find a Hamilton vector field X_H and thus y is a skeleton for the homotopy $X_\alpha = (1 - \alpha)X + \alpha X_H$, for all $\alpha \in [0, 1]$. Consequently, $\Omega = [x] \text{rel } y$ isolates for all $\alpha \in [0, 1]$ and therefore $\deg_{LS}(\Phi, \Omega, 0) = \deg_{LS}(\Phi_H, \Omega, 0)$, where $\Phi_H(x) = 0$ is the associated Leray-Schauder equation with X_H . This allows us to determine $\deg_{LS}(\Phi, \Omega, 0)$ via Hamiltonian systems. This step is established in Section 2, Equation (7). The Leray-Schauder degree is a sum of the local degrees (generically):

$$\deg_{LS}(\Phi, \Omega, 0) = \deg_{LS}(\Phi_H, \Omega, 0) = \sum_x \deg_{LS}(\Phi_H, B_\varepsilon(x), 0),$$

see Section 3, Equation (9).

Parity. Choose a path $\eta \mapsto \mathcal{A}(\eta)$ of operators connecting θId and $D_x \Phi_H(x)$, which is not a path of self-adjoint operators in general. Then,

$$\deg_{LS}(\Phi_H, B_\varepsilon(x), 0) = \text{parity}(\mathcal{A}(\eta), I),$$

where $\text{parity}(\mathcal{A}(\eta), I)$ is an invariant of paths with invertible ends. See Section 3, Proposition 5.

Spectral flow. We can construct another path $\eta \mapsto \mathcal{C}(\eta)$ of self-adjoint operators which has the same parity as $\mathcal{A}(\eta)$, i.e. $\text{parity}(\mathcal{A}(\eta), I) = \text{parity}(\mathcal{C}(\eta), I)$. For self-adjoint operators spectral flow is defined and

$$\text{parity}(\mathcal{A}(\eta), I) = \text{parity}(\mathcal{C}(\eta), I) = (-1)^{\text{specflow}(\mathcal{C}(\eta), I)}.$$

These steps are established in Section 4, Equation (21).

The Conley-Zehnder index. We consider yet another path of unbounded self-adjoint operators $\eta \mapsto \mathcal{B}(\eta)$ with the property that

$$\text{specflow}(\mathcal{C}(\eta), I) = \text{specflow}(\mathcal{B}(\eta), I),$$

see Section 6, Equation (30). The Conley-Zehnder index is related to the spectral flow of $\mathcal{B}(\eta)$:

$$-(-1)^{\text{specflow}(\mathcal{B}(\eta), I)} = (-1)^{\mu^{\text{CZ}}(x)},$$

see Proposition 6 and Lemma 5.

The Euler-Floer characteristic. In [12] it is proved that generically

$$\sum_x (-1)^{\mu^{\text{CZ}}(x)} = \chi(x\text{rel}y),$$

see Proposition 7. If we combine all the steps we obtain

$$\chi(x\text{rel}y) = -\deg_{LS}(\Phi, \Omega, 0),$$

see Proposition 9.

The index. In Section 7 we introduce the index $\mathfrak{t}(x)$ given in Equation (3) for closed integral curves x , which does not depend on the choices we used to define $\Phi(x)$. The index relates directly to the Leray-Schauder degree. We use the index $\mathfrak{t}(x)$ to formulate the Poincaré-Hopf Formula and prove Theorems 1 and 2.

Legendrian braids. In order to compute the Euler-Floer characteristic we represent braid classes via Legendrian braids of the form $x = (q_t, q)$, where q is a 1-periodic function, see Section 8. Legendrian braids can be realized via mechanical Hamiltonian systems for which the Conley-Zehnder index equals the Morse index of q , see Lemma 13. In Equation (40) we express the Euler-Floer characteristic by $\chi(x\text{rel}y) = \sum_q (-1)^{\gamma(q)}$, where $\gamma(q)$ is the Morse index of a critical point.

Discrete braid classes. The final step toward proving Theorem 3 uses yet another representation of braids. Via the method of broken geodesics, Legendrian braids are discretized and represented via parabolic recurrence relations of conservative type, see [8]. For such braid classes the Conley index of proper relative braid classes is well-defined, see Section 10. In Lemma 15 we show that the Morse index of the discretized critical points is equal to the Morse index $\gamma(q)$. The theory in [8] then equates the Euler-Floer characteristic to the Euler characteristic of the Conley index, see Equation (45), which proves Theorem 3.

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2 Closed integral curves

In this section we rephrase the existence of a closed integral curves in terms of zeroes of appropriate mapping on $C^0(\mathbb{R}/\mathbb{Z})$. Let $X \in \mathcal{X}(\mathbb{D}^2 \times \mathbb{R}/\mathbb{Z})$, then closed integral curves of X of period 1 satisfy the differential equation

$$\begin{cases} \frac{dx}{dt} = X(x, t), & x \in \mathbb{D}^2, t \in \mathbb{R}/\mathbb{Z}, \\ x(0) = x(1). \end{cases} \quad (5)$$

Consider the unbounded operator $L_\mu : C^1(\mathbb{R}/\mathbb{Z}) \subset C^0(\mathbb{R}/\mathbb{Z}) \rightarrow C^0(\mathbb{R}/\mathbb{Z})$, defined by

$$L_\mu := -J \frac{d}{dt} + \mu, \quad \mu \in \mathbb{R}.$$

The operator is invertible for $\mu \neq 2\pi k, k \in \mathbb{Z}$ and the inverse $L_\mu^{-1} : C^0(\mathbb{R}/\mathbb{Z}) \rightarrow C^0(\mathbb{R}/\mathbb{Z})$ is compact. Transforming Equation (5), using L_μ^{-1} , yields the equation $\Phi(x) = 0$, where

$$\Phi(x) := x - L_\mu^{-1}(-JX(x, t) + \mu x).$$

If we set

$$K(x) := L_\mu^{-1}(-JX(x, t) + \mu x),$$

then Φ is of the form $\Phi(x) = x - K(x)$, where K is a (non-linear) compact operator on $C^0(\mathbb{R}/\mathbb{Z})$. Since X is a smooth vector field the mapping Φ is a smooth mapping on $C^0(\mathbb{R}/\mathbb{Z})$.

Proposition 1 *A function $x \in C^0(\mathbb{R}/\mathbb{Z})$, with $|x(t)| \leq 1$ for all t , is a solution of $\Phi(x) = 0$ if and only if $x \in C^1(\mathbb{R}/\mathbb{Z})$ and x satisfies Equation (5).*

Proof If $x \in C^1(\mathbb{R}/\mathbb{Z}; \mathbb{D}^2)$ is a solution of Equation (5), then $\Phi(x) = 0$ is obviously satisfied. On the other hand, if $x \in C^0(\mathbb{R}/\mathbb{Z}; \mathbb{D}^2)$ is a zero of Φ , then $x = K(x) \in C^1(\mathbb{R}/\mathbb{Z})$, since $R(L_\mu^{-1}) \subset C^1(\mathbb{R}/\mathbb{Z})$. Applying L_μ to both sides shows that x satisfies Equation (5).

Note that the zero set $\Phi^{-1}(0)$ does not depend on the parameter μ . In order to apply the Leray-Schauder degree theory we consider appropriate bounded, open subsets $\Omega \subset C^0(\mathbb{R}/\mathbb{Z})$, which have the property that $\Phi^{-1}(0) \cap \partial\Omega = \emptyset$. Let y be a skeleton for X consisting of closed integral curves and consider a proper relative braid class $[x \text{ rel } y]$. Due to properness of $[x \text{ rel } y]$ all fibers $[x] \text{ rel } y$ are isolating neighborhoods for the Cauchy-Riemann equations, cf. [12], and in particular for Equation (5). Therefore, we consider $\Omega = [x] \text{ rel } y$.

Proposition 2 *Let $[x \text{ rel } y]$ be a proper relative braid class and let $\Omega = [x] \text{ rel } y$ be the fiber given by y . Then, there exists an $0 < r < 1$ such that*

$$|x(t)| < r, \text{ and } |x(t) - y^j(t)| > 1 - r, \quad \forall j = 1, \dots, m, \quad \forall t \in \mathbb{R},$$

and for all $x \in \Phi^{-1}(0) \cap \Omega$.

Proof Since $\Omega \subset C^0(\mathbb{R}/\mathbb{Z})$ is a bounded set and K is compact, the solution set $\Phi_\mu^{-1}(0) \cap \Omega$ is compact. Indeed, let $x_n = K(x_n)$ be a sequence in $\Phi_\mu^{-1}(0) \cap \Omega$, then $K(x_{n_k}) \rightarrow x$, and thus $x_{n_k} \rightarrow x$, which, by continuity, implies that $K(x_{n_k}) \rightarrow K(x)$, and thus $x \in \Phi^{-1}(0) \cap \Omega$.

Let $x_n \in \Phi^{-1}(0) \cap \Omega$ and assume that such an $0 < r < 1$ does not exist. Then, by the compactness of $\Phi^{-1}(0) \cap \Omega$, there is a subsequence $x_{n_k} \rightarrow x$ such that one, or both of the following two possibilities hold: (i) $|x(t_0)| = 1$ for some t_0 . By the uniqueness of solutions of Equation (5) and the invariance of the boundary $\partial \mathbb{D}^2(X(x, t))$ is tangent to the boundary, $|x(t)| = 1$ for all t , which is impossible since $[x] \text{ rel } y$ is proper; (ii) $x(t_0) = y^j(t_0)$ for some t_0 and some j . As before, by the uniqueness of solutions of Equation (5), then $x(t) = y^j(t)$ for all t , which again contradicts the fact that $[x] \text{ rel } y$ is proper.

By Proposition 2 the Leray-Schauder degree $\deg_{LS}(\Phi, \Omega, 0)$ is well-defined. Consider the Hamiltonian vector field

$$X_H = J \nabla H, \quad J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad (6)$$

where $H(x, t)$ is a smooth Hamiltonian satisfying (i)-(ii) in Section 1.1, and therefore, $X_H \in \mathcal{X}(\mathbb{D}^2 \times \mathbb{R}/\mathbb{Z})$. Assume that y is a skeleton for X_H . Such Hamiltonians can always be constructed, see [12], and the set of Hamiltonians meeting these requirements is denoted by $\mathcal{H}(y)$. Associate with the vector field X_H we write $\Phi_H(x) := x - L_\mu^{-1}(-JX_H(x, t) + \mu x)$.

Since y is a skeleton for both X and X_H , it is also a skeleton for the linear homotopy $X_\alpha = (1 - \alpha)X + \alpha X_H$, $\alpha \in [0, 1]$. Associated with the homotopy X_α of vector fields we define the homotopy

$$\Phi_\alpha(x) := x - L_\mu^{-1}(-JX_\alpha(x, t) + \mu x) = x - K_\alpha(x), \quad \alpha \in [0, 1],$$

with $K_\alpha(x) = L_\mu^{-1}(-JX_\alpha(x, t) + \mu x)$. Proposition 2 applies for all $\alpha \in [0, 1]$, i.e. by compactness there exists a uniform $0 < r < 1$ such that

$$|x(t)| < r, \text{ and } |x(t) - y^j(t)| > 1 - r,$$

for all $t \in \mathbb{R}$, for all j and for all $x \in \Phi_\alpha^{-1}(0) \cap \Omega = \{x \in \Omega \mid x = K_\alpha(x)\}$ and all $\alpha \in [0, 1]$. By the homotopy invariance of the Leray-Schauder degree we have

$$\deg_{LS}(\Phi, \Omega, 0) = \deg_{LS}(\Phi_\alpha, \Omega, 0) = \deg_{LS}(\Phi_H, \Omega, 0), \quad \forall \alpha \in [0, 1], \quad (7)$$

where $\Phi_0 = \Phi$ and $\Phi_1 = \Phi_H$. Note that the zeroes of Φ_H correspond to critical point of the functional

$$\mathcal{A}_H(x) = \int_0^1 \frac{1}{2} Jx \cdot x_t - H(x, t) dt, \quad (8)$$

and are denoted by $\text{Crit}_{\mathcal{A}_H}([x] \text{ rely})$. The Braid Floer homology groups $\text{HB}_*([x] \text{ rely})$, defined [12], provide information about $\Phi_H^{-1}(0) \cap \Omega = \text{Crit}_{\mathcal{A}_H}([x] \text{ rely})$. In the next section we examine spectral properties of the solutions of $\Phi_\alpha^{-1}(0) \cap \Omega$ in order to compute $\deg_{LS}(\Phi_H, \Omega, 0)$ and thus $\deg_{LS}(\Phi, \Omega, 0)$.

Remark 3 There is obviously more room for choosing appropriate operators L_μ and therefore functions Φ . In Section 7 this issue will be discussed in more detail.

3 The Leray-Schauder degree and parity

The Leray-Schauder degree of an isolated zero x of $\Phi(x) = 0$ is called the local degree. A zero $x \in \Phi^{-1}(0)$ is non-degenerate if $1 \notin \sigma(D_x K(x))$, where $D_x K(x) : C^0(\mathbb{R}/\mathbb{Z}) \rightarrow C^0(\mathbb{R}/\mathbb{Z})$ is the (compact) linearization at x and is given by $D_x K(x) = L_\mu^{-1}(-JD_x X(x, t) + \mu)$. If x is a non-degenerate zero, then it is an isolated zero and the degree can be determined from spectral information.

3.1 The Leray-Schauder degree

Proposition 3 *Let $x \in C^0(\mathbb{R}/\mathbb{Z})$ be a non-degenerate zero of Φ and let $\varepsilon > 0$ be sufficiently small such that $B_\varepsilon(x) = \{\tilde{x} \in C^0(\mathbb{R}/\mathbb{Z}) \mid |\tilde{x}(t) - x(t)| < \varepsilon, \forall t\}$ is a neighborhood in which x is the only zero. Then*

$$\deg_{LS}(\Phi, B_\varepsilon(x), 0) = \deg_{LS}(\text{Id} - D_x K(x), B_\varepsilon(x), 0) = (-1)^{\beta(x)}$$

where

$$\beta(x) = \sum_{\sigma_j > 1, \sigma_j \in \sigma(D_x K(x))} \beta_j, \quad \beta_j = \dim \left(\bigcup_{i=1}^{\infty} \ker(\sigma_j \text{Id} - D_x K(x))^i \right),$$

which will be referred to as the Morse index of x , or alternatively the Morse index of linearized operator $D_x \Phi(x)$.

Proof See [9].

The functions $\Phi_\alpha(x) = x - K_\alpha(x)$ are of the form ‘identity + compact’ and Proposition 3 can be applied to non-degenerate zeroes of $\Phi_\alpha(x) = 0$. If we choose the Hamiltonian ‘generically’, then the zeroes of $\Phi_1 = \Phi_H$ are non-degenerate, i.e. $1 \notin \sigma(D_x K_H(x))$, where $D_x K_H(x) = D_x K_1(x)$. By compactness there are only finitely many zeroes in a fiber $\Omega = [x] \text{ rely}$.

Lemma 1 *Let $x \in \Phi_H^{-1}(0) \cap \Omega$. Then following criteria for non-degeneracy are equivalent:*

- (i) $1 \notin \sigma(D_x K_H(x))$;
- (ii) the operator $\mathcal{B} = -J \frac{d}{dt} - D_x^2 H(x(t), t)$ is invertible;
- (iii) let $\Psi(t)$ be defined by $\mathcal{B}\Psi(t) = 0$, $\Psi(0) = \text{Id}$, then $\det(\Psi(1) - \text{Id}) \neq 0$.

Proof A function ψ satisfies $D_x K_H(x)\psi = \psi$ if and only if $\mathcal{B}\psi = 0$, which shows the equivalence between (i) and (ii). The equivalence between (ii) and (iii) is proved in [12].

The generic choice of H follows from Proposition 7.1 in [12] based on criterion (iii). Hamiltonians for which the zeroes of Φ_H are non-degenerate are denoted by $\mathcal{H}_{\text{reg}}(y)$. Note that *no* genericity is needed for $\alpha \in [0, 1]$! For the Leray-Schauder degree this yields

$$\deg_{LS}(\Phi_\alpha, \Omega, 0) = \deg_{LS}(\Phi_H, \Omega, 0) = \sum_{x \in \text{Crit}_{\mathcal{H}_H}([x] \text{ rely})} (-1)^{\beta_H(x)}, \quad (9)$$

for all $\alpha \in [0, 1]$ and where $\beta_H(x)$ is the Morse index of $\text{Id} - D_x K_H(x)$.

The goal is to determine the Leray-Schauder degree $\deg_{LS}(\Phi, \Omega, 0)$ from information contained in the Braid Floer homology groups $\text{HB}_*([x] \text{ rely})$. In order to do so we examine the Hamiltonian case. In the Hamiltonian case the linearized operator $D_x \Phi_H(x)$ is given by

$$\mathcal{A} := D_x \Phi_H(x) = \text{Id} - D_x K_H(x) = \text{Id} - L_\mu^{-1}(D_x^2 H(x(t), t) + \mu),$$

which is a bounded operator on $C^0(\mathbb{R}/\mathbb{Z})$. The operator $\mathcal{A}$ extends to a bounded operator on $L^2(\mathbb{R}/\mathbb{Z})$. Consider a path $\eta \mapsto \mathcal{A}(\eta)$, $\eta \in I = [0, 1]$, given by

$$\mathcal{A}(\eta) = \text{Id} - L_\mu^{-1}(\mathcal{S}(t; \eta) + \mu) \quad (10)$$

where $\eta \mapsto \mathcal{S}(t; \eta)$ a smooth path of t -dependent symmetric matrices with the ends satisfying

$$\mathcal{S}(t; 0) = \theta \text{Id}, \quad \mathcal{S}(t; 1) = D_x^2 H(x(t), t),$$

where $\theta \neq 2\pi k$, for some $k \in \mathbb{Z}$ and $D_x^2 H(x(t), t)$ is the Hessian of H at a critical point in $\text{Crit}_{\mathcal{H}_H}([x] \text{ rely})$. The path of $\eta \mapsto \mathcal{A}(\eta)$ is a path bounded linear Fredholm operators on $L^2(\mathbb{R}/\mathbb{Z})$ of Fredholm index 0, which are compact perturbations of the identity and whose ends are invertible.

Lemma 2 *A path $\eta \mapsto \mathcal{A}(\eta)$ as defined in (10) is a smooth path of bounded linear Fredholm operators in $H^s(\mathbb{R}/\mathbb{Z})$ of index 0, with invertible ends.*

Proof By the smoothness of $\mathcal{S}(t; \eta)$ we have that $\|\mathcal{S}(t; \eta)x\|_{H^m} \leq C\|x\|_{H^m}$, for any $x \in H^m(\mathbb{R}/\mathbb{Z})$ and any $m \in \mathbb{N} \cup \{0\}$. By interpolation the same holds for all $x \in H^s(\mathbb{R}/\mathbb{Z})$ and the claim follows from the fact that $L_\mu^{-1} : H^s(\mathbb{R}/\mathbb{Z}) \rightarrow H^{s+1}(\mathbb{R}/\mathbb{Z}) \hookrightarrow H^s(\mathbb{R}/\mathbb{Z})$ is compact.

In the Hamiltonian case the Conley-Zehnder indices can be related to the spectral flow of self-adjoint operators. However, the paths $\mathcal{A}(\eta)$ are not self-adjoint however, and thus spectral flow cannot be used in general. We therefore need cruder tool to link the Morse indices $\beta_H(x)$ to the Conley-Zehnder indices. Parity is such an invariant for paths of Fredholm operators and is related to spectral flow.

3.2 Parity of paths of linear Fredholm operators

Let $\eta \mapsto \Lambda(\eta)$ be a smooth path of bounded linear Fredholm operators of index 0 on a Hilbert space $\mathcal{H}$. A crossing $\eta_0 \in I$ is a number for which the operator $\Lambda(\eta_0)$ is not invertible. A crossing is simple if $\dim \ker \Lambda(\eta_0) = 1$. A path $\eta \mapsto \Lambda(\eta)$ between invertible ends can always be perturbed to have only simple crossings. Such paths are called generic. Following [4, 3, 5, 6], we define the *parity* of a generic path $\eta \mapsto \Lambda(\eta)$ by

$$\text{parity}(\Lambda(\eta), I) := \prod_{\ker \Lambda(\eta_0) \neq 0} (-1) = (-1)^{\text{cross}(\Lambda(\eta), I)}, \quad (11)$$

where $\text{cross}(\Lambda(\eta), I) = \#\{\eta_0 \in I : \ker \Lambda(\eta_0) \neq 0\}$. The parity is a homotopy invariant with values in $\mathbb{Z}_2$. In [4, 3, 5, 6] an alternative characterization of parity is given via the Leray-Schauder degree. For any Fredholm path $\eta \mapsto \Lambda(\eta)$ there exists a path $\eta \mapsto \mathcal{M}(\eta)$, called a *parametrix*, such that $\eta \mapsto \mathcal{M}(\eta)\Lambda(\eta)$ is of the form ‘identity + compact’. For parity this gives:

$$\text{parity}(\Lambda(\eta), I) = \deg_{LS}(\mathcal{M}(0)\Lambda(0)) \cdot \deg_{LS}(\mathcal{M}(1)\Lambda(1)),$$

where $\deg_{LS}(\mathcal{M}(\eta)\Lambda(\eta)) = \deg_{LS}(\mathcal{M}(\eta)\Lambda(\eta), \mathcal{H}, 0)$, for $\eta = 0, 1$, and the expression is independent of the choice of parametrix. The latter extends the above definition to arbitrary paths with invertible ends. For a list of properties of parity see [4, 3, 5, 6].

Proposition 4 *Let $\eta \mapsto \mathcal{A}(\eta)$ be a path of bounded linear Fredholm operators on $H^s(\mathbb{R}/\mathbb{Z})$ defined by (10). Then*

$$\text{parity}(\mathcal{A}(\eta), I) = (-1)^{\beta_{\mathcal{A}(0)}} \cdot (-1)^{\beta_{\mathcal{A}(1)}} = (-1)^{\beta_{\mathcal{A}(0)} - \beta_{\mathcal{A}(1)}}. \quad (12)$$

where $\beta_{\mathcal{A}(0)}$ and $\beta_{\mathcal{A}(1)}$ are the Morse indices of $\mathcal{A}(0)$ and $\mathcal{A}(1)$ respectively.

Proof For $\eta \mapsto \mathcal{A}(\eta)$ the parametrix is the constant path $\eta \mapsto \mathcal{M}(\eta) = \text{Id}$. From Proposition 3 we derive that

$$\deg_{LS}(\mathcal{A}(0)) = (-1)^{\beta_{\mathcal{A}(0)}}, \quad \text{and} \quad \deg_{LS}(\mathcal{A}(1)) = (-1)^{\beta_{\mathcal{A}(1)}},$$

which proves the first part of the formula. Since $\beta(\mathcal{A}(0)) - \beta(\mathcal{A}(1)) = [\beta(\mathcal{A}(0)) + \beta(\mathcal{A}(1))] \pmod{2}$, the second identity follows.

Lemma 3 *For $\theta > 0$, the Morse index for $\mathcal{A}(0)$ is given by $\beta_{\mathcal{A}(0)} = 2 \left\lceil \frac{\mu + \theta}{2\pi} \right\rceil$.*

Proof The eigenvalues of the operator $\mathcal{A}(0)$ are given by $\lambda = \frac{-\theta + 2k\pi}{\mu + 2k\pi}$ and all have multiplicity 2. Therefore number of integers k for which $\lambda < 0$ is equal to $\left\lceil \frac{\mu + \theta}{2\pi} \right\rceil$ and consequently $\beta_{\mathcal{A}(0)} = 2 \left\lceil \frac{\mu + \theta}{2\pi} \right\rceil$.

If $x \in \Phi_H^{-1}(0)$ is a non-degenerate zero, then its local degree can be expressed in terms of the parity of $\mathcal{A}(\eta)$.

Proposition 5 *Let $x \in \Phi_H^{-1}(0)$ be a non-degenerate zero, then*

$$\deg_{LS}(\Phi_H, B_\varepsilon(x), 0) = \text{parity}(\mathcal{A}(\eta), I), \quad (13)$$

where $\eta \mapsto \mathcal{A}(\eta)$ is given by (10).

Proof From Proposition 3 we have that $\deg_{LS}(\Phi_H, B_\varepsilon(x), 0) = (-1)^{\beta_{\mathcal{A}(1)}}$ and by Equation (12), $\text{parity}(\mathcal{A}(\eta), I) = (-1)^{\beta_{\mathcal{A}(0)}} \cdot (-1)^{\beta_{\mathcal{A}(1)}} = (-1)^{\beta_{\mathcal{A}(1)}}$, which completes the proof.

In the next subsection we establish yet another path of operators such that we can link the local degree to the Conley-Zehnder indices of the critical points.

4 Parity and spectral flow

The spectral flow for paths of selfadjoint operators is a more refined invariant than parity. Using the Fourier expansion $x = \sum_{k \in \mathbb{Z}} e^{2\pi J k t} x_k$ we can characterize $x \in H^s(\mathbb{R}/\mathbb{Z})$ by $\sum_{k \in \mathbb{Z}} |k|^{2s} |x_k|^2 < \infty$. From the functional calculus of the selfadjoint operator

$$-J \frac{d}{dt} x = \sum_{k \in \mathbb{Z}} (2\pi k) e^{2\pi J k t} x_k,$$

we define the selfadjoint operators

$$N_\mu x = \sum_{k \in \mathbb{Z}} (2\pi |k| + \mu) e^{2\pi J k t} x_k, \quad \text{and} \quad P_\mu x = \sum_{k \in \mathbb{Z}} \frac{2\pi k + \mu}{2\pi |k| + \mu} e^{2\pi J k t} x_k. \quad (14)$$

For $\mu > 0$ and $\mu \neq 2\pi k$, $k \in \mathbb{Z}$, the operators P_μ are isomorphisms on $H^s(\mathbb{R}/\mathbb{Z})$, for all $s \geq 0$.³ Consider the path

$$\mathcal{C}(\eta) = P_\mu \mathcal{A}(\eta) = P_\mu - N_\mu^{-1}(\mathcal{S}(t; \eta) + \mu), \quad (15)$$

which is a path of operators of Fredholm index 0. The constant path $\eta \mapsto \mathcal{M}_\mu(\eta) = P_\mu^{-1}$ is a parametrix for $\eta \mapsto \mathcal{C}(\eta)$ (see [5, 6]) and since $\mathcal{M}_\mu \mathcal{C}(\eta) = \mathcal{A}(\eta)$, the parity of $\mathcal{C}(\eta)$ is given by

$$\text{parity}(\mathcal{C}(\eta), I) = \text{parity}(\mathcal{A}(\eta), I). \quad (16)$$

Using N_μ , with $\mu > 0$ and $\mu \neq 2\pi k$, we define an equivalent norms on the Sobolev spaces $H^s(\mathbb{R}/\mathbb{Z})$:

$$(x, y)_{H^s} := (N_\mu^s x, N_\mu^s y)_{L^2}, \quad \forall x, y \in H^s(\mathbb{R}/\mathbb{Z}),$$

where $N_\mu^s x = \sum_{k \in \mathbb{Z}} |2\pi |k| + \mu|^s e^{2\pi J k t} x_k$.

Lemma 4 *The operators $\mathcal{C}(\eta)$ are selfadjoint on $(H^{1/2}(\mathbb{R}/\mathbb{Z}), (\cdot, \cdot)_{H^{1/2}})$ for all $\eta \in I$, and $\eta \mapsto \mathcal{C}(\eta)$ is a smooth path of selfadjoint operators on $H^{1/2}(\mathbb{R}/\mathbb{Z})$.*

³ As before $\|P_\mu x\|_{H^s} \leq \|x\|_{H^s}$ and $\|P_\mu^{-1} x\|_{H^s} \leq C(\mu) \|x\|_{H^s}$, $\mu > 0$ and $\mu \neq 2\pi k$.

Proof From the functional calculus we derive that

$$(P_\mu x, y)_{H^s} = \sum_{k \in \mathbb{Z}} p_\mu(k) n_\mu^{2s}(k) x_k y_k = (x, P_\mu y)_{H^s},$$

where $n_\mu(k) = 2\pi|k| + \mu$ and $p_\mu(k) = \frac{2\pi k + \mu}{2\pi|k| + \mu}$. For $s = 1/2$ we have that

$$\begin{aligned} (N_\mu^{-1}(S(t; \eta) + \mu)x, y)_{H^{1/2}} &= ((S(t; \eta) + \mu)x, y)_{L^2} = (x, (S(t; \eta) + \mu)y)_{L^2} \\ &= (x, N_\mu^{-1}(S(t; \eta) + \mu)y)_{H^{1/2}}, \end{aligned}$$

which completes the proof.

For a path $\eta \mapsto \Lambda(\eta)$ of *selfadjoint* operators on a Hilbert space $\mathcal{H}$, which is continuously differentiable in the (strong) operator topology we define the crossing operator $\Gamma(\Lambda, \eta) = \pi \frac{d}{d\eta} \Lambda(\eta) \pi|_{\ker \Lambda(\eta)}$, where π is the orthogonal projection onto $\ker \Lambda(\eta)$. A crossing $\eta_0 \in I$ is a number for which the operator $\Lambda(\eta_0)$ is not invertible. A crossing is regular if $\Gamma(\Lambda, \eta_0)$ is non-singular. A point η_0 for which $\dim \ker \Lambda(\eta_0) = 1$, is called a simple crossing. A path $\eta \mapsto \Lambda(\eta)$ is called generic if all crossings are simple. A path $\eta \mapsto \Lambda(\eta)$ with invertible ends can always be chosen to be generic by a small perturbation. At a simple crossing η_0 , there exists a C^1 -curve $\lambda(\eta)$, for η near η_0 , and $\lambda(\eta)$ is an eigenvalue of $\Lambda(\eta)$, with $\lambda(\eta_0) = 0$ and $\lambda'(\eta_0) \neq 0$, see [10, 11]. The spectral flow for a generic path is defined by

$$\text{specflow}(\Lambda(\eta), I) = \sum_{\lambda(\eta_0)=0} \text{sgn}(\lambda'(\eta_0)). \quad (17)$$

For a simple crossing η_0 the crossing operator is simply multiplication by $\lambda'(\eta_0)$ and

$$\Gamma(\Lambda, \eta) \psi(\eta_0) = \left(\frac{d}{d\eta} \Lambda(\eta_0) \psi(\eta_0), \psi(\eta_0) \right)_{\mathcal{H}} \psi(\eta_0) = \lambda'(\eta_0) \psi(\eta_0), \quad (18)$$

where $\psi(\eta_0)$ is normalized in $\mathcal{H}$, and

$$\lambda'(\eta_0) = \left(\frac{d}{d\eta} \Lambda(\eta_0) \psi(\eta_0), \psi(\eta_0) \right)_{\mathcal{H}}. \quad (19)$$

The spectral flow is defined any for continuously differentiable path $\eta \mapsto \Lambda(\eta)$ with invertible ends. From the theory in [6] there is a connection between the spectral flow of $\Lambda(\eta)$ and its parity:

$$\text{parity}(\Lambda(\eta), I) = (-1)^{\text{specflow}(\Lambda(\eta), I)}, \quad (20)$$

which in view of Equation (11) follows since $\text{cross}(\Lambda(\eta), I) = \text{specflow}(\Lambda(\eta), \eta) \bmod 2$ in the generic case.

The path $\eta \mapsto \mathcal{C}(\eta)$ defined in (15) is a continuously differentiable path of operators on $\mathcal{H} = H^{1/2}(\mathbb{R}/\mathbb{Z})$ with invertible ends, and therefore both parity and spectral flow are well-defined. If we combine Equations (13) and (16) with Equation (20) we obtain

$$\deg_{LS}(\Phi_H, B_\varepsilon(x), 0) = \text{parity}(\mathcal{A}(\eta), I) = (-1)^{\text{specflow}(\mathcal{C}(\eta), I)}. \quad (21)$$

In the next section we link the spectral flow of $\mathcal{C}(\eta)$ to the Conley-Zehnder indices of non-degenerate zeroes and therefore to the Euler-Floer characteristic.

5 The Conley-Zehnder index

For a non-degenerate 1-periodic solution $x(t)$ of the Hamilton equations the Conley-Zehnder index can be defined as follows. The linearized flow Ψ is given by

$$\begin{cases} -J \frac{d\Psi}{dt} - D_x^2 H(x(t), t) \Psi = 0 \\ \Psi(0) = \text{Id}, \end{cases}$$

By Lemma 1(iii), a 1-periodic solution is non-degenerate if $\Psi(1)$ has no eigenvalues equal to 1. The Conley-Zehnder index is defined using the symplectic path $\Psi(t)$. Following [11], consider the crossing form $\Gamma(\Psi, t)$, defined for vectors $\xi \in \ker(\Psi(t) - \text{Id})$,

$$\Gamma(\Psi, t)\xi = \omega(\xi, \frac{d}{dt}\Psi(t)\xi) = (\xi, D_x^2 H(x(t), t)\xi). \quad (22)$$

A crossing $t_0 > 0$ is defined by $\det(\Psi(t_0) - \text{Id}) = 0$. A crossing is regular if the crossing form is non-singular. A path $t \mapsto \Psi(t)$ is regular if all crossings are regular. Any path can be approximated by a regular path with the same endpoints and which is homotopic to the initial path, see [10] for details. For a regular path $t \mapsto \Psi(t)$ the Conley-Zehnder index is given by

$$\mu^{CZ}(\Psi) = \frac{1}{2} \text{sgn} D_x^2 H(x(0), 0) + \sum_{\substack{t_0 > 0, \\ \det(\Psi(t_0) - \text{Id}) = 0}} \text{sgn} \Gamma(\Psi, t_0). \quad (23)$$

For a non-degenerate 1-periodic solution $x(t)$ we define the Conley-Zehnder index as $\mu^{CZ}(x) := \mu^{CZ}(\Psi)$, and the index is integer valued.

Let x be a 1-periodic solution and consider the associated path $\eta \mapsto \mathcal{B}(\eta) = -J \frac{d}{dt} - \mathcal{S}(t; \eta)$, where, as before, $\mathcal{S}(t; \eta)$ is a smooth path of symmetric matrices with ends $\mathcal{S}(t; 0) = \theta \text{Id}$ and $\mathcal{S}(t; 1) = D_x^2 H(x(t), t)$ where $\theta \neq 2\pi k, k \in \mathbb{Z}$ as defined in (10). The operators $\mathcal{B}(\eta)$ are unbounded operators on $L^2(\mathbb{R}/\mathbb{Z})$, with domain $H^1(\mathbb{R}/\mathbb{Z})$. A path $\eta \mapsto \mathcal{B}(\eta)$ is continuously differentiable in the (weak) operator topology of $\mathcal{B}(H^1, L^2)$ and Hypotheses (A1)-(A3) in [11] are satisfied. We now repeat the definition of spectral flow for a path of unbounded operators as developed in [11]. The crossing operator for a path $\eta \mapsto \mathcal{B}(\eta)$ is given by $\Gamma(\mathcal{B}, \eta) = \pi \frac{d}{d\eta} \mathcal{B}(\eta) \pi|_{\ker \mathcal{B}(\eta)}$, where π is the orthogonal projection onto $\ker \mathcal{B}(\eta)$. A crossing $\eta_0 \in I$ is a number for which the operator $\mathcal{B}(\eta_0)$ is not invertible. A crossing is regular if $\Gamma(\mathcal{B}, \eta_0)$ is non-singular. A point η_0 for which $\dim \ker \mathcal{B}(\eta_0) = 1$, is called a simple crossing. A path $\eta \mapsto \mathcal{B}(\eta)$ is called generic if all crossing are simple. A path $\eta \mapsto \mathcal{B}(\eta)$ can always be chosen to be generic. At a simple crossing η_0 there exists a C^1 -curve $\ell(\eta)$, for η near η_0 , and $\ell(\eta)$ is an eigenvalue of $\mathcal{B}(\eta)$ with $\ell(\eta_0) = 0$ and $\ell'(\eta_0) \neq 0$. The spectral flow for a generic path is defined by

$$\text{specflow}(\mathcal{B}(\eta), I) = \sum_{\ell(\eta_0)=0} \text{sgn}(\ell'(\eta_0)), \quad (24)$$

and at simple crossings η_0 ,

$$\Gamma(\mathcal{B}, \eta)\phi(\eta_0) = \left(\frac{d}{d\eta} \mathcal{B}(\eta_0)\phi(\eta_0), \phi(\eta_0) \right)_{L^2} \phi(\eta_0) = \ell'(\eta_0)\phi(\eta_0), \quad (25)$$

after normalizing $\phi(\eta_0)$ in $L^2(\mathbb{R}/\mathbb{Z})$. As before the derivative of ℓ at η_0 is given by

$$\ell'(\eta_0) = -(\partial_\eta \mathcal{S}(t; \eta_0) \phi(\eta_0), \phi(\eta_0))_{L^2}. \quad (26)$$

Proposition 6 *Let $\eta \mapsto \mathcal{B}(\eta)$, $\eta \in I$, as defined above, be a generic path of unbounded self-adjoint operators with invertible endpoints, and let $\eta \mapsto \Psi(\eta; t)$ be the associated path of symplectic matrices defined by*

$$\begin{cases} -J \frac{d\Psi}{dt}(t; \eta) - \mathcal{S}(t; \eta) \Psi(t; \eta) = 0 \\ \Psi(0; \eta) = \text{Id}, \end{cases}$$

Then

$$\text{specflow}(\mathcal{B}(\eta), I) = \mu_{\mathcal{B}(0)}^{CZ} - \mu_{\mathcal{B}(1)}^{CZ} \quad (27)$$

where $\mu_{\mathcal{B}(0)}^{CZ} = \mu^{CZ}(\Psi(t; 0))$, $\mu_{\mathcal{B}(1)}^{CZ} = \mu^{CZ}(\Psi(t; 1))$.

Proof The expression for the spectral flow follows from [11] and [12].

In the case $\eta = 0$, the Conley-Zehnder index $\mu_{\mathcal{B}(0)}^{CZ}$ can be computed explicitly. Recall that $\mathcal{B}(0) = -J \frac{d}{dt} - \mathcal{S}(0) = -J \frac{d}{dt} - \theta \text{Id}$.

Lemma 5 *Let $\theta > 0$ (fixed) and $\theta \neq 2\pi k$, then $\mu_{\mathcal{B}(0)}^{CZ} = 1 + 2 \lfloor \frac{\theta}{2\pi} \rfloor$.*

Proof The solution to $\mathcal{B}(0)\Psi(t) = 0$ is given by $\Psi(t) = e^{\theta J t}$ and $\det(\Psi(1) - \text{Id}) = 0$ exactly when $t = t_0 = \frac{2\pi k}{\theta}$. By (22) and (23) we have that $\Gamma(\Psi, t)\xi = \theta|\xi|^2$ and therefore $\mu_{\mathcal{B}(0)}^{CZ} = 1 + 2 \lfloor \frac{\theta}{2\pi} \rfloor$, which proves the lemma.

The zeroes $x \in \Phi_H^{-1}(0)$ in $\Omega = [x] \text{rel} y$ can be estimated by Braid Floer homology $\text{HB}_*([x] \text{rel} y)$. Recall that the *Euler-Floer* characteristic of $\text{HB}_*([x] \text{rel} y)$, as defined in (1), is given by $\chi(x \text{rel} y) = \sum_{k \in \mathbb{Z}} (-1)^k \dim \text{HB}_k([x] \text{rel} y)$, which is also the Euler-Floer characteristic for $\text{HB}_*([x] \text{rel} y)$. In [12] the following analogue of the Poincaré-Hopf formula is proved.

Proposition 7 *For a proper braid class fiber $[x] \text{rel} y$ and a generic Hamiltonian $H \in \mathcal{H}_{\text{reg}}(y)$, it holds that*

$$\chi(x \text{rel} y) = \sum_{x \in \text{Crit}_{\mathcal{A}_H}([x] \text{rel} y)} (-1)^{\mu^{CZ}(x)}.$$

Since the Conley-Zehnder index is linked to spectral flow we obtain:

Proposition 8 *For a proper braid class fiber $[x] \text{rel} y$ and a generic Hamiltonian $H \in \mathcal{H}_{\text{reg}}(y)$, we have that*

$$\chi(x \text{rel} y) = - \sum_{x \in \text{Crit}_{\mathcal{A}_H}([x] \text{rel} y)} (-1)^{-\text{specflow}(\mathcal{B}(\eta; x), I)}, \quad (28)$$

where $\eta \mapsto \mathcal{B}(\eta; x) = -J \frac{d}{dt} - \mathcal{S}(t; \eta)$, for $x \in \text{Crit}_{\mathcal{A}_H}([x] \text{rel} y)$.

Proof By Proposition 6 and Lemma 5 the spectral flow satisfies,

$$\begin{aligned}\mu^{CZ}(x) &= \mu_{\mathcal{B}(1;x)}^{CZ} = \mu_{\mathcal{B}(0)}^{CZ} - \text{specflow}(\mathcal{B}(\eta;x), I) \\ &= 1 + 2\left\lfloor \frac{\theta}{2\pi} \right\rfloor - \text{specflow}(\mathcal{B}(\eta;x), I).\end{aligned}$$

This implies

$$(-1)^{\mu^{CZ}(x)} = -(-1)^{-\text{specflow}(\mathcal{B}(\eta;x), I)},$$

which completes the proof.

In order to prove that the Euler-Floer characteristic $\chi(x_{\text{rely}})$ and $\deg_{LS}(\Phi_H, \Omega, 0)$ are related we need to investigate the relation between the spectral flows of $\mathcal{B}(\eta)$ and $\mathcal{C}(\eta)$.

6 The spectral flows are the same

In order to show that the spectral flows are the same we use the fact that the paths $\eta \mapsto \mathcal{C}(\eta)$ and $\eta \mapsto \mathcal{B}(\eta)$ for a non-degenerate zero $x_i \in \Phi_H^{-1}(0) \cap \Omega$ are chosen to have only simple crossings for their crossing operators, i.e. zero eigenvalues are simple. In this case the spectral flows are determined by the signs of the derivatives of the eigenvalues at the crossings. For $\eta \mapsto \mathcal{B}(\eta)$ the expression given by Equation (26) and from Equation (19) a similar expression for $\eta \mapsto \mathcal{C}(\eta)$ can be derived and is given by

$$\mathcal{L}'(\eta_0) = -(N_\mu^{-1} \partial_\eta S(t; \eta_0) \psi(\eta_0), \psi(\eta_0))_{H^{1/2}} = -(\partial_\eta S(t; \eta_0) \psi(\eta_0), \psi(\eta_0))_{L^2} \quad (29)$$

Lemma 6 *The sets $\{\eta \in [0, 1] : \mathcal{C}(\eta) \psi(\eta) = 0\}$ and $\{\eta \in [0, 1] : \mathcal{B}(\eta) \phi(\eta) = 0\}$ are the same, and the operators $\mathcal{C}(\eta)$ and $\mathcal{B}(\eta)$ have the same eigenfunctions at crossings η_0 . In particular, $\eta \mapsto \mathcal{B}(\eta)$ is generic if and only if $\eta \mapsto \mathcal{C}(\eta)$ is generic.*

Proof Given $\eta_0 \in [0, 1]$ such that $\mathcal{C}(\eta_0) \psi(\eta_0) = 0$, then

$$P_\mu \psi(\eta_0) - N_\mu^{-1} (S(\eta_0; t) + \mu) \psi(\eta_0) = 0,$$

and thus $\psi(\eta_0) - L_\mu^{-1} (S(\eta_0; t) + \mu) \psi(\eta_0) = 0$, which is equivalent to the equation $(-J \frac{d}{dt} - S(t; \eta_0)) \psi(\eta_0) = 0$, i.e. $\mathcal{B}(\eta_0) \psi(\eta_0) = 0$.

Lemma 7 *For all $\mu > 0$, with $\mu \neq 2\pi k$, $k \in \mathbb{Z}$, $\text{sgn} \mathcal{L}'(\eta_0) = \text{sgn} \ell'(\eta_0)$ for all crossings at η_0 .*

Proof The eigenfunctions $\psi(\eta_0)$ in Equation (29) for $\mathcal{L}'(\eta_0)$ are normalized in $H^{1/2}(\mathbb{R}/\mathbb{Z})$ and therefore they relate to the eigenfunctions $\phi(\eta_0)$ in Equation (26) for $\ell'(\eta_0)$ as follows:

$$\psi(\eta_0) = \frac{\phi(\eta_0)}{\|\phi(\eta_0)\|_{H^{1/2}}}, \quad \|\phi(\eta_0)\|_{L^2} = 1.$$

Combining Equations (26) and (29) then gives

$$\begin{aligned}\lambda'(\eta_0) &= -(\partial_\eta S(t; \eta_0) \psi(\eta_0), \psi(\eta_0))_{L^2} \\ &= -\frac{1}{\|\phi(\eta_0)\|_{H^{1/2}}^2} (\partial_\eta S(t; \eta_0) \phi(\eta_0), \phi(\eta_0))_{L^2} = \frac{\ell'(\eta_0)}{\|\phi(\eta_0)\|_{H^{1/2}}^2},\end{aligned}$$

which proves the lemma.

Lemma 7 implies that for any non-degenerate $x \in \Phi_H^{-1}(0) \cap \Omega$

$$\text{specflow}(\mathcal{C}(\eta; x), I) = \text{specflow}(\mathcal{B}(\eta; x), I), \quad (30)$$

where $\mathcal{B}(\eta; x)$ and $\mathcal{C}(\eta; x)$ are the above described path associated with x . Therefore

$$\text{parity}(\mathcal{A}(\eta; x), I) = (-1)^{\text{specflow}(\mathcal{C}(\eta; x), I)} = (-1)^{\text{specflow}(\mathcal{B}(\eta; x), I)}, \quad (31)$$

which yields the following proposition.

Proposition 9 *The Leray-Schauder degree satisfies*

$$\deg_{LS}(\Phi_H, \Omega, 0) = -\chi(x \text{ rel } y).$$

Proof For any Hamiltonian $H \in \mathcal{H}(y)$ there exists a generic Hamiltonian $\tilde{H} \in \mathcal{H}_{\text{reg}}(y)$ such all zeroes $x_i \in \Phi_{\tilde{H}}^{-1}(0) \cap \Omega$ are non-degenerate. Since $\Omega = [x] \text{ rel } y$ is isolating for all Hamiltonians in $\mathcal{H}(y)$, the invariance of the Leray-Schauder degree yields $\deg_{LS}(\Phi_H, \Omega, 0) = \deg_{LS}(\Phi_{\tilde{H}}, \Omega, 0)$. From the Propositions 5 and 8 and Equation (31) we conclude that

$$\begin{aligned}\deg_{LS}(\Phi_H, \Omega, 0) &= \deg_{LS}(\Phi_{\tilde{H}}, \Omega, 0) \\ &= \sum_{x \in \Phi_{\tilde{H}}^{-1}(0) \cap \Omega} \deg_{LS}(\Phi_{\tilde{H}}, B_\varepsilon(x), 0) = \sum_{x \in \Phi_{\tilde{H}}^{-1}(0) \cap \Omega} \text{parity}(\mathcal{A}(\eta; x), I) \\ &= \sum_{x \in \Phi_{\tilde{H}}^{-1}(0) \cap \Omega} (-1)^{\text{specflow}(\mathcal{B}(\eta; x), I)} = \sum_{x \in \Phi_{\tilde{H}}^{-1}(0) \cap \Omega} (-1)^{-\text{specflow}(\mathcal{B}(\eta; x), I)} \\ &= -\chi(x \text{ rel } y),\end{aligned}$$

which completes the proof.

Remark 4 As $\mu \gg 1$ it holds that $\ell'(\eta_0) \sim \mu \lambda'(\eta_0)$. Indeed, $\|\phi(\eta_0)\|_{H^{1/2}}^2 = \sum_k (2\pi|k| + \mu) a_k^2$, where a_k are the Fourier coefficients of $\phi(\eta_0)$ and $\sum_k a_k^2 = 1$. Since $\phi(\eta_0)$ are smooth functions the Fourier coefficients satisfy the following properties. For any $\delta > 0$ and any $s > 0$, there exists $N_{\delta, s} > 0$ such that $\sum_{|k| \geq N} |k|^{2s} |a_k|^2 \leq \delta$, for all $N \geq N_{\delta, s}$. From the latter it follows that $\sum_k 2\pi|k| a_k^2 \leq C$, with $C > 0$ independent of η_0 and μ . We derive that $\mu \leq \|\phi(\eta_0)\|_{H^{1/2}}^2 \leq C + \mu$ and

$$1 \leftarrow \frac{\mu}{C + \mu} \leq \frac{\mu \lambda'(\eta_0)}{\ell'(\eta_0)} = \frac{\mu}{\|\phi(\eta_0)\|_{H^{1/2}}^2} \leq \frac{\mu}{\mu} = 1,$$

as $\mu \rightarrow \infty$, which proves our statement.

7 The proof of Theorems 1 and 2

We start with the proof of Theorem 2. Since $\text{HB}_*([x \text{ rel } y])$ is an invariant of the proper braid class $[x \text{ rel } y]$ it does not depend on a particular fiber $[x] \text{ rel } y$. Recall the homotopy invariance of the Leray-Schauder degree as expressed in Equation (7)

$$\deg_{LS}(\Phi, \Omega, 0) = \deg_{LS}(\Phi_\alpha, \Omega, 0) = \deg_{LS}(\Phi_H, \Omega, 0).$$

By Proposition 9 we have that

$$\deg_{LS}(\Phi, \Omega, 0) = \deg_{LS}(\Phi_H, \Omega, 0) = -\chi(x \text{ rel } y),$$

and $\chi(x \text{ rel } y) \neq 0$, then implies that $\Phi^{-1}(0) \cap \Omega \neq \emptyset$. Therefore there exist a closed integral curves in any relative braid class fiber $[x] \text{ rel } y$, whenever $\chi(x \text{ rel } y) \neq 0$. This completes the proof of Theorem 2.

The remainder of this section is to prove the Poincaré-Hopf Formula in Theorem 1 for closed integral curves in proper relative braid fibers. The mapping

$$\mathcal{E} : C^1(\mathbb{R}/\mathbb{Z}) \rightarrow C^0(\mathbb{R}/\mathbb{Z}), \quad \mathcal{E}(x) = \frac{dx}{dt} - X(x, t),$$

is smooth (nonlinear) Fredholm mapping of index 0. Let $\mathcal{M} \in \text{GL}(C^0, C^1)$ be an isomorphism such that $\mathcal{M}\mathcal{E}(x)$ is of the form $\mathcal{M}\mathcal{E}(x) = \Phi_{\mathcal{M}}(x) = x - K_{\mathcal{M}}(x)$, with $K_{\mathcal{M}} : C^1(\mathbb{R}/\mathbb{Z}) \rightarrow C^1(\mathbb{R}/\mathbb{Z})$ compact. Such isomorphisms $\mathcal{M}$ (constant parametrices) obviously exist. For example $\mathcal{M} = \left(\frac{d}{dt} + 1\right)^{-1}$, or $\mathcal{M} = -JL_\mu^{-1}$. The mappings $\Phi_{\mathcal{M}} : C^1(\mathbb{R}/\mathbb{Z}) \rightarrow C^1(\mathbb{R}/\mathbb{Z})$ are Fredholm mappings of index 0.

Let $x \in C^1(\mathbb{R}/\mathbb{Z})$ be a non-degenerate zero of $\mathcal{E}$ and recall, using the theory in [4, 3, 5, 6], that the index $\mathfrak{I}(x)$ defined in (3) is equivalent to:

$$\mathfrak{I}(x) = -\text{sgn}(\det(\Theta))(-1)^{\beta_{\mathcal{M}}(\Theta)} \deg_{LS}(\Phi_{\mathcal{M}}, B_{\mathcal{E}}(x), 0). \quad (32)$$

where $\Theta \in \text{M}_{2 \times 2}(\mathbb{R})$, with $\sigma(\Theta) \cap 2\pi k i \mathbb{R} = \emptyset$, $k \in \mathbb{Z}$ and $\beta_{\mathcal{M}}(\Theta)$ is the Morse index of $\text{Id} - K_{\mathcal{M}}(0)$.

Lemma 8 *The index $\mathfrak{I}(x)$ for a non-degenerate zero of $\mathcal{E}$ is well-defined, i.e. independent of the choices of $\mathcal{M} \in \text{GL}(C^0, C^1)$ and $\Theta \in \text{M}_{2 \times 2}(\mathbb{R})$.*

Proof Consider smooth paths $\eta \mapsto \mathcal{F}_\Theta(\eta)$, defined by $\mathcal{F}_\Theta(\eta) = \frac{d}{dt} - \mathcal{R}(t; \eta)$, where $\mathcal{R}(t; 0) = \Theta$ and $\mathcal{R}(t; 1) = \mathcal{D}_x X(x(t), t)$. The path

$$\mathcal{F}_\Theta : [0, 1] \rightarrow \text{Fred}_0(C^1, C^0)$$

has invertible end points, and by the theory in [4, 3] we have that the parity of $\eta \mapsto \mathcal{F}_\Theta(\eta)$ is well-defined and independent of $\mathcal{M}$, i.e.

$$\begin{aligned} \text{parity}(\mathcal{F}_\Theta(\eta), I) &= \text{parity}(\mathcal{D}_{\mathcal{M}, \Theta}(\eta), I) = (-1)^{\beta_{\mathcal{M}}(\Theta)} (-1)^{\beta_{\mathcal{M}}(x)} \\ &= (-1)^{\beta_{\mathcal{M}}(\Theta)} \deg_{LS}(\Phi_{\mathcal{M}}, B_{\mathcal{E}}(x), 0), \end{aligned}$$

where $\mathcal{D}_{\mathcal{M}, \Theta}(\eta) = \mathcal{M}\mathcal{F}_\Theta(\eta)$ and $\beta_{\mathcal{M}}(x)$ is the Morse index of $\mathcal{D}_{\mathcal{M}, \Theta}(1) = \text{Id} - K_{\mathcal{M}}(1)$. It remains to show that the index $\mathfrak{I}(x)$ is independent with respect to

Θ . Let Θ and Θ' be admissible matrices and let $\eta \mapsto \mathcal{G}(\eta)$ be a path connecting $\mathcal{G}(0) = \frac{d}{dt} - \Theta$ and $\mathcal{G}(1) = \frac{d}{dt} - \Theta'$. For the parities it holds that

$$\text{parity}(\mathcal{F}_\Theta(\eta), I) = \text{parity}(\mathcal{G}(\eta), I) \cdot \text{parity}(\mathcal{F}_{\Theta'}(\eta), I).$$

To compute $\text{parity}(\mathcal{G}(\eta), I)$ we consider a special parametrix $\mathcal{M}_\mu$, given by $\mathcal{M}_\mu = \left(\frac{d}{dt} + \mu\right)^{-1}$, $\mu > 0$. From the definition of parity we have that

$$\text{parity}(\mathcal{G}(\eta), I) = \text{parity}(\mathcal{M}_\mu \mathcal{G}(\eta), I) = \deg_{LS}(\mathcal{M}_\mu \mathcal{G}(0)) \cdot \deg_{LS}(\mathcal{M}_\mu \mathcal{G}(1)).$$

We now compute the Leray-Schauder degrees of $\mathcal{M}_\mu \mathcal{G}(0)$ and $\mathcal{M}_\mu \mathcal{G}(1)$. We start with Θ and in order to compute the degree we determine the Morse index. Consider the eigenvalue problem

$$\mathcal{M}_\mu \mathcal{G}(0)\psi = \lambda\psi, \quad \lambda \in \mathbb{R},$$

which is equivalent to $(1 - \lambda)\frac{d\psi}{dt} = (\Theta + \lambda\mu)\psi$. Non-trivial solutions are given by $\psi(t) = \exp\left(\frac{\Theta + \lambda\mu}{1 - \lambda}t\right)\psi_0$, which yields the condition $\frac{\Theta + \lambda\mu}{1 - \lambda} = 2\pi ki$, $k \in \mathbb{Z}$, where θ is an eigenvalues of Θ . We now consider three cases:

(i) $\theta_\pm = a \pm ib$. In case of a negative eigenvalue λ we have $\frac{a + \lambda\mu}{1 - \lambda} = 0$ and $\frac{b}{1 - \lambda} = 2\pi k$. The same $\lambda < 0$ also suffices for the conjugate eigenvalue via $\frac{-b}{1 - \lambda} = -2\pi k$. This implies that any eigenvalue $\lambda < 0$ has multiplicity 2, and thus $\deg_{LS}(\mathcal{M}_\mu \mathcal{G}(0)) = 1$.

(ii) $\theta_\pm \in \mathbb{R}$, $\theta_- \cdot \theta_+ > 0$. In case of a negative eigenvalue λ we have $\frac{\theta_\pm + \lambda\mu}{1 - \lambda} = 0$ and thus $\lambda_\pm = -\frac{\theta_\pm}{\mu}$, which yields two negative or two positive eigenvalues. As before $\deg_{LS}(\mathcal{M}_\mu \mathcal{G}(0)) = 1$.

(iii) $\theta_\pm \in \mathbb{R}$, $\theta_- \cdot \theta_+ < 0$. From case (ii) we easily derive that there exist two eigenvalues $\lambda_\pm$, one positive and one negative, and therefore $\deg_{LS}(\mathcal{M}_\mu \mathcal{G}(0)) = -1$.

These cases combined imply that $\deg_{LS}(\mathcal{M}_\mu \mathcal{G}(0)) = \text{sgn}(\det(\Theta))$ and

$$\text{parity}(\mathcal{G}(\eta), I) = \text{sgn}(\det(\Theta)) \cdot \text{sgn}(\det(\Theta')).$$

From the latter we derive:

$$\begin{aligned} & \text{sgn}(\det(\Theta)) \cdot \text{parity}(\mathcal{F}_\Theta(\eta), I) \\ &= \text{sgn}(\det(\Theta)) \cdot \text{sgn}(\det(\Theta)) \cdot \text{sgn}(\det(\Theta')) \cdot \text{parity}(\mathcal{F}_{\Theta'}(\eta), I) \\ &= \text{sgn}(\det(\Theta')) \cdot \text{parity}(\mathcal{F}_{\Theta'}(\eta), I), \end{aligned}$$

which proves the independence of Θ .

Lemmas 8 shows that the index of a non-degenerate zero of $\mathcal{E}$ is well-defined. We now show that the same holds for isolated zeroes.

Lemma 9 *The index $\iota(x)$ for an isolated zero of $\mathcal{E}$ is well-defined and for a fixed choice of $\mathcal{M}$ and Θ the index is given by*

$$\iota(x) = -\text{sgn}(\det(\Theta))(-1)^{\beta_{\mathcal{M}}(\Theta)} \deg_{LS}(\Phi_{\mathcal{M}}, B_\varepsilon(x), 0),$$

where $\varepsilon > 0$ is small enough such that x is the only zero of $\mathcal{E}$ in $B_\varepsilon(x)$.

Proof By the Sard-Smale Theorem one can choose an arbitrarily small $h \in C^0(\mathbb{R}/\mathbb{Z})$, $\|h\|_{C^0} < \epsilon'$, such that h is a regular value of $\mathcal{E}$ and $\mathcal{E}^{-1}(h) \cap B_\epsilon(x)$ consists of finitely many non-degenerate zeroes in x_h . Set $\tilde{\mathcal{E}}(x) = \mathcal{E}(x) - h$ and define

$$\mathfrak{u}(x) = \sum_{x_h \in \tilde{\mathcal{E}}^{-1}(0) \cap B_\epsilon(x)} \mathfrak{u}(x_h). \quad (33)$$

We now show that $\mathfrak{u}(x)$ is well-defined. Choose a fixed parametrix $\mathcal{M}$ (for $\mathcal{E}$) and fixed $\Theta \in \mathbf{M}_{2 \times 2}(\mathbb{R})$, and let $\tilde{\Phi}_{\mathcal{M}} = \mathcal{M}\mathcal{E}$, then

$$\sum_{x_h} \mathfrak{u}(x_h) = -\operatorname{sgn}(\det(\Theta))(-1)^{\beta_{\mathcal{M}}(\Theta)} \sum_{x_h} \deg_{LS}(\tilde{\Phi}_{\mathcal{M}}, B_{\epsilon_h}(x_h), 0),$$

where $B_{\epsilon_h}(x_h)$ are sufficiently small neighborhoods containing only one zero. From Leray-Schauder degree theory we derive that

$$\sum_{x_h} \deg_{LS}(\tilde{\Phi}_{\mathcal{M}}, B_{\epsilon_h}(x_h), 0) = \deg_{LS}(\tilde{\Phi}_{\mathcal{M}}, B_\epsilon(x), 0) = \deg_{LS}(\Phi_{\mathcal{M}}, B_\epsilon(x), 0),$$

which proves the lemma.

Theorem 1 now follows from the Leray-Schauder degree. Suppose all zeroes of $\mathcal{E}$ in $\Omega = [x]\operatorname{rel} y$ are isolated, then Lemma 9 implies that

$$\begin{aligned} \sum_{x \in \mathcal{E}^{-1}(0) \cap \Omega} \mathfrak{u}(x) &= -\operatorname{sgn}(\det(\Theta))(-1)^{\beta_{\mathcal{M}}(\Theta)} \sum_x \deg_{LS}(\Phi_{\mathcal{M}}, B_\epsilon(x), 0) \\ &= -\operatorname{sgn}(\det(\Theta))(-1)^{\beta_{\mathcal{M}}(\Theta)} \deg_{LS}(\Phi_{\mathcal{M}}, \Omega, 0) \end{aligned}$$

Since the latter expression is independent of $\mathcal{M}$ and Θ we choose $\mathcal{M} = L_\mu^{-1}$ and $\Theta = \theta J$. Then, $\Phi_{\mathcal{M}} = \Phi$, and for the indices we have $\operatorname{sgn}(\det(\theta J)) = 1$ and by Lemma 3, $(-1)^{\beta_{L_\mu^{-1}}(\theta J)} = 1$. By Proposition 9, $\deg_{LS}(\Phi, \Omega, 0) = -\chi(x \operatorname{rel} y)$, which, by substitution of these choices into the index formula, yields

$$\sum_{x \in \mathcal{E}^{-1}(0) \cap \Omega} \mathfrak{u}(x) = \chi(x \operatorname{rel} y),$$

completing the proof Theorem 1.

8 Legendrian braids

In this section we prove Theorem 3 and show that the Euler-Floer characteristic can be determined via a discrete topological invariant.

8.1 Hyperbolic Hamiltonians on $\mathbb{R}^2$

Consider Hamiltonians of the form

$$H(x, t) = \frac{1}{2}p^2 - \frac{1}{2}q^2 + h(x, t), \quad (34)$$

where h satisfies the following hypotheses:

- (h1) $h \in C^\infty(\mathbb{R}^2 \times \mathbb{R}/\mathbb{Z})$;
- (h2) $\text{supp}(h) \subset \mathbb{R} \times [-R, R] \times \mathbb{R}/\mathbb{Z}$, for some $R > 0$;
- (h3) $\|h\|_{C_b^2(\mathbb{R}^2 \times \mathbb{R}/\mathbb{Z})} \leq c$.

Lemma 10 *Let H be given by (34), with h satisfying (h1)-(h3). Then, there exists a constant $R' \geq R > 0$, such any 1-periodic solution of x of $x' = X_H(x, t)$ satisfies the estimate*

$$|x(t)| \leq R', \quad \text{for all } t \in \mathbb{R}/\mathbb{Z}.$$

Proof The Hamilton equation in local coordinates are given by

$$p_t = q - h_q(p, q, t), \quad q_t = p + h_p(p, q, t).$$

Since h is smooth we can rewrite the equations as

$$q_{tt} = h_{pq}(p, q, t)q_t + (1 + h_{pp}(p, q, t))(q - h_q(p, q, t)) + h_{pt}(p, q, t). \quad (35)$$

If $x(t)$ is a 1-periodic solution to the Hamilton equations, and suppose there exists an interval $I = [t_0, t_1] \subset [0, 1]$ such that $|q(t)| > R$ on $\text{int}(I)$ and $|q(t)||_{\partial I} = R$. The function $q|_I$ satisfies the equation $q_{tt} - q = 0$, and obviously such solutions do not exist. Indeed, if $q|_I \geq R$, then $q_t(t_0) \geq 0$ and $q_t(t_1) \leq 0$ and thus $0 \geq q_t|_{\partial I} = \int_I q \geq R|I| > 0$, a contradiction. The same holds for $q|_I \leq -R$. We conclude that

$$|q(t)| < R, \quad \text{for all } t \in \mathbb{R}/\mathbb{Z}.$$

We now use the a priori q -estimate in combination with Equation (35) and Hypothesis (h3). Multiplying Equation (35) by q and integrating over $[0, 1]$ gives:

$$\begin{aligned} \int_0^1 q_t^2 &= - \int_0^1 h_{pq} q_t q - \int_0^1 (1 + h_{pp})(q - h_q)q - \int_0^1 h_{pt} q \\ &\leq C \int_0^1 |q_t| + C \leq \epsilon \int_0^1 q_t^2 + C_\epsilon, \end{aligned}$$

which implies that $\int_0^1 q_t^2 \leq C(R)$. The L^2 -norm of the right hand side in (35) can be estimated using the L^∞ estimate on q and the L^2 -estimate on q_t , which yields $\int_0^1 q_{tt}^2 \leq C(R)$. Combining these estimates we have that $\|q\|_{H^2(\mathbb{R}/\mathbb{Z})} \leq C(R)$ and thus $|q_t(t)| \leq C(R)$, for all $t \in \mathbb{R}/\mathbb{Z}$. From the Hamilton equations it follows that $|p(t)| \leq |q_t(t)| + C$, which proves the lemma.

Lemma 11 *If $H(x, t; \alpha)$, $\alpha \in [0, 1]$ is a (smooth) homotopy of Hamiltonians satisfying (h1)-(h3) with uniform constants $R > 0$ and $c > 0$, then $|x_\alpha(t)| \leq R'$, for all 1-periodic solutions and for all $\alpha \in [0, 1]$.*

Proof The a priori H^2 -estimates in Lemma 10 hold with uniform constants with respect to $\alpha \in [0, 1]$. This then proves the lemma.

8.2 Braids on $\mathbb{R}^2$ and Legendrian braids

In Section 1 we defined braid classes as path components of closed loops in $\mathcal{L}\mathbf{C}_n(\mathbb{D}^2)$, denoted by $[x]$. If we consider closed loops in $\mathbf{C}_n(\mathbb{R}^2)$, then the braid classes will be denoted by $[x]_{\mathbb{R}^2}$. The same notation applies to relative braid classes $[x \text{ rel } y]_{\mathbb{R}^2}$. A relative braid class is proper if components $x_c \subset x$ cannot be deformed onto (i) itself, or other components $x'_c \subset x$, or (ii) components $y_c \subset y$. A fiber $[x]_{\mathbb{R}^2} \text{ rel } y$ is *not* bounded!

In order to compute the Euler-Floer characteristic of $[x \text{ rel } y]$ we assume without loss of generality that $x \text{ rel } y$ is a positive representative. If not we compose $x \text{ rel } y$ with a sufficient number of positive full twists such that the resulting braid is positive, i.e. only positive crossings, see [12] for more details. The Euler-Floer characteristic remains unchanged. We denote a positive representative $x^+ \text{ rel } y^+$ again by $x \text{ rel } y$.

Define an augmented skeleton y^* by adding the constant strands $y_-(t) = (0, -1)$ and $y_+(t) = (0, 1)$. For proper braid classes it holds that $[x \text{ rel } y] = [x \text{ rel } y^*]$. For notational simplicity we denote the augmented skeleton again by y . We also choose the representative $x \text{ rel } y$ with the additional property that $\pi_2 x \text{ rel } \pi_2 y$ is a relative braid diagram, i.e. there are no tangencies between the strands, where π_2 the projection onto the q -coordinate. We denote the projection by $q \text{ rel } Q$, where $q = \pi_2 x$ and $Q = \pi_2 y$. Special braids on $\mathbb{R}^2$ can be constructed from (smooth) positive braids. Define $x_L = (q_t, q)$ and $y_L = (Q_t, Q)$, where the subscript t denotes differentiating with respect to t . These are called *Legendrian braids* with respect to $\theta = p dt - dq$.

Lemma 12 *For positive braid $x \text{ rel } y$ with only transverse, positive crossings, the braids $x_L \text{ rel } y_L$ and $x \text{ rel } y$ are isotopic as braids on $\mathbb{R}^2$. Moreover, if $x_L \text{ rel } y_L$ and $x'_L \text{ rel } y'_L$ are isotopic Legendrian braids, then they are isotopic via a Legendrian isotopy.*

Proof By assumption $x \text{ rel } y$ is a representative for which the braid diagram $q \text{ rel } Q$ has only positive transverse crossings. Due to the transversality of intersections the associated Legendrian braid $x_L \text{ rel } y_L$ is a braid $[x \text{ rel } y]_{\mathbb{R}^2}$. Consider the homotopy

$$\zeta^j(t, \tau) = \tau p^j(t) + (1 - \tau) q_t^j,$$

for every strand q^j . At q -intersections, i.e. times t_0 such that $q^j(t_0) = q^{j'}(t_0)$ for some $j \neq j'$, it holds that $p^j(t_0) - p^{j'}(t_0)$ and $q_t^j(t_0) - q_t^{j'}(t_0)$ are non-zero and have the same sign since all crossings in $x \text{ rel } y$ are positive! Therefore, $\zeta^j(t_0, \tau) \neq \zeta^{j'}(t_0, \tau)$ for any intersection t_0 and any $\tau \in [0, 1]$, which shows that $x \text{ rel } y$ and $x_L \text{ rel } y_L$ are isotopic. Since $x_L \text{ rel } y_L$ and $x'_L \text{ rel } y'_L$ have only positive crossings, a smooth Legendrian isotopy exists.

The associated equivalence class of Legendrian braid diagrams is denoted by $[q \text{ rel } Q]$ and its fibers by $[q] \text{ rel } Q$.

9 Mechanical Hamiltonian systems

Legendrian braids can be described via Hamiltonian of mechanical type, i.e. $H(x, t) = \frac{1}{2}p^2 + V(q, t)$. Due to the special form we can also use the Lagrangian formalism for

such systems. In the next subsection we investigate the relation between the Conley-Zehnder index and the Lagrangian Morse index of closed integral curves.

9.1 The Lagrangian Morse index

A mechanical system is defined as the Euler-Lagrange equations of the Lagrangian density $L(q, t) = \frac{1}{2} \dot{q}^2 - V(q, t)$. The linearization at a critical points $q(t)$ of the Lagrangian action is given by the unbounded operator

$$-\frac{d^2}{dt^2} - D_q^2 V(q(t), t) : H^2(\mathbb{R}/\mathbb{Z}) \subset L^2(\mathbb{R}/\mathbb{Z}) \rightarrow L^2(\mathbb{R}/\mathbb{Z}).$$

Consider a path of unbounded self-adjoint operators on $L^2(\mathbb{R}/\mathbb{Z})$ given by $\eta \mapsto \mathcal{D}(\eta) = -\frac{d^2}{dt^2} - \mathcal{Q}(t; \eta)$, with $\mathcal{Q}(t; \eta)$ smooth. If $\mathcal{D}(0)$ and $\mathcal{D}(1)$ are invertible, then the spectral flow is well-defined. Define the ‘Lagrangian’ Morse indices $\gamma_{\mathcal{D}(0)}$ and $\gamma_{\mathcal{D}(1)}$ as the number of negative eigenvalues of $\mathcal{D}(0)$ and $\mathcal{D}(1)$ respectively.⁴

Proposition 10 *Assume that the endpoints of $\eta \mapsto \mathcal{D}(\eta)$ are invertible. Then*

$$\text{specflow}(\mathcal{D}(\eta), I) = \gamma_{\mathcal{D}(0)} - \gamma_{\mathcal{D}(1)}. \quad (36)$$

Proof In [11] the concatenation property of the spectral flow is proved. We use concatenation as follows. Let $c > 0$ be a sufficiently large constant such that $\mathcal{D}(0) + c \text{Id}$ and $\mathcal{D}(1) + c \text{Id}$ are positive definite self-adjoint operators on $L^2(\mathbb{R}/\mathbb{Z})$. Consider the paths $\eta \mapsto \mathcal{D}_1(\eta) = \mathcal{D}(0) + \eta c \text{Id}$ and $\eta \mapsto \mathcal{D}_2(\eta) = \mathcal{D}(1) + (1 - \eta) c \text{Id}$. Their concatenation $\mathcal{D}_1 \# \mathcal{D}_2$ is a path from $\mathcal{D}(0)$ to $\mathcal{D}(1)$ and $\eta \mapsto \mathcal{D}_1 \# \mathcal{D}_2$ is homotopic to $\eta \mapsto \mathcal{D}(\eta)$. Using the homotopy invariance and the concatenation property of the spectral flow we obtain

$$\text{specflow}(\mathcal{D}(\eta), I) = \text{specflow}(\mathcal{D}_1 \# \mathcal{D}_2, I) = \text{specflow}(\mathcal{D}_1, I) + \text{specflow}(\mathcal{D}_2, I).$$

Since $\mathcal{D}(0)$ is invertible, the regular crossings of $\mathcal{D}_1(\eta)$ are given by $\eta_i^1 = -\frac{\lambda_i}{c}$, where λ_i are negative eigenvalues of $\mathcal{D}(0)$. By the positive definiteness of $\mathcal{D}(0) + c \text{Id}$, the negative eigenvalues of $\mathcal{D}(0)$ satisfy $0 > \lambda_i > -c$. For the crossing η_i this implies

$$0 < \eta_i = -\frac{\lambda_i}{c} < 1,$$

and therefore the number of crossings equals the number of negative eigenvalues of $\mathcal{D}(0)$ counted with multiplicity. By the choice of c , we also have that $\frac{d}{d\eta} \mathcal{D}_1(\eta) = c \text{Id}$ is positive definite and therefore the signature of the crossing operator of $\mathcal{D}_1(\eta)$ is exactly the number of negative eigenvalues of $\mathcal{D}(0)$, i.e. $\text{specflow}(\mathcal{D}_1, I) = \gamma_{\mathcal{D}(0)}$. For $\mathcal{D}_2(\eta)$ we obtain, $\text{specflow}(\mathcal{D}_2, I) = -\gamma_{\mathcal{D}(1)}$. This proves that $\text{specflow}(\mathcal{D}(\eta), I) = \gamma_{\mathcal{D}(0)} - \gamma_{\mathcal{D}(1)}$.

⁴ We use the adjective ‘Lagrangian’ to distinguish the latter from the Morse index in Proposition 3.

For a mechanical system the Hamiltonian is given by $H(x, t) = \frac{1}{2}p^2 + V(q, t)$. As such the Conley-Zehnder index of a critical point q can be defined as the Conley-Zehnder index of $x = (q_t, q)$, see also [1] and [2]. The Morse index of a critical point is defined as $\gamma(q) = \gamma_{\mathcal{D}}$, where $\mathcal{D} = -\frac{d^2}{dt^2} - D_q^2 V(q(t), t)$.

Lemma 13 *Let q be a critical point of the mechanical Lagrangian action, then the associated Conley-Zehnder index $\mu^{CZ}(x)$ is well-defined, and $\mu^{CZ}(x) = \gamma(q)$.*

Proof As before, consider the curves $\eta \mapsto \mathcal{B}(\eta)$ and $\eta \mapsto \mathcal{D}(\eta)$, $\eta \in I = [0, 1]$ given by

$$\mathcal{B}(\eta) = -J \frac{d}{dt} - \begin{pmatrix} 1 & 0 \\ 0 & \mathcal{Q}(t; \eta) \end{pmatrix}, \quad \mathcal{D}(\eta) = -\frac{d^2}{dt^2} - \mathcal{Q}(t; \eta).$$

The crossing forms of the curves are the same — $\Gamma(\mathcal{B}, \eta) = \Gamma(\mathcal{D}, \eta)$ — and therefore also the crossings η_0 are identical. Indeed, $\mathcal{B}(\eta_0)$ is non-invertible if and only if $\mathcal{D}(\eta_0)$ is non-invertible. Consequently, $\text{specflow}(\mathcal{B}(\eta), I) = \text{specflow}(\mathcal{D}(\eta), I)$ and the Propositions 6 and 10 then imply that

$$\gamma_{\mathcal{D}(0)} - \gamma_{\mathcal{D}(1)} = \mu_{\mathcal{B}(0)}^{CZ} - \mu_{\mathcal{B}(1)}^{CZ}. \quad (37)$$

Now choose $\mathcal{Q}(t; \eta)$ such that $\mathcal{Q}(t; 0) = d^2 V(q(t), t) + c$ and $\mathcal{Q}(t; 1) = D_q^2 V(q(t), t)$ and such that $\eta \mapsto \mathcal{B}(\eta)$ and $\eta \mapsto \mathcal{D}(\eta)$ are regular curves. If $c \ll 0$, then $\gamma_{\mathcal{D}(0)} = 0$. In order to compute $\mu_{\mathcal{B}(0)}^{CZ}$ we invoke the crossing from $\Gamma(\Psi, t)$ for the associated symplectic path $\Psi(t)$ as explained in Section 5. Crossings at $t_0 \in (0, 1]$ correspond to non-trivial solutions of the equation $\mathcal{D}(0)\psi = 0$ on $[0, t_0]$, with periodic boundary conditions. To be more precise, let $\Psi = (\phi, \psi)$, then $\mathcal{B}(0)\Psi = 0$ is equivalent to $\psi_t = \phi$ and $-\phi_t - (D_q^2 V(q(t), t) + c)\psi = 0$, which yields the equation $\mathcal{D}(0)\psi = 0$. For the latter the kernel is trivial for any $t_0 \in (0, 1]$. Indeed, if ψ is a solution, then $\int_0^{t_0} |\psi_t|^2 = \int_0^{t_0} (D_q^2 V(q(t), t) + c)\psi^2 < 0$, which is a contradiction. Therefore, there are no crossing $t_0 \in (0, 1]$. As for $t_0 = 0$ we have that $D_q^2 V(q(0), 0) + c \text{Id} < 0$, which implies that $\text{sgn} \mathcal{S}(0; 0) = 0$ and therefore $\mu_{\mathcal{B}(0)}^{CZ} = 0$, which proves the lemma.

9.2 The Poincaré-Hopf Formula and the Morse index

Legendrian braids can be described with Lagrangian systems and Hamiltonians of the form $H_L(x, t) = \frac{1}{2}p^2 - \frac{1}{2}q^2 + g(q, t)$. On the potential functions g we impose the following hypotheses:

- (g1) $g \in C^\infty(\mathbb{R} \times \mathbb{R}/\mathbb{Z})$;
- (g2) $\text{supp}(g) \subset [-R, R] \times \mathbb{R}/\mathbb{Z}$, for some $R > 1$.

In order to have a straightforward construction of a mechanical Lagrangian we may consider a special representation of y . The Euler-Floer characteristic $\chi(x \text{ rely})$ does not depend on the choice of the fiber $[x] \text{ rely}$ and therefore also not on the skeleton y . We assume that y has linear crossings in y_L . Let $t = t_0$ be a crossing and let $I(t_0)$

be the set of labels defined by: $i, j \in I(t_0)$, if $i \neq j$ and $Q^i(t_0) = Q^j(t_0)$. A crossing at $t = t_0$ is *linear* if

$$Q_t^i(t) = \text{constant}, \quad \forall i \in I(t_0), \text{ and } \quad \forall t \in (-\varepsilon + t_0, \varepsilon + t_0),$$

for some $\varepsilon = \varepsilon(t_0) > 0$. Every skeleton Q with transverse crossings is isotopic to a skeleton with linear crossings via a small local deformation at crossings. For Legendrian braids $x_L \text{ rel } y_L \in [x \text{ rel } y]_{\mathbb{R}^2}$ with linear crossings the following result holds:

Lemma 14 *Let y_L be a Legendrian skeleton with linear crossings. Then, there exists a Hamiltonian of the form $H_L(x, t) = \frac{1}{2}p^2 - \frac{1}{2}q^2 + g(q, t)$, with g satisfying Hypotheses (g1)-(g2), and $R > 0$ sufficiently large, such that y_L is a skeleton for $X_{H_L}(x, t)$.*

Proof Due to the linear crossings in y_L we can follow the construction in [12]. For each strand Q^i we define the potentials $g^i(t, x) = -Q_t^i(t)q$. By construction Q^i is a solution of the equation $Q_t^i = -g_q^i(t, Q^i)$. Now choose small tubular neighborhoods of the strands Q^i and cut-off functions ω^i that are equal to 1 near Q^i and are supported in the tubular neighborhoods. If the tubular neighborhoods are narrow enough, then $\text{supp}(\omega^i g^i) \cap \text{supp}(\omega^j g^j) = \emptyset$, for all $i \neq j$, due to the fact that at crossings the functions g^i in question are zero. This implies that all strands Q^i satisfy the differential equation $Q_t^i = -\sum_i \omega^i(t) g_q^i(Q^i, t)$ and on $[-1, 1] \times \mathbb{R}/\mathbb{Z}$, the function is $\sum_i \omega^i(t) g^i(q, t)$ is compactly supported. The latter follows from the fact that for the constant strands $Q^i = \pm 1$, the potentials g^i vanish. Let $R > 1$ and define

$$\tilde{g}^i(t, q) = \begin{cases} g^i(t, q) & \text{for } |q| \leq 1, t \in \mathbb{R}/\mathbb{Z}, \\ -\frac{1}{2m}q^2 & \text{for } |q| \geq R, t \in \mathbb{R}/\mathbb{Z}. \end{cases}$$

where $m = \#Q$, which yields smooth functions $\tilde{g}^i$ on $\mathbb{R} \times \mathbb{R}/\mathbb{Z}$. Now define

$$g(q, t) = \frac{1}{2}q^2 + \sum_{i=1}^m \tilde{g}^i(q, t).$$

By construction $\text{supp}(g) \subset [-R, R] \times \mathbb{R}/\mathbb{Z}$, for some $R > 1$ and the strands Q^i all satisfy the Euler-Lagrange equations $Q_t^i = Q^i - g_{qq}(Q^i, t)$, which completes the proof.

The Hamiltonian H_L given by Lemma 14 gives rise to a Lagrangian system with the Lagrangian action given by

$$\mathcal{L} = \int_0^1 \frac{1}{2}q_t^2 + \frac{1}{2}q^2 - g(q, t) dt. \quad (38)$$

The braid class $[q] \text{ rel } Q$ is bounded due to the special strands ± 1 and all free strands q satisfy $-1 \leq q(t) \leq 1$. Therefore, the set of critical points of $\mathcal{L}$ in $[q] \text{ rel } Q$ is a compact set. The critical points of $\mathcal{L}$ in $[q] \text{ rel } Q$ are in one-to-one correspondence with the zeroes of the equation

$$\Phi_{H_L}(x) = x - L_\mu^{-1}(\nabla H_L(x, t) + \mu x) = 0,$$

in the set $\Omega_{\mathbb{R}^2} = [x_L]_{\mathbb{R}^2} \text{rel } y_L$, which implies that Φ_{H_L} is a proper mapping on $\Omega_{\mathbb{R}^2}$. From Lemma 10 we derive that the zeroes of Φ_{H_L} are contained in ball in $\mathbb{R}^2$ with radius $R' > 1$, and thus $\Phi_{H_L}^{-1}(0) \cap \Omega_{\mathbb{R}^2} \subset B_{R'}(0) \subset C^1(\mathbb{R}/\mathbb{Z})$. Therefore the Leray-Schauder degree is well-defined and in the generic case Lemma 13 and Equations (21), (27) and (30) yield

$$\deg_{LS}(\Phi_{H_L}, \Omega_{\mathbb{R}^2}, 0) = - \sum_{x \in \Phi_{H_L}^{-1}(0) \cap \Omega_{\mathbb{R}^2}} (-1)^{\mu^{CZ}(x)} = - \sum_{q \in \text{Crit}(\mathcal{L}) \cap ([q] \text{rel } Q)} (-1)^{\gamma(q)}. \quad (39)$$

We are now in a position to use a homotopy argument. We can scale y to a braid ρy such that the rescaled Legendrian braid ρy_L is supported in $\mathbb{D}^2$. By Lemma 12, y is isotopic to y_L and scaling defines an isotopy between y_L and ρy_L . Denote the isotopy from y to ρy_L by y_α . By Proposition 9 we obtain that for both skeletons y and ρy_L it holds that

$$\deg_{LS}(\Phi_H, \Omega, 0) = -\chi(x \text{rel } y) = \deg_{LS}(\Phi_{H_\rho}, \Omega_\rho, 0),$$

where $\Omega_\rho = [\rho x_L] \text{rel } \rho y_L \subset [x \text{rel } y]$ and $H_\rho \in \mathcal{H}(\rho y_L)$. Now extend H_ρ to $\mathbb{R}^2 \times \mathbb{R}/\mathbb{Z}$, such that Hypotheses (h1)-(h3) are satisfied for some $R > 1$. We denote the Hamiltonian again by H_ρ . By construction all zeroes of Φ_{H_ρ} in $[\rho x_L] \text{rel } \rho y_L$ are supported in $\mathbb{D}^2$ and therefore the zeroes of Φ_{H_ρ} in $[\rho x_L]_{\mathbb{R}^2} \text{rel } \rho y_L$ are also supported in $\mathbb{D}^2$. Indeed, any zero intersects $\mathbb{D}^2$, since the braid class is proper and since $\partial \mathbb{D}^2$ is invariant for the Hamiltonian vector field, a zero is either inside or outside $\mathbb{D}^2$. Combining these facts implies that a zero lies inside $\mathbb{D}^2$. This yields

$$\deg_{LS}(\Phi_{H_\rho}, \Omega_{\rho, \mathbb{R}^2}, 0) = \deg_{LS}(\Phi_{H_\rho}, \Omega_\rho, 0) = -\chi(x \text{rel } y),$$

where $\Omega_{\rho, \mathbb{R}^2} = [\rho x_L]_{\mathbb{R}^2} \text{rel } \rho y_L$. For the next homotopy we keep the skeleton ρy_L fixed as well as the domain $\Omega_{\rho, \mathbb{R}^2}$. Consider the linear homotopy of Hamiltonians

$$H_1(x, t; \alpha) = \frac{1}{2}p^2 - \frac{1}{2}q^2 + (1 - \alpha)h_\rho(x, t) + \alpha g_\rho(q, t),$$

where $H_{\rho, L}(t, x) = \frac{1}{2}p^2 - \frac{1}{2}q^2 + g_\rho(q, t)$ given by Lemma 14. This defines an admissible homotopy since ρy_L is a skeleton for all $\alpha \in [0, 1]$. The uniform estimates are obtained, as before, by Lemma 11, which allows application of the Leray-Schauder degree:

$$\deg_{LS}(\Phi_{H_{\rho, L}}, \Omega_{\rho, \mathbb{R}^2}, 0) = \deg_{LS}(\Phi_{H_\rho}, \Omega_{\rho, \mathbb{R}^2}, 0) = -\chi(x \text{rel } y).$$

Finally, we scale ρy_L to y_L via $y_{\alpha, L} = (1 - \alpha)\rho y_L + \alpha y_L$ and we consider the homotopy

$$H_2(x, t; \alpha) = \frac{1}{2}p^2 - \frac{1}{2}q^2 + g(q, t; \alpha),$$

between H_L and $H_{\rho, L}$, where $g(q, t; \alpha)$ is found by applying Lemma 14 to $y_{\alpha, L}$. The uniform estimates from Lemma 11 allows us to apply the Leray-Schauder degree:

$$\deg_{LS}(\Phi_{H_L}, \Omega_{\mathbb{R}^2}, 0) = \deg_{LS}(\Phi_{H_{\rho, L}}, \Omega_{\rho, \mathbb{R}^2}, 0) = -\chi(x \text{rel } y).$$

Combining the equalities for the various Leray-Schauder degrees with (39) yields:

$$-\deg_{LS}(\Phi_{H_L}, \Omega_{\mathbb{R}^2}, 0) = \chi(x \text{rel } y) = \sum_{q \in \text{Crit}(\mathcal{L}) \cap ([q] \text{rel } Q)} (-1)^{\gamma(q)}. \quad (40)$$

10 The proof of Theorem 3

The Lagrangian setting introduced in the previous section allows for another simplification via finite dimensional systems.

10.1 Discretized braid classes

The Lagrangian problem (38) can be treated by using a variation on the method of broken geodesics. If we choose $1/d > 0$ sufficiently small, the integral

$$S_i(q_i, q_{i+1}) = \min_{\substack{q(t) \in E_i(q_i, q_{i+1}) \\ |q(t)| \leq 1}} \int_{\tau_i}^{\tau_{i+1}} \frac{1}{2} \dot{q}^2 + \frac{1}{2} q^2 - g(q, t) dt, \quad (41)$$

has a unique minimizer q^i , where $E_i(q_i, q_{i+1}) = \{q \in H^1(\tau_i, \tau_{i+1}) \mid q(\tau_i) = q_i, q(\tau_{i+1}) = q_{i+1}\}$, and $\tau_i = i/d$. Moreover, if $1/d$ is small, then the minimizers are non-degenerate and S_i is a smooth function of q_i and q_{i+1} . Critical points q of $\mathcal{L}$ with $|q(t)| \leq 1$ correspond to sequences $q_D = (q_0, \dots, q_d)$, with $q_0 = q_d$, which are critical points of the discrete action

$$\mathcal{W}(q_D) = \sum_{i=0}^{d-1} S_i(q_i, q_{i+1}). \quad (42)$$

A concatenation $\#_i q^i$ of minimizers q^i is continuous and is an element in the function space $H^1(\mathbb{R}/\mathbb{Z})$, and is referred to as a *broken geodesic*. The set of broken geodesics $\#_i q^i$ is denoted by $E(q_D)$ and standard arguments using the non-degeneracy of minimizers q^i show that $E(q_D) \hookrightarrow H^1(\mathbb{R}/\mathbb{Z})$ is a smooth, d -dimensional submanifold in $H^1(\mathbb{R}/\mathbb{Z})$. The submanifold $E(q_D)$ is parametrized by sequences $D_d = \{q_D \in \mathbb{R}^d \mid |q_i| \leq 1\}$ and yields the following commuting diagram:

$$\begin{array}{ccc} E(q_D) & \xrightarrow{\mathcal{L}} & \mathbb{R} \\ \uparrow \#_i & \nearrow \mathcal{W} & \\ D_n & & \end{array}$$

In the above diagram $\#_i$ is regarded as a mapping $q_D \mapsto \#_i q^i$, where the minimizers q_i are determined by q_D . The tangent space to $E(q_D)$ at a broken geodesic $\#_i q^i$ is identified by

$$\begin{aligned} T_{\#_i q^i} E(q_D) &= \left\{ \psi \in H^1(\mathbb{R}/\mathbb{Z}) \mid -\psi_{tt} + \psi - g_{qq}(q^i(t), t)\psi = 0, \right. \\ &\quad \left. \psi(\tau_i) = \delta q_i, \psi(\tau_{i+1}) = \delta q_{i+1}, \delta q_i \in \mathbb{R}, \forall i \right\}, \end{aligned}$$

and $\#_i q^i + T_{\#_i q^i} E(q_D)$ is the tangent hyperplane at $\#_i q^i$. For $H^1(\mathbb{R}/\mathbb{Z})$ we have the following decomposition for any broken geodesic $\#_i q^i \in E(q_D)$:

$$H^1(\mathbb{R}/\mathbb{Z}) = E' \oplus T_{\#_i q^i} E(q_D), \quad (43)$$

where $E' = \{\eta \in H^1(\mathbb{R}/\mathbb{Z}) \mid \eta(\tau_i) = 0, \forall i\}$. To be more specific the decomposition is orthogonal with respect to the quadratic form

$$D^2 \mathcal{L}(q) \phi \tilde{\phi} = \int_0^1 \phi_t \tilde{\phi}_t + \phi \tilde{\phi} - g_{qq}(q(t), t) \phi \tilde{\phi} dt, \quad \phi, \tilde{\phi} \in H^1(\mathbb{R}/\mathbb{Z}).$$

Indeed, let $\eta \in E'$ and $\psi \in T_{\#_i q^i} E(q_D)$, then

$$\begin{aligned} D^2 \mathcal{L}(\#_i q^i) \eta \psi &= \sum_i \int_{\tau_i}^{\tau_{i+1}} \eta_t \psi_t + \eta \psi - g_{qq}(q^i(t), t) \xi \eta dt \\ &= \sum_i \psi_t \eta|_{\tau_i}^{\tau_{i+1}} - \sum_i \int_{\tau_i}^{\tau_{i+1}} [-\psi_t + \psi + g_{qq}(q^i(t), t) \psi] \eta dt = 0. \end{aligned}$$

Let $\phi = \eta + \psi$, then

$$D^2 \mathcal{L}(\#_i q^i) \phi \tilde{\phi} = D^2 \mathcal{L}(\#_i q^i) \eta \tilde{\eta} + D^2 \mathcal{L}(\#_i q^i) \psi \tilde{\psi},$$

by the above orthogonality. By construction the minimizers q^i are non-degenerate and therefore $D^2 \mathcal{L}|_{E'}$ is positive definite. This implies that the Morse index of a (stationary) broken geodesic is determined by $D^2 \mathcal{L}|_{T_{\#_i q^i} E(q_D)}$. By the commuting diagram for $\mathcal{W}$ this implies that the Morse index is given by quadratic form $D^2 \mathcal{W}(q_D)$. We have now proved the following lemma that relates the Morse index of critical points of the discrete action $\mathcal{W}$ to Morse index of the ‘full’ action $\mathcal{L}$.

Lemma 15 *Let q be a critical point of $\mathcal{L}$ and q_D the corresponding critical point of $\mathcal{W}$, then the Morse indices are the same i.e. $\gamma(q) = \gamma(q_D)$.*

For a 1-periodic function $q(t)$ we define the mapping

$$q \xrightarrow{D_d} q_D = (q_0, \dots, q_d), \quad q_i = q(i/d), \quad i = 0, \dots, d,$$

and q_D is called the discretization of q . The linear interpolation

$$q_D \mapsto \ell_{q_D}(t) = \#_i \left[q_i + \frac{q_{i+1} - q_i}{d} t \right],$$

reconstructs a piecewise linear 1-periodic function. For a relative braid diagram $q \text{ rel } Q$, let $q_D \text{ rel } Q_D$ be its discretization, where Q_D is obtained by applying D_d to every strand in Q . A discretization $q_D \text{ rel } Q_D$ is *admissible* if $\ell_{q_D} \text{ rel } \ell_{Q_D}$ is homotopic to $q \text{ rel } Q$, i.e. $\ell_{q_D} \text{ rel } \ell_{Q_D} \in [q \text{ rel } Q]$. Define the *discrete* relative braid class $[q_D \text{ rel } Q_D]$ as the set of ‘discrete relative braids’ $q'_D \text{ rel } Q'_D$, such that $\ell_{q'_D} \text{ rel } \ell_{Q'_D} \in [q \text{ rel } Q]$. The associated fibers are denoted by $[q_D] \text{ rel } Q_D$. It follows from [8], Proposition 27, that $[q_D \text{ rel } Q_D]$ is guaranteed to be connected when

$$d > \#\{\text{crossings in } q \text{ rel } Q\},$$

i.e. for any two discrete relative braids $q_D \text{ rel } Q_D$ and $q'_D \text{ rel } Q'_D$, there exists a homotopy $q_D^\alpha \text{ rel } Q_D^\alpha$ (discrete homotopy) such that $\ell_{q_D^\alpha} \text{ rel } \ell_{Q_D^\alpha}$ is a path in $[q \text{ rel } Q]$. Note that fibers are not necessarily connected! For a braid classes $[q \text{ rel } Q]$ the associated discrete braid class $[q_D \text{ rel } Q_D]$ may be connected for a smaller choice of d .

We showed above that if $1/d > 0$ is sufficiently small, then the critical points of $\mathcal{L}$, with $|q| \leq 1$, are in one-to-one correspondence with the critical points of $\mathcal{W}$, and their Morse indices coincide by Lemma 15. Moreover, if $1/d > 0$ is small enough, then for all critical points of $\mathcal{L}$ in $[q] \text{ rel } Q$, the associated discretizations are admissible and $[q_D] \text{ rel } Q_D$ is a connected set. The discretizations of the critical points of $\mathcal{L}$ in $[q] \text{ rel } Q$ are critical points of $\mathcal{W}$ in the discrete braid class fiber $[q_D] \text{ rel } Q_D$.

Now combine the index identity with (40), which yields

$$\chi(x \text{ rel } y) = \sum_{q \in \text{Crit}(\mathcal{L}) \cap ([q] \text{ rel } Q)} (-1)^{\gamma(q)} = \sum_{q_D \in \text{Crit}(\mathcal{W}) \cap ([q_D] \text{ rel } Q_D)} (-1)^{\gamma(q_D)}. \quad (44)$$

10.2 The Conley index for discrete braids

In [8] an invariant for discrete braid classes $[q_D] \text{ rel } Q_D$ is defined based on the Conley index. The invariant $\text{HC}_*([q_D] \text{ rel } Q_D)$ is independent of the fiber and can be described as follows. A fiber $[q_D] \text{ rel } Q_D$ is a finite dimensional cube complex with a finite number of connected components. Denote the closures of the connected components by N_j . The faces of the hypercubes N_j can be co-oriented in direction of decreasing the number of crossing in $q_D \text{ rel } Q_D$, and define N_j^- as the closure of the set of faces with outward pointing co-orientation. The sets N_j^- are called *exit sets*. The invariant is given by

$$\text{HC}_*([q_D] \text{ rel } Q_D) = \bigoplus_j H_*(N_j, N_j^-).$$

The invariant is well-defined for any $d > 0$ for which there exist admissible discretizations and is independent of both the fiber and the discretization size. From [8] we have for any Morse function $\mathcal{W}$ on a proper braid class fiber $[q_D] \text{ rel } Q_D$,

$$\sum_{q_D \in \text{Crit}(\mathcal{W}) \cap ([q_D] \text{ rel } Q_D)} (-1)^{\gamma(q_D)} = \chi(\text{HC}_*([q_D] \text{ rel } Q_D)) =: \chi(q_D \text{ rel } Q_D). \quad (45)$$

The latter can be computed for any admissible discretization and is an invariant for $[q \text{ rel } Q]$. Combining Equations (44) and (45) gives

$$\chi(x \text{ rel } y) = \chi(q_D \text{ rel } Q_D). \quad (46)$$

In this section we assumed without loss of generality that $x \text{ rel } y$ is augmented and since the Euler-Floer characteristic is a braid class invariant, an admissible discretization is construction for an appropriate augmented, Legendrian representative $x_L \text{ rel } y_L$. Summarizing

$$\chi(x \text{ rel } y) = \chi(x_L \text{ rel } y_L^*) = \chi(q_D \text{ rel } Q_D^*).$$

Since $\chi(q_D \text{ rel } Q_D^*)$ is the same for any admissible discretization, the Euler-Floer characteristic can be computed using any admissible discretization, which proves Theorem 3.

Remark 5 The invariant $\chi(q_D \text{ rel } Q_D)$ is a true Euler characteristic of a topological pair. To be more precise

$$\chi(q_D \text{ rel } Q_D) = \chi([q_D] \text{ rel } Q_D, [q_D]^- \text{ rel } Q_D),$$

where $[q_D]^- \text{ rel } Q_D$ is the exit set as described above. A similar characterization does not a priori exist for $[x] \text{ rely}$. Firstly, it is more complicated to designate the equivalent of an exit set $[x]^- \text{ rely}$ for $[x] \text{ rely}$, and secondly it is not straightforward to develop a (co)-homology theory that is able to provide meaningful information about the topological pair $([x] \text{ rely}, [x]^- \text{ rely})$. This problem is circumvented by considering Hamiltonian systems and carrying out Floer's approach towards Morse theory (see [7]), by using the isolation property of $[x] \text{ rely}$. The fact that the Euler characteristic of Floer homology is related to the Euler characteristic of topological pair indicates that Floer homology is a good substitute for a suitable (co)-homology theory.

11 Examples

We will illustrate by means of two examples that the Euler-Floer characteristic is computable and can be used to find closed integral curves of vector fields on the 2-disc.

11.1 Example

Figure 4[left] shows the braid diagram $q \text{ rel } Q$ of a positive relative braid $x \text{ rely}$. The discretization with $q_D \text{ rel } Q_D$, with $d = 2$, is shown in Figure 4[right]. The chosen discretization is admissible and defines the relative braid class $[q_D \text{ rel } Q_D]$. There are five strands, one is free and four are fixed. We denote the points on the free strand by $q_D = (q_0, q_1)$ and on the skeleton by $Q_D = \{Q^1, \dots, Q^4\}$, with $Q^i = (Q_0^i, Q_1^i)$, $i = 1, \dots, 4$.

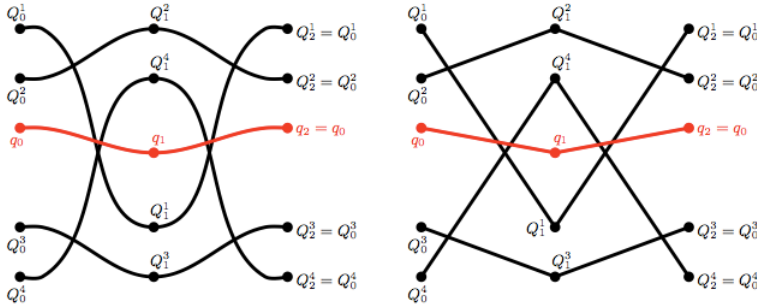


Fig. 4 A positive braid diagram [left] and an admissible discretization [right].

In Figure 5[left] the braid class fiber $[q_D] \text{ rel } Q_D$ is depicted. The coordinate q_0 is allowed to move between Q_0^3 and Q_0^2 and q_1 remains in the same braid class if it

varies between Q_1^1 and Q_1^4 . For the values $q_0 = Q_0^3$ and $q_0 = Q_0^2$ the relative braid becomes singular and if q_0 crosses these values two intersections are created. If q_1 crosses the values Q_1^1 or Q_1^4 two intersections are destroyed. This provides the desired co-orientation, see Figure 5[middle]. The braid class fiber $[q_D] \text{rel } Q_D$ consists of 1 component and we have that

$$N = \text{cl}([q_D] \text{rel } Q_D) = \{(q_0, q_1) : Q_0^3 \leq q_0 \leq Q_0^2, Q_1^1 \leq q_1 \leq Q_1^4\},$$

and the exit set is

$$N^- = \{(q_0, q_1) : q_1 = Q_1^1, \text{ or } q_1 = Q_1^4\}.$$

For the Conley index this gives:

$$\text{HC}_k([q_D] \text{rel } Q_D) = H_k(N, N^-; \mathbb{Z}) \cong \begin{cases} \mathbb{Z} & k = 1 \\ 0 & \text{otherwise} \end{cases}$$

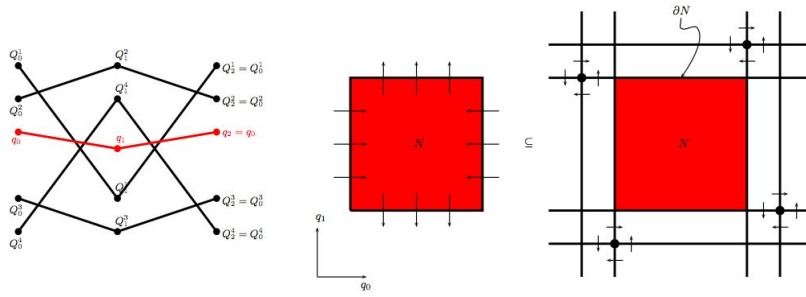


Fig. 5 The relative braid fiber $[q_D] \text{rel } Q_D$ and $N = \text{cl}([q_D] \text{rel } Q_D)$.

The Euler characteristic of $([q_D] \text{rel } Q_D, [q_D]^- \text{rel } Q_D)$ can be computed now and the Euler-Floer characteristic $(x \text{rel } y)$ is given by

$$\chi(x \text{rel } y) = \chi([q_D] \text{rel } Q_D, [q_D]^- \text{rel } Q_D) = -1 \neq 0$$

From Theorem 2 we derive that any vector field for which y is a skeleton has at least 1 closed integral curve $x_0 \text{rel } y \in [x] \text{rel } y$. Theorem 2 also implies that any orientation preserving diffeomorphism f on the 2-disc which fixes the set of four points A_4 , whose mapping class $[f; A_4]$ is represented by the braid y has an additional fixed point.

11.2 Example

The theory can also be used to find additional closed integral curves by concatenating the skeleton y . As in the previous example y is given by Figure 4. Glue ℓ copies of the skeleton y to its ℓ -fold concatenation and reparametrize time by $t \mapsto \ell \cdot t$. Denote the rescaled ℓ -fold concatenation of y by $\#_\ell y$. Choose $d = 2\ell$ and discretize $\#_\ell y$ as

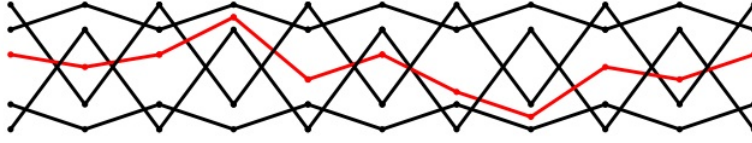


Fig. 6 A discretization of a braid class with a 5-fold concatenation of the skeleton y . The number of odd anchor points in middle position is $\mu = 3$.

in the previous example. For a given braid class $[x \text{ rel } \#_\ell y]$, Figure 6 below shows a discretized representative $q_D \text{ rel } \#_\ell Q_D$, which is admissible. For the skeleton $\#_\ell Q_D$ we can construct $3^\ell - 2$ proper relative braid classes in the following way: the even anchor points of the free strand q_D are always in the middle and for the odd anchor points we have 3 possible choices: bottom, middle, top (2 braids are not proper). We now compute the Conley index of the $3^\ell - 2$ different proper discrete relative braid classes and show that the Euler-Floer characteristic is non-trivial for these relative braid classes.

The configuration space $N = \text{cl}([q_D] \text{ rel } \#_\ell Q_D)$ in this case is given by a cartesian product of 2ℓ closed intervals, and therefore a 2ℓ -dimensional hypercube. We now proceed by determining the exit set N^- . As in the previous example the co-orientation is found by a union of faces with an outward pointing co-orientation. Due to the simple product structure of N , the set N^- is determined by the odd anchor points in the middle position. Denote the number of middle positions at odd anchor points by μ . In this way N^- consists of opposite faces at at odd anchor points in middle position, see Figure 6. Therefore

$$\text{HC}_k([q_D] \text{ rel } \#_\ell Q_D) = H_k(N, N^-) = \begin{cases} \mathbb{Z}_2 & k = \mu \\ 0 & k \neq \mu, \end{cases}$$

and the Euler-Floer characterisc is given by

$$\chi(x \text{ rel } \#_\ell y) = (-1)^\mu \neq 0.$$

Let $X(x, t)$ be a vector field for which y is a skeleton of closed integral curves, then $\#_\ell y$ is a skeleton for the vector field $X^\ell(x, t) := \ell X(x, \ell t)$. From Theorem 2 we derive that there exists a closed integral curve in each of the $3^\ell - 2$ proper relative classes $[x] \text{ rel } y$ described above. For the original vector field X this yields $3^\ell - 2$ distinct closed integral curves. Using the arguments in [13] one can find a compact invariant set for X with positive topological entropy, which proves that the associated flow is ‘chaotic’ whenever y is a skeleton of given integral curves

11.3 Example

So far we have not addressed the question whether the closed integral curves $x \text{ rel } y$ are non-trivial, i.e. not equilibrium points of X . The theory can also be extended in order to find non-trivial closed integral curves. This paper restricts to relative braids

where x consists of just one strand. Braid Floer homology for relative braids with x consisting of n strands is defined in [12]. To illustrate the importance of multi-strand braids we consider the discrete braid class in Figure 7.

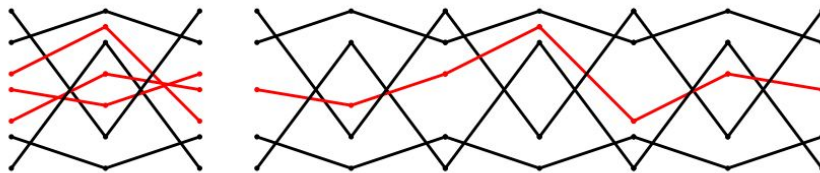


Fig. 7 A discretization of a braid class with a 3-fold concatenation of the skeleton y . The number of odd anchor points in middle position is $\mu = 2$ [right]. If we represent all translates of x we obtain a proper relative braid class where x is a 3-strand braid [left]. The latter provides additional linking information.

The braid class depicted in Figure 7[right] is discussed in the previous example and the Euler-Floer characteristic is equal to 1. By considering all translates of x on the circle $\mathbb{R}/\mathbb{Z}$, we obtain the braids in Figure 7[left]. The latter braid class is proper and encodes extra information about q_D relative to Q_D . The braid class fiber is a 6-dimensional cube with the same Conley index as the braid class in Figure 7[right]. Therefore,

$$\chi(q_D \text{ rel } Q_D) = (-1)^2 = 1.$$

As in the 1-strand case, the discrete Euler characteristic can be used to compute the associated Euler-Floer characteristic of $x \text{ rel } y$ and $\chi(x \text{ rel } y) = 1$. The skeleton y thus forces solutions $x \text{ rel } y$ of the above described type. The additional information we obtain this way is that for braid classes $[x \text{ rel } y]$, the associated closed integral curves for X cannot be constant and therefore represent non-trivial closed integral curves.

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