

RABINOWITZ FLOER HOMOLOGY FOR TENTACULAR HAMILTONIANS

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ABSTRACT. This paper extends the definition of Rabinowitz Floer homology to non-compact hypersurfaces. We present a general framework for the construction of Rabinowitz Floer homology in the non-compact setting under suitable compactness assumptions on the periodic orbits and the moduli spaces of Floer trajectories. We introduce a class of hypersurfaces arising as the level sets of specific Hamiltonians: strongly tentacular Hamiltonians for which the compactness conditions are satisfied, cf. [19], thus enabling us to define the Rabinowitz Floer homology for this class. Rabinowitz Floer homology in turn serves as a tool to address the Weinstein conjecture and establish existence of closed characteristics for non-compact contact manifolds.

1. INTRODUCTION

Rabinowitz Floer homology was defined in [6] as an invariant of exact contact hypersurfaces in exact convex symplectic manifolds. This paper presents the first result in extending the definition of Rabinowitz Floer homology to include non-compact hypersurfaces. It is the homology of a complex generated by the critical points of the Rabinowitz action functional and, by construction, it reduces to the singular homology of the hypersurface if the latter does not carry any closed characteristic. Because of this property it is a suitable tool for studying the question of existence of closed characteristics of contact type hypersurfaces, a question known as the (non-compact) Weinstein Conjecture. The Weinstein conjecture for compact contact manifolds has a rich history in symplectic topology: it has been the driving force behind many important developments in symplectic topology, most notably the study of pseudo-holomorphic curves in symplectizations of contact manifolds. The majority of the literature on the Weinstein Conjecture is devoted to the case of compact hypersurfaces. The approach in this paper applies to both compact and non-compact hypersurfaces. In the case of non-compact hypersurfaces the problem is much harder and additional topological and geometrical conditions are needed as the partial results in [3] suggest. The goal of the current paper is to show that Rabinowitz Floer homology can be defined for a class of non-compact hypersurfaces in the standard symplectic space $(\mathbb{R}^{2n}, \omega_0)$. Some of the results in this work are presented in a more general setting so as to serve as a first step towards a more general Floer theoretic treatment of the non-compact Weinstein conjecture. The present work builds on previous results concerning L^∞ -bounds for moduli spaces of Floer trajectories corresponding to a class of Hamiltonians with non-compact regular level sets, cf.

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[19]. In [19] such Hamiltonians are referred to as *tentacular Hamiltonians*: in fact, in order for the Rabinowitz Floer homology to be well-defined, we need to restrict to Hamiltonians in a subclass, which will be referred to *strongly tentacular Hamiltonians* and which will be defined below.

A vector field X is called a Liouville vector field if $d\iota_X\omega_0 = \omega_0$ and is called *asymptotically regular* if $\|DX(x)\| \leq c$. The asymptotically regular Liouville vector fields on $\mathbb{R}^{2n}$ are denoted by $\mathcal{L}(\mathbb{R}^{2n})$. The Poisson bracket of two functions F and H is defined as $\{F, H\} := \omega(X_F, X_H)$, where X_F, X_H are Hamiltonian vector fields associated to H and F respectively, i.e. $dH = \omega(\cdot, X_H)$ and $dF = \omega(\cdot, X_F)$. A continuous function F is called *coercive* if all its sublevel sets $F^{-1}((-\infty, a])$, $a \in \mathbb{R}$ are compact.

Definition 1.1. A (smooth) Hamiltonian $H: \mathbb{R}^{2n} \rightarrow \mathbb{R}$ is called *strongly tentacular* if the following axioms are satisfied:

- (h1) there exists a Liouville vector field $X^\dagger \in \mathcal{L}(\mathbb{R}^{2n})$ such that $dH(X^\dagger)(x) \geq c|x|^2 - c'$, for all $x \in \mathbb{R}^{2n}$;
- (h2) (sub-quadratic growth) $\sup_{x \in \mathbb{R}^{2n}} \|D^3H(x)\| \cdot |x| < \infty$;
- (h3) (contact type) there exists a Liouville vector field $X^\ddagger \in \mathcal{L}(\mathbb{R}^{2n})$ such that $dH(X^\ddagger)(x) > 0$, for all $x \in H^{-1}(0)$;
- (h4) in the neighborhood of $H^{-1}(0)$ there exists a coercive function F , such that for all $x \in H^{-1}(0)$ either $\{H, F\}(x) \neq 0$ or $\{H, \{H, F\}\}(x) > 0$, as $|x| \rightarrow \infty$.

Remark 1.2. Hamiltonians for which Condition (h4) is relaxed by omitting the coercivity condition on F are called *tentacular*. This class is used in [19] to establish L^∞ -bounds. Hamiltonians which satisfies axioms (h1)-(h3) are called *admissible*.

Remark 1.3. Strongly tentacular Hamiltonians include a variety of quadratic functions with hyperboloids as their 0-level sets. Their compact perturbations are also strongly tentacular provided the contact type property is preserved.

Conditions (h1)-(h3) in the definition of tentacular Hamiltonians, together with suitable a priori bounds on the set of non-degenerate periodic orbits contained in a fixed action window guarantee the appropriate bounds for the Rabinowitz Floer equations, based on the estimates in [19]. In order to guarantee that the action functional is of Morse-Bott type, condition (h4) is employed to ensure a priori bounds on all the non-degenerate periodic orbits (also under perturbations). Rabinowitz Floer homology can be defined generically for tentacular Hamiltonians. For strongly tentacular Hamiltonian we show that the definition is stable under suitable perturbations needed to define Rabinowitz Floer homology.

The main result of this paper is the definition of Rabinowitz Floer homology for strongly tentacular Hamiltonians:

Theorem 1.4. *Rabinowitz Floer homology of strongly tentacular Hamiltonians is well-defined. Moreover, if $\{H_s\}$ is a one-parameter family of tentacular Hamiltonians in the affine space of compactly supported perturbations of a given Hamiltonian H , then Rabinowitz Floer homology is constant along $\{H_s\}$.*

The set of strongly tentacular Hamiltonians is not invariant under the action of the group of symplectomorphisms, and yet the conditions sufficient to define Rabinowitz Floer homology for non-compact hypersurfaces are invariant under symplectomorphisms. As a result, we can extend the definition of Rabinowitz Floer homology to the image of the set of the strongly tentacular Hamiltonians under the action of the group of symplectomorphisms:

Corollary 1.5. *For every strongly tentacular Hamiltonian H and every symplectomorphism φ , the Rabinowitz Floer homology of $H \circ \varphi$ is well-defined and isomorphic to the Rabinowitz Floer homology of H .*

Our construction ensures that the invariant we define still has the following property: in the absence of periodic orbits of the Hamiltonian vector field on $\Sigma := H^{-1}(0)$, Rabinowitz Floer homology is isomorphic to the Morse (or singular) homology of Σ . This property, combined with the invariance under small compactly supported perturbations, makes it into a useful tool to detect periodic orbits.

Corollary 1.6. *Let H be a strongly tentacular Hamiltonian and suppose that its Rabinowitz Floer homology is not isomorphic to the singular homology of the level set $\Sigma := H^{-1}(0)$, then Σ carries periodic orbits of the Hamiltonian vector field X_H .*

In order to actually apply this result to the (non-compact) Weinstein conjecture, one needs to be able to compute Rabinowitz Floer homology: this is a highly non-trivial task, even in simple examples. In the compact case one can sometime argue that the Rabinowitz Floer homology vanishes by relying on the concept of *displaceability*: if a hypersurface is displaceable, then its Rabinowitz Floer homology necessarily vanishes. Unfortunately, in the non-compact case, we do not have the notion of displaceability at our disposal, so we need to be able to carry out explicit computations, involving a careful study of Conley-Zehnder indices and connecting flow lines with cascades. A powerful tool for computing Floer homologies in various contexts, such as regular Floer homology, symplectic homology, contact homology, etc. is to have a suitable continuation principle for the homology theory at hand. In this paper we prove that some special homotopies induce isomorphisms of the associated Rabinowitz Floer homology groups, and we use this result to show that the homology is well-defined, cf. Section 7. We strongly believe that Rabinowitz Floer homology allows a more global continuation principle (i.e., generic homotopies should induce isomorphisms of the homology groups) and that this continuation principle can be used to establish the homology in various cases, such as the quadratic tentacular hypersurfaces and their perturbations. The extension of the continuation principle will be a subject of future research.

The paper is organized as follows: in Section 2 we describe our setting and recall some definitions. In Sections 3 to 7 we discuss the general framework of the definition of non-compact Rabinowitz Floer homology. More precisely, we consider a pair (H, J) consisting of a Hamiltonian function and a compatible almost complex structure on an arbitrary exact symplectic manifold with the following properties: H satisfies the Morse-Bott property and property (PO) in [19] and the moduli spaces of Floer trajectories between different components of the critical set are bounded. In this setting we show that the Rabinowitz

Floer homology $RFH(H, J)$ is well defined. If uniform bounds on the moduli spaces of the perturbed Floer trajectories can be obtained, then the Rabinowitz Floer homology is independent of the choice of almost complex structure and invariant under sufficiently small compactly supported perturbations of H .

In the last two sections of the paper we concentrate on strongly tentacular Hamiltonians. In Section 8 we present the proof of Theorem 1.4. Using the bounds obtained in [19] and given that the Morse-Bott condition is generic in a suitable neighborhood of the affine space of compactly supported perturbations of a given strongly tentacular Hamiltonians we define Rabinowitz Floer homology for every Hamiltonian in this class.

Finally, in Section 9, we use the Hörmander's classification of quadratic forms [16] to show that a large number of quadratic Hamiltonians are strongly tentacular and thus the associated Rabinowitz Floer homology is well defined. In analogy with the case of positive definite quadratic forms and symplectic ellipsoids, we also introduce the notion of *symplectic hyperboloid* as a sublevelset of a Hamiltonian coming from a quadratic form with at least one positive and one negative eigenvalue. Our hope is that Rabinowitz Floer homology can eventually be applied to obtain a classification of symplectic hyperboloids, similar to the classification of symplectic ellipsoids obtained as an application of symplectic homology in [13].

2. SETTING AND DEFINITIONS

We recall the definition of Rabinowitz Floer homology in the setting of hypersurfaces in an exact symplectic manifold $(M, \omega = d\lambda)$. In the last part of the paper we will specialize to the case of the standard symplectic manifold $(\mathbb{R}^{2n}, \omega)$. For a (smooth) Hamiltonian function $H: M \rightarrow \mathbb{R}$ we define the Hamiltonian vector field X_H by the relation $\iota_{X_H}\omega = -dH$. Assume without loss of generality that 0 is a regular value for H and denote by $\Sigma = H^{-1}(0)$ the regular hypersurface at level 0. Periodic solutions of the Hamilton equations $\dot{x} = X_H(x)$ restricted to the hypersurface Σ are called closed characteristics. The objective is to find geometric and topological conditions on Σ and/or H that guarantee the existence of closed characteristics. Central in this approach is the variational formulation of the problem. Any periodic solution of the Hamilton equations is a critical point of the Hamilton action functional. In the variational formulation one can either fix the period η , in which case the energy of the Hamiltonian is undetermined, or one can fix the energy, in which case variations in the period are needed. In order to have an appropriate variational principle, we scale the period such that all periodic solutions are defined as mappings $\mathbb{R}/\mathbb{Z} \rightarrow M$, that is, we define $v(t) = x(\eta t)$. For the Hamilton equations this yields $\partial_t v = \eta \dot{x} = \eta X_H(x) = \eta X_H(v)$. The Hamilton action $\mathcal{A}^H$ is defined on the variables $(v, \eta) \in C^\infty(\mathbb{R}/\mathbb{Z}; M) \times \mathbb{R}$ and is given by

$$\mathcal{A}^H(v, \eta) = \int_0^1 \lambda(\partial_t v) - \eta \int_0^1 H(v),$$

where λ is a primitive of ω and is called a Liouville form. Critical points of $\mathcal{A}^H$ with respect to variations in both v and η force closed characteristics v to lie on the hypersurface $\Sigma = H^{-1}(0)$ — energy $H(v) = 0$. The set of critical points of $\mathcal{A}^H$ is denoted by $\text{Crit}(\mathcal{A}^H)$. Other energy values e can be chosen by replacing the Hamiltonian H by $H - e$. We allow η to be positive and negative. If (v, η) is a critical point of $\mathcal{A}^H$, then $(v, -\eta)$ is also a

critical point. The pairs $(v, -\eta)$ and (v, η) have opposite orientation. Pairs of the form $(v, 0)$ correspond to constant loops and are points on $\Sigma = H^{-1}(0)$. The Hamilton equations for a non-trivial closed characteristic are $\partial_t v = \eta X_H(v)$, $\eta \neq 0$ and $H(v(t)) = 0$. Via $v(t) := x(\sigma)$ we have $\dot{x} = X_H(x)$ where $\sigma = \eta t$, $\dot{x}$ the derivative with respect to σ and η is the period. The Rabinowitz action for pairs $u = (v, \eta)$ records every periodic orbit infinitely many times, i.e. given x a non-trivial closed characteristic with period $\tau > 0$, then

$$u_k = (v_k, \eta_k), \quad v_k(t) = x(\eta_k t), \quad \eta_k = k\tau, k \in \mathbb{Z},$$

are critical points on $\mathcal{A}^H$ which correspond to traversing the give closed characteristics k times. If k is negative then an orbit is traversed in the opposite direction (opposite orientation) and the action has the opposite sign.

We study critical points of $\mathcal{A}^H$ by considering the appropriate positive gradient flow equation: $\partial_s u = \nabla \mathcal{A}^H(u)$, where ∇ is a gradient with respect to a Riemannian structure and $u = (v, \eta)$. In order to do so we now define natural Riemannian structures. An almost complex structure compatible with ω is a smooth field of complex structures on the tangent bundle TM , i.e. for every $x \in M$, $J(x): T_x M \rightarrow T_x M$ is a linear mapping such that $J(x)^2 = -\text{id}$ and the assignment $g(\cdot, \cdot) := \omega(\cdot, J(x)\cdot)$ defines a Riemannian metric on M . Let $\mathcal{J}(M, \omega)$ be the set of ω -compatible, almost complex structures on M . Then, $\mathcal{J}(M, \omega)$ is contractible as shown in in [7, Prop. 13.1]. Let

$$(t, \eta) \mapsto J(\cdot, \eta, t) \in \mathcal{J}(M, \omega),$$

be a smooth function such that outside an open set $V \subseteq M \times \mathbb{R}$, J is equal to a fixed, ω -compatible, almost complex structure J_0 . In addition we assume that

$$\sup_{(t, \eta) \in \mathbb{R}/\mathbb{Z} \times \mathbb{R}} \|J(\cdot, \eta, t)\|_{C^k} < +\infty, \quad \forall k \in \mathbb{N}.$$

The set of such functions of ω -compatible almost complex structures we denote by $\mathcal{J}^\infty(M, \omega, V)$. Note that for every open set $V \subseteq M \times \mathbb{R}$ the corresponding set $\mathcal{J}^\infty(M, \omega, V)$ is also contractible. For $V = M \times \mathbb{R}$ the latter is denoted by $\mathcal{J}^\infty(M, \omega)$. Each $J \in \mathcal{J}^\infty(M, \omega)$ induces a metric g on $C^\infty(\mathbb{R}/\mathbb{Z}; M) \times \mathbb{R}$ defined as follows. For every $(\xi, \sigma), (\xi', \sigma') \in T_{(v, \eta)} C^\infty(\mathbb{R}/\mathbb{Z}; M) \times \mathbb{R} = C^\infty(\mathbb{R}/\mathbb{Z}; v^* TM) \times \mathbb{R}$ we define the metric g by

$$g_{(v, \eta)}((\xi, \sigma), (\xi', \sigma')) = \int_0^1 \omega(\xi(t), J(v(t), \eta, t) \xi'(t)) + \sigma \sigma'. \quad (2.1)$$

The gradient of $\mathcal{A}^H$ with respect to the metric g will be denoted by $\nabla_J \mathcal{A}^H$ and is given by

$$\nabla_J \mathcal{A}^H(v, \eta) = \begin{pmatrix} -J(v, \eta, t) [\partial_t v - \eta X_H(v)] \\ -\int_0^1 H(v) \end{pmatrix}. \quad (2.2)$$

The gradient flow equations $\partial_s u = \nabla_J \mathcal{A}^H(u)$ are known as the Rabinowitz Floer equations, and we study solutions of these equations, according to the following definition.

Definition 2.1. Let $J \in \mathcal{J}^\infty(M, \omega)$. A Floer trajectory is a solution $u \in C^\infty(\mathbb{R} \times \mathbb{R}/\mathbb{Z}; M \times \mathbb{R})$ that satisfies the Rabinowitz Floer equations

$$\partial_s u = \nabla_J \mathcal{A}^H(u) = \begin{pmatrix} -J(v, \eta, t) [\partial_t v - \eta X_H(v)] \\ -\int_0^1 H(v) \end{pmatrix}, \quad (2.3)$$

and for which

$$\int_{-\infty}^{\infty} \|\partial_s u\|_{L^2(\mathbb{R}/\mathbb{Z})}^2 ds < \infty. \quad (2.4)$$

For a pair of connected components Λ^-, Λ^+ of the critical set $\mathcal{A}^H$, the set of Floer trajectories with

$$\lim_{s \rightarrow \pm\infty} u(s) \in \Lambda^\pm$$

is called a moduli space of Floer trajectories and it is denoted by $\mathcal{M}(\Lambda^-, \Lambda^+)$.

In our study of the Rabinowitz Floer equations, and more precisely when discussing invariance or Rabinowitz Floer homology, it is necessary to include 1-parameter families of Hamiltonians and almost complex structures. We denote smooth 1-parameter families of Hamiltonians and almost complex structures by $\{H_s\}_{s \in \mathbb{R}}$ and $\{J_s\}_{s \in \mathbb{R}}$ respectively. Such families are referred to as homotopies. We will always assume our homotopies $\Gamma = \{(H_s, J_s)\}_{s \in \mathbb{R}}$ to be constant outside the interval $[0, 1]$, i.e.

$$(H_s, J_s) = \begin{cases} (H_0, J_0) & s \leq 0 \\ (H_1, J_1) & s \geq 1 \end{cases} \quad (2.5)$$

and compactly supported in the Hamiltonian component, i.e., $H_s - H_0 \in C_0^\infty(M; \mathbb{R})$. Define

$$\|H_s\|_{1,\infty} := \max_{\substack{x \in M \\ s \in [0,1]}} \|\partial_s H_s(x)\|, \quad \|J_s\|_\infty = \max_{\substack{x \in M, t \in \mathbb{R}/\mathbb{Z} \\ \eta \in \mathbb{R}, s \in [0,1]}} \|J_s(x, \eta, t)\|$$

The perturbed Rabinowitz Floer equations are now found by replacing H by H_s and J by J_s in Eqn. (2.3). Floer trajectories for the perturbed Rabinowitz Floer equations $\partial_s u = \nabla_{J_s} \mathcal{A}^{H_s}$ are solutions satisfying the integral bound in (2.4). The associated moduli spaces will be denoted by $\mathcal{M}_\Gamma(\Lambda^-, \Lambda^+)$ with Λ^- a connected component of $\text{Crit}(\mathcal{A}^{H_0})$ and Λ^+ of $\text{Crit}(\mathcal{A}^{H_1})$.

As mentioned in the introduction, in this paper we show that Rabinowitz Floer homology is well-defined for strongly tentacular Hamiltonians. We will show in Section 8 that if we restrict ourselves to asymptotically regular vector fields, then the conditions in Definition 1.1 are equivalent to the properties (H1)-(H4) introduced in [21]. One can immediately notice that hypothesis (H1), (H2) and (H4) are identical to (h1), (h2) and (h4) respectively.

3. MORSE-BOTT PROPERTY

Many of the results we prove in this paper, hold in fact under more general, albeit more abstract conditions, which are defined below. Roughly speaking, these conditions require the non-degenerate periodic orbits (all of them or the ones in a prescribed action window) to be confined to a compact set. We will use the expression *uniformly continuous* in order to indicate that these properties persist under compact perturbations.

Definition 3.1. Let H be a Hamiltonian on an exact, symplectic manifold (M, ω) , having 0 as a regular value. The following conditions concern the non-degenerate periodic orbits of H and certain families of perturbations of H .

- (PO) We say that H satisfies property (PO) if for any fixed action window, all the non-degenerate periodic orbits are contained in a compact subset of M . Moreover, we say that property (PO) is *uniformly continuous* at H if there exists an open

neighborhood $B(H)$ of 0 in $C_c^\infty(M)$ and an exhaustion $\{K_n\}_{n \in \mathbb{N}}$ of M by compact sets K_n , such that for every $n \in \mathbb{N}$ and every $h \in B(K_n) = B(H) \cap C_0^\infty(K_n)$, whenever

$$(v, \eta) \in \text{Crit}(\mathcal{A}^{H+h}) \quad \text{and} \quad 0 < |\mathcal{A}^{H+h}(v, \eta)| \leq n$$

then $v(S^1) \subseteq K_n$.

(PO+) We say that H satisfies property (PO+) if all the non-degenerate periodic orbits of the Hamiltonian flow of H are contained in a compact subset. Moreover, we say that property (PO+) is *uniformly continuous* at H if there exists an open neighborhood $B(H)$ of 0 in $C_c^\infty(M)$ and an exhaustion $\{K_n\}_{n \in \mathbb{N}}$ of M by compact sets K_n , such that for every $n \in \mathbb{N}$ and every $h \in B(K_n) = B(H) \cap C_0^\infty(K_n)$, whenever

$$(v, \eta) \in \text{Crit}(\mathcal{A}^{H+h}) \quad \text{and} \quad |\mathcal{A}^{H+h}(v, \eta)| > 0,$$

then $v(S^1) \subseteq K_n$. In other words, all the nondegenerate periodic orbits of the perturbed Hamiltonian, are contained in K_n .

Condition (PO+) (and its persistence under perturbations) are necessary in this paper for the construction of the homology, but this depends on the specific construction. In a following paper we will show that (PO) and its uniform continuity are in fact sufficient for the definition of RFH, by constructing this homology as a (direct and inverse) limit of truncated homologies, taken over an increasing, nested family of intervals.

In order to be able to define Rabinowitz Floer for tentacular Hamiltonians, we will at first assume that the following Morse-Bott condition holds for H .

(MB) The associated Rabinowitz functional $\mathcal{A}^H$ is Morse-Bott and all the periodic orbits are of Morse-Bott type.

In fact we will prove below that whenever the (PO+) is uniformly continuous at the Hamiltonian H , the Morse-Bott assumption is generically satisfied in the affine space of compactly supported perturbations of H , and therefore not a restrictive condition. The Morse-Bott condition on the periodic orbits has the following meaning. Consider the projection

$$\begin{aligned} \pi_{\mathcal{A}^H} : \text{Crit}(\mathcal{A}^H) &\rightarrow H^{-1}(0), \\ (v, \eta) &\mapsto v(0). \end{aligned}$$

Definition 3.2. Let ϕ be the Hamiltonian flow on $H^{-1}(0)$. We say that the closed orbits of the Hamiltonian flow are of *Morse-Bott type* if η is constant on every connected component $\Lambda \subseteq \text{Crit}(\mathcal{A}^H)$, the image of Λ under the projection

$$\mathcal{P}^\Lambda := \pi_{\mathcal{A}^H}(\Lambda)$$

is a closed submanifold of $H^{-1}(0)$ and for all $p \in \mathcal{P}^\Lambda$

$$T_p \mathcal{P}^\Lambda = \text{Ker}(D_p \phi^\eta - \text{Id}) \subseteq T_p H^{-1}(0).$$

We will first show how these two conditions are related by proving the following lemma:

Lemma 3.3. *If $\mathcal{A}^H$ is Morse-Bott and η is constant on every connected component of $\text{Crit}(\mathcal{A}^H)$, then all the closed orbits of the Hamiltonian flow ϕ on $H^{-1}(0)$ are of Morse-Bott type.*

Observe that the assumption that η is constant on every connected component of $\text{Crit}(\mathcal{A}^H)$ is not very restrictive. In particular it is satisfied whenever the critical set consists of a union of disjoint circles and the hypersurface $H^{-1}(0) \times \{0\}$, or when we assume H to be a defining Hamiltonian: in our setting the first condition is satisfied generically, whereas [6] and [10] work with defining Hamiltonians.

Proof. Fix a connected component $\Lambda \subseteq \text{Crit}(\mathcal{A}^H)$ and let $\mathcal{P}^\Lambda$ be the corresponding projection on $H^{-1}(0)$. By assumption η is constant on Λ . We want to prove that for all $p \in \mathcal{P}^\Lambda$

$$T_p \mathcal{P}^\Lambda = \text{Ker}(D_p \phi^\eta - \text{Id}).$$

Part 1. ($\subseteq$)

Take $\mathbf{v} \in T_p \mathcal{P}^\Lambda$. Then there exists a path $p(s) \in \mathcal{P}^\Lambda, p(0) = p$, such that

$$\mathbf{v} = \left. \frac{d}{ds} \right|_{s=0} p(s).$$

Since η is constant on Λ , we have

$$(\phi^{\eta t}(p(s)), \eta) \in \Lambda, \quad \text{and} \quad \phi^{\eta t}(p(s)) \in \mathcal{P}^\Lambda \quad \forall t \in S^1.$$

Define

$$\xi(t) := \left. \frac{d}{ds} \right|_{s=0} \phi^{\eta t}(p(s)) = D_p \phi^{\eta t}(\mathbf{v}).$$

Then

$$\xi(1) = \left. \frac{d}{ds} \right|_{s=0} \phi^\eta(p(s)) = \left. \frac{d}{ds} \right|_{s=0} p(s) = \xi(0).$$

Therefore

$$\mathbf{v} = \xi(0) = \xi(1) = D_p \phi^\eta(\mathbf{v}),$$

which proves the first inclusion.

Part 2. ($\supseteq$)

Now we want to prove the second inclusion. Take $\mathbf{v} \in \text{Ker}(D_p \phi^{\eta t} - \text{Id}) \subseteq T_p H^{-1}(0)$ and define

$$\xi(t) := D_p \phi^{\eta t}(\mathbf{v}).$$

Then

$$\xi(1) = D_p \phi^\eta(\mathbf{v}) = \mathbf{v} = \xi(0).$$

Since, by assumption,

$$(v, \eta) := (\phi^{\eta t}(p), \eta) \in \Lambda,$$

it follows that

$$(\xi, 0) \in T_{(v, \eta)}(C^\infty(S^1, M) \times \mathbb{R}).$$

Now we would like to calculate $\nabla^2 \mathcal{A}^H_{(v,\eta)}(\xi, 0)$. First observe that since $\mathbf{v} \in T_p H^{-1}(0)$, we have

$$\int dH_{v(t)}(D_p \phi^{\eta t}(\mathbf{v})) = \int dH_p(\mathbf{v}) = 0.$$

On the other hand, note that we can express the first term of $\nabla^2 \mathcal{A}^H_{(v,\eta)}(\xi, 0)$ using the Lie bracket and Lie derivative as follows:

$$\begin{aligned} \partial_t \xi - \eta J_0 \text{Hess}_v H(\xi) &= \nabla_{\partial_t v} \xi - \eta \nabla_\xi X_H = \nabla_{\eta X_H} \xi - \eta \nabla_\xi X_H = [\eta X_H, \xi] \\ &= \mathcal{L}_{\eta X_H} \xi = \frac{d}{ds} \Big|_{s=0} D\phi^{-\eta s} \xi(t+s) = \frac{d}{ds} \Big|_{s=0} D\phi^{-\eta s} D_p \phi^{\eta(t+s)}(\mathbf{v}) \\ &= \frac{d}{dt} \Big|_{s=0} D_p \phi^{\eta t}(\mathbf{v}) = 0. \end{aligned}$$

This proves that

$$(\xi, 0) \in \text{Ker}(\nabla^2 \mathcal{A}^H_{(v,\eta)}) = T_{(v,\eta)} \Lambda,$$

in view of the Morse-Bott property of the action functional. Hence $\xi(0) = \mathbf{v} \in T_p \mathcal{P}^\Lambda$, which proves the second inclusion. $\square$

If H be a Hamiltonian on an exact, symplectic manifold (M, ω) , having 0 as a regular value, the corresponding Rabinowitz action functional will always be Morse-Bott along the component of the critical set consisting of constant loops, but it may in general not be Morse-Bott along the other components. Nevertheless, one can prove that the Morse-Bott property is generic under certain assumptions. The present proof is inspired by Appendix B of [6], which applies to compactly supported Hamiltonians. However, since we consider possibly non-compact energy level sets, we will restrict ourselves to compact perturbations $h \in C_0^\infty(M)$ and Hamiltonians such that uniform continuity of (PO+) is satisfied.

Theorem 3.4. *Let (M, ω) be a symplectic manifold and $H : M \rightarrow \mathbb{R}$ a smooth Hamiltonian, such that 0 is a regular value of H and (PO+) is uniformly continuous at H . Then there exists an open neighborhood $B(H)$ of 0 in $C_c^\infty(M)$, such that the set*

$$A(H) := \{h \in B(H) \mid \mathcal{A}^{H+h} \text{ satisfies (MB)}\} \quad (3.1)$$

is comeager in $(B(H), \|\cdot\|_{C^\infty})$. Moreover, if $H^{-1}(0)$ is of contact type, then we can assume that for all $h \in A(H)$ the set of critical values of $\mathcal{A}^{H+h}$ is closed and discrete.

The first part of the proof is just an adaptation of Theorem B.1 in [6] to the non-compact setting. In order to prove the second part, though, we will first have to prove a couple of lemmas. The proof of Theorem 3.4 can be found at the end of this section.

The first lemma shows how the (PO) and (PO+) conditions guarantee that the value of η on the non-degenerate components of $\text{Crit}(\mathcal{A}^H)$ is (uniformly) bounded away from 0. This will be a crucial step in proving that the set of critical values of $\mathcal{A}^H$ is discrete and, later on, in the construction of a suitable set of perturbations when we prove transversality for tentacular Hamiltonians.

Lemma 3.5. *Let (M, ω) be an exact symplectic manifold and $H : M \rightarrow \mathbb{R}$ a Hamiltonian. Assume 0 is a regular value of H and denote the hypersurface $\Sigma := H^{-1}(0)$. For all $n \in \mathbb{N}$*

denote

$$\eta(H)_\infty^n := \inf\{|\eta| \mid (v, \eta) \in \text{Crit}(\mathcal{A}^H) \cap (\mathcal{A}^H)^{-1}([-n, 0) \cup (0, n])\}.$$

Then

- (1) Whenever H satisfies condition (PO), then for all $n \in \mathbb{N}$, $\eta(H)_\infty^n > 0$.
- (2) Whenever H satisfies condition (PO+), then $\inf_{n \in \mathbb{N}} \eta(H)_\infty^n > 0$.

Proof. Fix $n \in \mathbb{N}$ and suppose that $\eta(H)_\infty^n = 0$. That means that there exists a sequence

$$(v_k, \eta_k) \in \text{Crit}(\mathcal{A}^H) \cap (\mathcal{A}^H)^{-1}([-n, 0) \cup (0, n]),$$

such that $\lim_{k \rightarrow \infty} \eta_k = 0$.

On the other hand, by property (PO) we have that there exists a compact set $K \subseteq \Sigma$, such that for all $k \in \mathbb{N}$, we have $v_k(S^1) \subseteq K$ and hence

$$\|\partial_t v_k(t)\|_{L^\infty(S^1)} \leq |\eta_k| \sup_K \|X_H\|.$$

Therefore, by the Arzelà-Ascoli theorem, there exists a convergent subsequence, which we will denote in the same way:

$$\lim_{k \rightarrow \infty} (v_k, \eta_k) = (v, \eta).$$

By continuity, $\eta = 0$ and $\partial_t v = 0$. Moreover, by continuity of $\nabla \mathcal{A}^H$, (v, η) is a critical point of $\mathcal{A}^H$. Therefore $(v, \eta) \in \Sigma \times \{0\}$. Yet, we this would contradict the fact that $\Sigma \times \{0\}$ is isolated in the $\text{Crit}(\mathcal{A}^H)$ due to the fact that for every Hamiltonian the corresponding action functional is Morse-Bott along $\Sigma \times \{0\}$ as proven in Theorem B.1, Step 4 in [6]. Indeed, by the arguments presented in the proof of Theorem 23. in [11] we can show that due to the Morse-Bott property for every $(v, 0) \in \Sigma \times \{0\}$ we can find a neighborhood of $(v, 0)$, which does not contain any critical points of $\mathcal{A}^H$ except those in $\Sigma \times \{0\}$. That gives us a contradiction and proves the first result.

To prove the second result we argue analogously just for a sequence $(v_k, \eta_k) \in \text{Crit}(\mathcal{A}^H) \setminus (\mathcal{A}^H)^{-1}(0)$ with $\lim_{k \rightarrow \infty} \eta_k = 0$. $\square$

The previous lemma shows that, under the (PO) assumption, η is bounded away from 0 on the nondegenerate components of $\text{Crit}(\mathcal{A}^H)$. Moreover, we will show below that $\text{CritVal}(\mathcal{A}^H)$ is closed and discrete.

Lemma 3.6. *Let $(M, \omega = d\lambda)$ be an exact symplectic manifold and $H : M \rightarrow \mathbb{R}$ a smooth Hamiltonian, such that*

- (1) $\Sigma := H^{-1}(0)$ is of contact type,
- (2) the corresponding action functional $\mathcal{A}^H$ is Morse-Bott,
- (3) or any fixed action window all the non-degenerate periodic orbits in in this action window are contained in a compact subset of Σ .

Then its set of critical values $\text{CritVal}(\mathcal{A}^H)$ is closed and discrete.

Proof. First we will prove that the set $\text{CritVal}(\mathcal{A}^H)$ is closed. Take a sequence $a_k \in \text{CritVal}(\mathcal{A}^H)$, such that $\lim_{k \rightarrow \infty} a_k = a$. Without loss of generality we can assume that $a_k \neq 0$. Therefore, there exist

$$(v_k, \eta_k) \in \text{Crit}(\mathcal{A}^H) \cap (\mathcal{A}^H)^{-1}([\inf_{k \in \mathbb{N}} a_k, \sup_{k \in \mathbb{N}} a_k] \setminus \{0\}), \quad \text{such that} \quad \mathcal{A}^H(v_k, \eta_k) = a_k.$$

In particular, by property (PO), there exists a compact subset $K \subseteq M$, such that for all $k \in \mathbb{N}$ we have $v_k(S^1) \subseteq K$. Since $H^{-1}(0)$ is of contact type, there exists a Liouville vector field Y for which

$$a_n = \mathcal{A}^H(v_n, \eta_n) = \eta_n \int dH_{v_n}(Y) \\ |a_n| \geq |\eta_n| \inf_K dH(Y).$$

That means that there is a convergent subsequence (which we will denote the same), such that

$$\lim_{k \rightarrow \infty} \eta_k = \eta.$$

Knowing that (v_k, η_k) are critical points, we get

$$v_k(t) \in K \quad \forall k \in \mathbb{N}, t \in S^1, \\ \|\partial_t v_k(t)\| \leq |\eta_k| \sup_K \|X_H\|,$$

we can use Arzelà-Ascoli theorem, to conclude that there exists a convergent subsequence

$$\lim_{k \rightarrow \infty} (v_k, \eta_k) = (v, \eta).$$

By the fact that $\mathcal{A}^H$ is C^2 , $(v, \eta) \in \text{Crit}(\mathcal{A}^H)$ and $\mathcal{A}^H(v, \eta) = a$. This proves that $\text{CritVal}(\mathcal{A}^H)$ is closed.

Suppose that 0 were an accumulation point in $\text{CritVal}(\mathcal{A}^H)$. Then by the argument above, we would have a sequence

$$\text{Crit}(\mathcal{A}^H) \cap (\mathcal{A}^H)^{-1}([\inf_{k \in \mathbb{N}} a_k, \sup_{k \in \mathbb{N}} a_k] \setminus \{0\}), \quad \text{such that} \quad \lim_{k \rightarrow \infty} \eta_k = 0.$$

But this would contradict the results of Lemma 3.5. Therefore 0 is not an accumulation point.

Now we would like to prove that $\text{CritVal}(\mathcal{A}^H) \setminus \{0\}$ has no accumulation points. We can apply the same argument presented in [11], Theorem 23, which is in fact the same argument we used in the proof of Lemma 3.5, to other connected components of $\text{Crit}(\mathcal{A}^H)$ to show that the Morse-Bott property of $\mathcal{A}^H$ guarantees that $\text{CritVal}(\mathcal{A}^H)$ is discrete and closed, even if $\text{Crit}(\mathcal{A}^H) \setminus (H^{-1}(0) \times \{0\})$ consists of manifolds of different dimension (as is the case in [11]).

Since all connected components of $\text{Crit}(\mathcal{A}^H)$ are isolated, therefore the corresponding critical values are also isolated. $\square$

We will conclude this section with the proof of Theorem 3.4.

Proof of Theorem 3.4: By uniform continuity of (PO+) at H , there exists an open neighborhood $B(H)$ of 0 in $C_c^\infty(M)$ and an exhaustion $\{K_n\}_{n \in \mathbb{N}}$ of M by compact sets K_n , such that for every $n \in \mathbb{N}$ and every $h \in B(K_n) = B(H) \cap C_0^\infty(K_n)$ all the nondegenerate periodic

orbits of the perturbed Hamiltonian $H + h$, are contained in K_n . By applying Theorem B.1 in [6], we can conclude that for every n the set

$$A(K_n) = \{h \in B(K_n) \mid \mathcal{A}^{H+h} \text{ is Morse-Bott}\}$$

is comeager in $B(K_n) \subseteq C_c^\infty(K_n)$. In order to prove genericity in $C_c^\infty(M)$, observe that $C_c^\infty(K_n)$ is Baire for every n and therefore $A(K_n)$ is dense in $B(K_n)$. This means that the union $\cup_{n \in \mathbb{N}} A(K_n)$ is dense in $B(M)$. In particular, for all $h \in \cup_{n \in \mathbb{N}} A(K_n)$ the following properties hold:

- (1) the Rabinowitz action functional $\mathcal{A}^{H+h}$ is Morse-Bott;
- (2) the critical set of $\mathcal{A}^{H+h}$ consists of the hypersurface $(H + h)^{-1}(0) \times \{0\}$ and a disjoint union of circles. In particular, η is constant on every connected component of $\text{Crit}(\mathcal{A}^{H+h})$.

By Lemma 3.3 the two conditions above imply that for all $h \in A(H)$, the non-degenerate periodic orbits of the Hamiltonian flow of X_{H+h} are of Morse-Bott type. In particular, $\cup_{n \in \mathbb{N}} A(K_n)$ is a subset of $A(H)$ and the claim follows.

Note that in that case $\text{Crit}(\mathcal{A}^{H+h}) \setminus (H + h)^{-1}(0) \times \{0\}$ is a 1-dimensional submanifold in $C^\infty(S^1, M)$. In particular all of its connected components are isolated, which implies that $a \in \text{CritVal}(\mathcal{A}^H)$, $a \neq 0$ cannot be an accumulation point of $\text{CritVal}(\mathcal{A}^H)$.

Now suppose that $H^{-1}(0)$ is of contact type. By possibly shrinking $B(H)$ we can assume that for every $h \in B(H)$ the associated hypersurface $(H + h)^{-1}(0)$ is of contact type. Then by Lemma 3.6 for every $h \in A(H)$ the corresponding set of critical values $\text{CritVal}(\mathcal{A}^{H+h})$ is closed and discrete. □

4. COMPACTNESS

In this section we will consider moduli spaces of flow lines with cascades, as defined in [14]. The setting is as follows: consider an exact, symplectic manifold $(M, \omega = d\lambda)$, a Hamiltonian function $H : M \rightarrow \mathbb{R}$, and let $\mathcal{A}^H$ be the associated Rabinowitz action functional. Fix a 2-parameter family of ω -compatible, almost complex structures $J = \{J_t\}_{t \in S^1} \in \mathcal{J}(M, \omega)$. Let (f, g) be a Morse-Smale pair on $\text{Crit}(\mathcal{A}^H)$. For a pair of critical points $p, q \in \text{Crit}(f)$, we will denote by $\overline{\mathcal{M}}_0(p, q)$ the set of flow lines with zero cascades from p to q and by $\overline{\mathcal{M}}_m(p, q)$ the set of flow lines with m cascades from p to q .

The group $\mathbb{R}$ acts by timeshift on the Morse trajectories of $\overline{\mathcal{M}}_0(p, q)$ and for $m \geq 1$ the group $\mathbb{R}^m$ acts on $\overline{\mathcal{M}}_m(p, q)$ by time shift on each cascade: we denote the quotients by $\mathcal{M}_0(p, q)$ and $\mathcal{M}_m(p, q)$, respectively, and define the *set of flow lines with cascades from p to q* by

$$\mathcal{M}(p, q) := \bigcup_{m \in \mathbb{N}_0} \mathcal{M}_m(p, q). \quad (4.1)$$

Since the action is increasing along the cascades, the following properties of the moduli spaces easily follow:

- if $\mathcal{A}^H(p) > \mathcal{A}^H(q)$, then $\mathcal{M}(p, q) = \emptyset$,
- if $\mathcal{A}^H(p) = \mathcal{A}^H(q)$, then $\mathcal{M}(p, q) = \mathcal{M}_0(p, q)$,
- if $\mathcal{A}^H(p) < \mathcal{A}^H(q)$, then $\mathcal{M}_0(p, q) = \emptyset$.

In order to prove compactness of the moduli spaces of flow lines with cascades, we will need compactness both of the Morse and Floer trajectories forming the cascades. One result which we will repeatedly need below is Gromov compactness of the Floer trajectories, that is, convergence of the Floer trajectories in the C_{loc}^∞ sense. Thanks to the bounds on the moduli spaces of Floer trajectories, the analogue of Proposition 3b in [12] holds, as one can prove by standard arguments (cf. for instance, Theorem 6.5.4 in [2]), after a suitable adaptation to the Rabinowitz Floer setting. The proof is even slightly simplified by the fact that our ambient symplectic manifold is exact, therefore bubbling and "cusps" can be excluded a priori.

4.1. The "shade" of an action window. In order to prove that the moduli spaces of cascades can be compactified, we will need the following result on moduli spaces of Floer trajectories. The setting is as follows: we consider an exact symplectic manifold (M, ω) , a Hamiltonian H satisfying the assumptions of Lemma 3.6 and an ω -compatible almost complex structure J , such that for every fixed action window, the moduli spaces of Floer trajectories between components within this action window are bounded. Let ev^- and ev^+ be the evaluation maps defined by

$$ev^-(u) := \lim_{s \rightarrow -\infty} u(s, t) \quad \text{and} \quad ev^+(u) := \lim_{s \rightarrow +\infty} u(s, t) \quad (4.2)$$

For a fixed value $b \in \mathbb{R}$, $b > 0$, define

$$\tilde{K}(b) := \left(\bigcup_{\substack{\Lambda \subseteq \text{Crit}(\mathcal{A}^H), \\ \mathcal{A}^H(\Lambda) \in (0, b]}} ev^-(\mathcal{M}(\Sigma, \Lambda)) \right).$$

This set can consists of limit points of trajectories between connected components of the critical set in the action window $(0, b]$ and the hypersurface Σ and can be thought of as the *shade* of moduli spaces with action within 0 and b on Σ .

Lemma 4.1. *Whenever a Hamiltonian H satisfies the assumptions of Lemma 3.6, for every $b > 0$ the corresponding set $\tilde{K}(b)$ is a compact subset of $\Sigma := H^{-1}(0) \times \{0\}$. In particular, it follows that for any coercive function f on Σ the set*

$$K(b) := f^{-1}((-\infty, \max_{\tilde{K}(b)} f]) \quad (4.3)$$

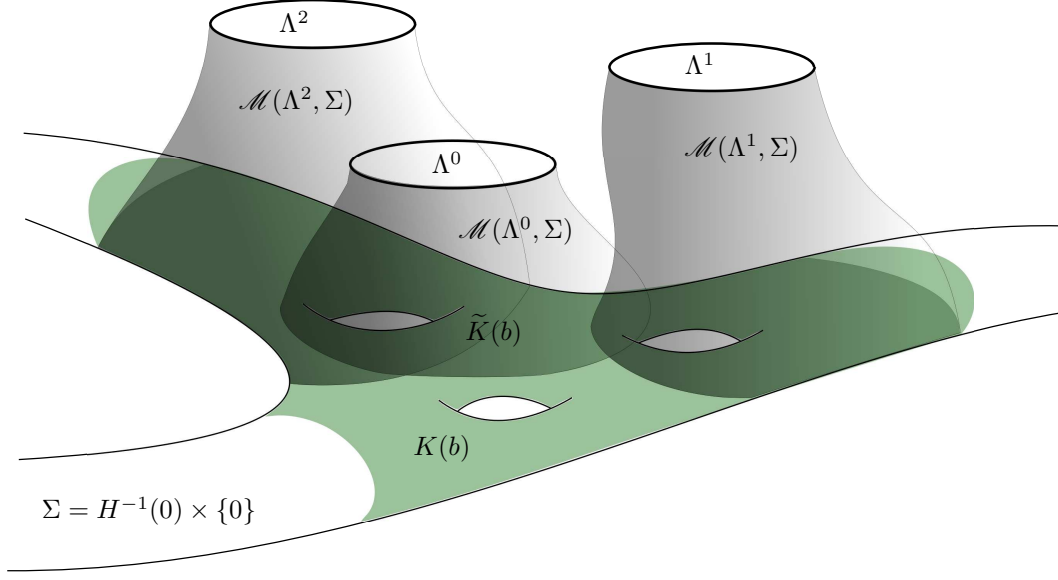
is also a compact subset of Σ .

In other words the set $\tilde{K}(b)$ is obtained by evaluating the endpoints of moduli spaces with action between 0 and b onto hypersurface $\Sigma = H^{-1}(0) \times \{0\}$, whereas the set $K(b)$ is the minimal sublevel set of f containing $\tilde{K}(b)$: see Figure 1.

Proof. Let $\{x_k\}_{k \in \mathbb{N}}$ be a sequence in $\tilde{K}(b)$. By assumption the Hamiltonian H satisfies the conditions of Lemma 3.6, hence the number of connected components of

$$\text{Crit}(\mathcal{A}^H) \cap (\mathcal{A}^H)^{-1}((0, b])$$

is finite. Therefore, without loss of generality we can assume that there exists a connected component $\Lambda \subseteq \text{Crit}(\mathcal{A}^H) \cap (\mathcal{A}^H)^{-1}((0, b])$ and a sequence $\{u_k\}_{k \in \mathbb{N}}$ corresponding to

FIGURE 1. Construction of the sets $\tilde{K}(b)$ and $K(b)$.

$\{x_k\}_{k \in \mathbb{N}}$, such that

$$u_k \in \mathcal{M}(\Lambda, \Sigma) \quad \text{and} \quad x_k = ev^-(u_k).$$

By assumption $\mathcal{M}(\Lambda, \Sigma)$ is contained in a compact subset of $M \times \mathbb{R}$, which only depends on b . Therefore we can apply standard compactness arguments (see for example Proposition 6.6.2 in [2]) to deduce that there exists another connected component $\tilde{\Lambda} \subseteq \text{Crit}(\mathcal{A}^H) \cap (\mathcal{A}^H)^{-1}((0, b])$ and a sequence $s_k \in \mathbb{R}$, such that for a subsequence

$$u_k(s_k + \cdot) \xrightarrow[k \rightarrow \infty]{C_{loc}^\infty} u \in \mathcal{M}(\tilde{\Lambda}, \Sigma).$$

Moreover, by the bounds on the moduli spaces of Floer trajectories, $ev^-(\mathcal{M}(\tilde{\Lambda}, \Sigma))$ is a pre-compact subset of Σ .

According to our assumptions, $\mathcal{A}^H$ is Morse-Bott and the associated moduli spaces are contained in a compact set of $M \times \mathbb{R}$, hence by Theorem 25 in [11], there exist $\alpha, \delta, \tilde{\alpha}, \tilde{\delta} > 0$ corresponding to $\Lambda, \tilde{\Lambda}$ respectively, such that equality (5.1) holds. As a result, for every $\varepsilon > 0$ we can choose $s_0 \in \mathbb{R}$ big enough to satisfy

$$\begin{aligned} \text{dist}(u_k(s_k + s_0, t), ev^-(u_k)) &\leq \alpha e^{-\delta s_0} < \frac{\varepsilon}{3} \quad \forall k \in \mathbb{N} \\ \text{dist}(u(s_0, t), ev^-(u)) &\leq \tilde{\alpha} e^{-\tilde{\delta} s_0} < \frac{\varepsilon}{3}. \end{aligned}$$

Now, since $u_k(s_k + \cdot) \xrightarrow[k \rightarrow \infty]{C_{loc}^\infty} u$, there exists $N \in \mathbb{N}$, such that for all $k \geq N$

$$\text{dist}(u_k(s_k + s_0, t), u(s_0, t)) < \frac{\varepsilon}{3}$$

Combining the above inequalities, we obtain that for every $\varepsilon > 0$ there exists $N \in \mathbb{N}$, such that for all $k \geq N$

$$\text{dist}(ev^-(u), ev^-(u_k)) < \varepsilon.$$

This ensures that $\{x_k\}_{k \in \mathbb{N}}$ has a converging subsequence and the limit is in fact in $\tilde{K}(b)$, which proves the first claim. The compactness of $K(b)$ follows directly from its definition and the coerciveness of f . $\square$

4.2. Compactness of the moduli spaces of cascades. We are now ready to prove that in the presence of bounds on the moduli spaces of Floer trajectories (as we obtained for tentacular Hamiltonians), the moduli spaces of cascades can be compactified by considering broken flow lines with cascades: these flow lines and the corresponding notion of convergence are defined in [14], section C.2.

Theorem 4.2. *Suppose the Hamiltonian H satisfies the assumptions of Lemma 3.6. Let (f, g) be a Morse-Smale pair on $\text{Crit}(\mathcal{A}^H)$ such that f is coercive on Σ , and let J be a compatible almost complex structure such that the moduli space of trajectories between any two connected component of the critical set of $\mathcal{A}^H$ is bounded. Fix $p, q \in \text{Crit}(f)$ and let $\{\mathbf{u}^n\}_{n \in \mathbb{N}}$ be a sequence of flow lines with cascades in the moduli space $\mathcal{M}(p, q)$. Then there exist a subsequence $\{\mathbf{u}^{n_j}\}_{j \in \mathbb{N}}$ and a broken flow line with cascades $\{\mathbf{w}_j\}_{j=1}^l$, such that $\{\mathbf{u}^{n_j}\}_{j \in \mathbb{N}}$ Floer-Gromov converges to $\{\mathbf{w}_j\}_{j=1}^l$.*

Proof. The scheme of the proof is the same as in the compact case: the compactness of the Floer trajectories follows from uniform bounds on the moduli spaces and standard compactness arguments (see Proposition 6.6.2 in [2] and Section 4.2 in [4]). Compactness of the spaces of Morse trajectories on the critical set and in particular on the non-compact hypersurface Σ needs both the uniform bounds on the moduli spaces of Floer trajectories and the coercivity of f .

Let $\mathcal{A}^H(p) = a$ and $\mathcal{A}^H(q) = b$. First observe that without loss of generality we can assume that all the connected components of $\text{Crit}(\mathcal{A}^H)$ with the only exception of Σ are compact, hence the restriction of the Morse flow to those components is compact up to breaking (cf. Lemma 2.38 in [20]). Therefore we only have to consider the case where $0 \in [a, b]$ and show that the gradient flow of f on Σ , which is part of the flowlines with cascades, is in fact restricted to compact sets.

We consider two cases:

- (1) $b = 0$.

Then the Morse parts of the cascades $\mathcal{M}(p, q)$ on Σ are contained in $W^s(q)$. The set $W^s(q)$ is a subset of $f^{-1}((-\infty, f(q)])$, which by assumption on f is a compact subset of Σ .

- (2) $0 \in [a, b)$.

Then by definition of $K(b)$ in (4.3) the Morse components of the cascades in $\mathcal{M}(p, q)$ passing through Σ are contained in $K(b)$, which by Lemma 4.1 is compact.

Knowing that the Morse parts of the flow lines with cascades are confined to compact subsets of Σ , we can apply the same arguments as in the compact case [6] to infer compactness of the Morse trajectories. Having established compactness both of the Morse trajectories and of the Floer trajectories, we can now directly apply the methods from [14], Appendix C.2, to show the compactness of the cascades.

□

5. TRANSVERSALITY

The aim of this section is to prove that moduli spaces of flow lines with cascades form smooth manifolds. In order to achieve this, we first consider spaces of Floer trajectories between different components of the critical set of $\mathcal{A}^H$. The following remark is in place: in the rest of this section, we will assume that the exact symplectic manifold we consider satisfies the condition $c_1(M) = 0$, which guarantees that the Conley-Zehnder index is integer-valued for all the periodic orbits and therefore the Rabinowitz Floer homology is defined with integer grading. This case suffices for us, since in the end we are going to concentrate on a class of Hamiltonians defined on the standard symplectic space $\mathbb{R}^{2n}$, which obviously satisfies the condition. If $c_1(M)|_{\pi_2(M)} \neq 0$, one can still define Rabinowitz Floer homology as a $\mathbb{Z}_2$ -graded invariant. For a discussion of the grading in that case we refer the reader to [15].

Theorem 5.1. *Consider a smooth Hamiltonian H on an exact symplectic manifold (M, ω) , such that $\Sigma = H^{-1}(0)$ is a regular level set and a smooth hypersurface of exact contact type. Additionally, H satisfies (PO) and (MB) and there exists an open subset $\mathcal{V}$ of $M \times \mathbb{R}$ such that for any $J \in \mathcal{J}(M, \omega, \mathcal{V})$ and any pair Λ^- and Λ^+ of connected components of the critical set the associated moduli space of Floer trajectories is contained in a bounded subset of $M \times \mathbb{R}$ and every Floer trajectory intersects $\mathcal{V}$.*

Then for a generic choice of $J \in \mathcal{J}(M, \omega, \mathcal{V})$ and every pair of connected submanifolds without boundary $\Lambda^-, \Lambda^+ \subseteq \text{Crit}(\mathcal{A}^H)$ the associated moduli space $\mathcal{M}_{J,H}(\Lambda^-, \Lambda^+)$ is a smooth manifold without boundary of dimension

$$\dim(\mathcal{M}_{J,H}(\Lambda^-, \Lambda^+)) = \mu_{CZ}(\Lambda^+) - \mu_{CZ}(\Lambda^-) + \frac{1}{2}(\dim(\Lambda^-) + \dim(\Lambda^+)).$$

Proof. The proof of this theorem is a direct adaptation of existing results to our framework. One first needs to show that there exists a residual subset $\mathcal{J}_{reg}(H, \mathcal{V}) \subset \mathcal{J}(M, \omega, \mathcal{V})$ such that if $J \in \mathcal{J}_{reg}(H, \mathcal{V})$, then for every pair Λ^+, Λ^- of components of $\text{Crit}(\mathcal{A}^H)$, the spaces $\mathcal{M}_{J,H}(\Lambda^-, \Lambda^+)$ are all smooth manifolds.

In order to be able to apply the implicit function theorem, we need to embed $\mathcal{M}_{J,H}(\Lambda^-, \Lambda^+)$ in a Banach manifold. The following identity is proved in Theorem 25, [11]:

$$\mathcal{M}_{J,H}(\Lambda^-, \Lambda^+) = \{u \in \mathcal{B}_\delta(\Lambda^-, \Lambda^+) \mid \partial_s u = \nabla_J \mathcal{A}^H(u)\}, \quad (5.1)$$

where $\mathcal{B}_\delta(\Lambda^-, \Lambda^+)$ denotes the Banach manifold

$$\mathcal{B}_\delta(\Lambda^-, \Lambda^+) := \left\{ u: \mathbb{R} \times S^1 \rightarrow M \times \mathbb{R} \left| \begin{array}{l} \exists (x^-, x^+) \in \Lambda^- \times \Lambda^+ \\ \text{dist}(u(t, s), x^\pm(t)) \leq \alpha e^{\mp \delta s} \\ \|\partial_s u(s, t)\| \leq \alpha e^{-\delta|s|} \end{array} \right. \right\}.$$

Define a Banach bundle $\mathcal{E}_\delta \rightarrow \mathcal{B}_\delta(\Lambda^-, \Lambda^+)$ by requiring the fiber over $u \in \mathcal{B}_\delta(\Lambda^-, \Lambda^+)$ to be

$$(\mathcal{E}_\delta)_u := L^p_\delta(\mathbb{R} \times S^1, u^*(TM \times \mathbb{R})).$$

For $l \in \mathbb{N}$, let $\mathcal{J}^l(M, \omega, \mathcal{V})$ be defined in the same way as $\mathcal{J}(M, \omega, \mathcal{V})$, with the only difference that we require J to be of class C^l instead of class C^∞ . For a fixed $J \in \mathcal{J}^l(M, \omega, \mathcal{V})$, we consider the section

$$\begin{aligned} \bar{\partial}_J : \mathcal{B}_\delta(\Lambda^-, \Lambda^+) &\rightarrow \mathcal{E}_\delta, \\ \bar{\partial}_J(v, \eta) &:= \begin{pmatrix} \partial_s v + J_t(v, \eta)(\partial_t v - \eta X_H(v)) \\ \partial_s \eta + \int_{S^1} H(v) dt \end{pmatrix} \end{aligned}$$

and extend it to a section

$$S : \mathcal{B}_\delta(\Lambda^-, \Lambda^+) \times \mathcal{J}^l(M, \omega, \mathcal{V}) \rightarrow \mathcal{E}_\delta, \quad S(u, J) = \bar{\partial}_J(u).$$

One needs to prove that for (u, J) in $S^{-1}(0)$, the vertical derivative

$$D_{(u, J)} S : W^{1,p}_\delta(\mathbb{R} \times S^1, u^*(TM \times \mathbb{R})) \times T_J \mathcal{J}^l(M, \omega, \mathcal{V}) \rightarrow \mathcal{E}_\delta$$

is surjective or, equivalently, any element of the dual space $(\mathcal{E}_\delta)_u^*$ vanishing on the image of $D_{(u, J)} S$ is, in fact, 0.

One can follow the proof of Theorem 4.10 in [1], and subsequently, in the proof of Theorem 4.11, modify the definition the set Ω as follows:

$$\Omega := \{(s, t) \in \mathcal{R}(u) \mid v(s, t) \notin \text{Crit}(H), v(s, t) \in \mathcal{V}\}.$$

We would like to show that $\Omega(u)$ is open and non-empty. Without loss of generality we can assume $\mathcal{A}^H(\Lambda^+) \neq 0$, so we have that

$$\lim_{s \rightarrow +\infty} u(s) \subseteq \mathcal{V} \cap (\Sigma \times \mathbb{R}).$$

Since, by assumption, $\mathcal{V}$ is open and Σ is a regular level set of H , it follows that

$$u^{-1}(\mathcal{V} \setminus (\text{Crit}(H) \times \mathbb{R}))$$

is open and non-empty. On the other hand, by Theorem 4.10 in [1], the set $\mathcal{R}(u)$ is an open and dense subset of the

$$\mathcal{C}(v) := \{(s, t) \in \mathbb{R} \times S^1 \mid \partial_s v(s, t) \neq 0\}.$$

Observe that $\mathcal{C}(v)$ is open, as a result of the continuity of $\partial_s v$. Therefore, to show that $\Omega(u)$ is open and non-empty, it suffices to show that

$$u^{-1}(\mathcal{V} \setminus (\text{Crit}(H) \times \mathbb{R})) \cap \mathcal{C}(v) \neq \emptyset. \quad (5.2)$$

Since (v, η) is a non-constant solution of the Rabinowitz Floer equation, we can apply a generalized version of Aronszajn's unique continuation theorem (cf. Proposition 3.3 in [5]), and conclude that the interior of the complement of $\mathcal{C}(v)$ in $\mathbb{R} \times S^1$ is empty. It follows that $\mathcal{C}(v)$ is dense in $\mathbb{R} \times S^1$ and therefore (5.2) is satisfied. Combining (5.2) with the fact that $\mathcal{R}(u)$ is open and dense in $\mathcal{C}(v)$, we can finally prove that $\Omega(u)$ is open and non-empty, as claimed.

Summarizing: since every element ρ of the dual space $(\mathcal{E}_\delta)_u^*$ vanishing on the image of $D_{(u,J)}S$ is in the kernel of a first order elliptic operator and vanishes on a non-empty, open set $\Omega(u)$, we conclude (once more by Aronszajn's unique continuation theorem) that $\rho = 0$ everywhere, thus proving that $D_{(u,J)}S$ is surjective for all $(u, J) \in S^{-1}(0)$ and $S^{-1}(0)$ is a smooth manifold of class C^l . Note that all the connected components $\Lambda^-, \Lambda^+ \subseteq \text{Crit}(\mathcal{A}^H)$ are open sets, therefore $S^{-1}(0)$ is in fact a smooth manifold without boundary.

Denote the set of regular almost complex structures by

$$\mathcal{J}_{reg}^l(\Lambda^-, \Lambda^+) := \left\{ J \in \mathcal{J}^l(M, \omega, \mathcal{V}) \mid (\mathcal{B}_\delta(\Lambda^-, \Lambda^+) \times \{J\}) \cap S^{-1}(0) \text{ is a } C^l \text{ manifold} \right\}.$$

Since $S^{-1}(0)$ is a manifold of class C^l , the set of regular values of the projection

$$\pi : S^{-1}(0) \rightarrow \mathcal{J}^l(M, \omega, \mathcal{V})$$

is in fact equal to $\mathcal{J}_{reg}^l(\Lambda^-, \Lambda^+)$. By the Sard's theorem (see Theorem A.5.1 in [18]) $\mathcal{J}_{reg}^l(\Lambda^-, \Lambda^+)$ is residual in $\mathcal{J}^l(M, \omega, \mathcal{V})$ with respect to the C^l -topology.

By assumption Σ is of contact type and H satisfies (PO+) and (MB), hence by Lemma 3.6 the set of connected components of $\text{Crit}(\mathcal{A}^H)$ is countable. Therefore, if we intersect over the set of all connected components of $\text{Crit}(\mathcal{A}^H)$ we obtain that

$$\mathcal{J}_{reg}^l(H, \mathcal{V}) := \bigcap_{\Lambda^-, \Lambda^+ \subseteq \text{Crit}(\mathcal{A}^H)} \mathcal{J}_{reg}^l(\Lambda^-, \Lambda^+)$$

is a countable intersection of residual sets and thus a residual set in $\mathcal{J}^l(M, \omega, \mathcal{V})$. Finally, one can get a result in the smooth category, by using the Taubes trick to show that the intersection over all $l \in \mathbb{N}$

$$\mathcal{J}_{reg}(H, \mathcal{V}) := \bigcap_{l \in \mathbb{N}} \mathcal{J}_{reg}^l(H, \mathcal{V})$$

is in fact residual in $\mathcal{J}(M, \omega, \mathcal{V})$.

The Conley-Zehnder index is invariant under homotopies, hence it is constant on the connected components of $\text{Crit}(\mathcal{A}^H)$. Let $J \in \mathcal{J}_{reg}(H, \mathcal{V})$ and let Λ^-, Λ^+ be a pair of connected components of $\text{Crit}(\mathcal{A}^H)$. Choose $(v^\pm, \eta^\pm) \in \Lambda^\pm$ with cappings $\bar{v}^\pm \in C^\infty(D, M)$, such that $\bar{v}^\pm(e^{i\pi t}) = v^\pm(t)$ and let $u = (v, \eta)$ be a Floer trajectory such that $\lim_{s \rightarrow \pm\infty} u(s) = (v^\pm, \eta^\pm)$. Then by Proposition 4.1 in [6] the local virtual dimension of the moduli space at u can be expressed in terms of the Conley-Zehnder indexes of the endpoint orbits and the first Chern class of the sphere obtained capping the cylinder v with v^- and v^+ . More precisely the local dimension of the moduli space is equal to

$$\mu_{CZ}(v^+, \eta^+) - \mu_{CZ}(v^-, \eta^-) + \frac{1}{2}(\dim(\Lambda^-) + \dim(\Lambda^+)) + c_1(\bar{v}^- \# v \# v^+).$$

In case $c_1(M)|_{\pi_2(M)} = 0$, the last term disappears, the Conley-Zehnder index does not depend on the capping or the choice of $(v^\pm, \eta^\pm) \in \Lambda^\pm$, hence we obtain that the dimension of the moduli space of Floer trajectories between Λ^- and Λ^+ is

$$\dim(\mathcal{M}_{J,H}(\Lambda^-, \Lambda^+)) = \mu_{CZ}(\Lambda^+) - \mu_{CZ}(\Lambda^-) + \frac{1}{2}(\dim(\Lambda^-) + \dim(\Lambda^+)). \quad (5.3)$$

□

We now concentrate on the smooth structure of moduli spaces of flow lines with cascades. We assume that f is a Morse function on the critical manifold of $\mathcal{A}^H$, and we introduce a grading on $\text{Crit}(f)$ in the following way. Given $x \in \text{Crit}(f)$, let $\mu_\sigma(x)$ be the signature index with respect to f , namely

$$\mu_\sigma(x) := \frac{1}{2}(\dim(W^s(x)) - \dim(W^u(x))), \quad (5.4)$$

where W^s and W^u are the stable and unstable manifolds at x with respect to flow of ∇f . Let $\mu_{CZ}(x)$ be the Conley-Zehnder index with respect to $\mathcal{A}^H$. We define the grading on the Rabinowitz Floer complex by

$$\mu(x) := \mu_\sigma(x) + \mu_{CZ}(x) + \frac{1}{2}. \quad (5.5)$$

Proposition 5.2. *Consider a smooth Hamiltonian H on an exact symplectic manifold (M, ω) , such that $\Sigma = H^{-1}(0)$ is a regular level set and a smooth hypersurface of exact contact type. Additionally, assume that H satisfies (PO) and (MB) and there exists $J \in \mathcal{J}(M, \omega)$, such that for every pair of connected submanifolds without boundary $\Lambda^-, \Lambda^+ \subseteq \text{Crit}(\mathcal{A}^H)$ the associated moduli space $\mathcal{M}_{J,H}(\Lambda^-, \Lambda^+)$ is a smooth manifold without boundary a smooth manifold, whose image in $M \times \mathbb{R}$ is bounded.*

Then for a Morse-Smale pair (f, g) on $\text{Crit}(\mathcal{A}^H)$ the associated moduli space $\mathcal{M}(x^-, x^+)$ of flowlines with cascades is a smooth, finite dimensional manifold without boundary. Its dimension is given by

$$\dim(\mathcal{M}(x^-, x^+)) = \mu(x^+) - \mu(x^-) - 1.$$

Proof. The first step is to show that for every pair $x^-, x^+ \in \text{Crit}(f)$ and every $m \in \mathbb{N}$ the corresponding moduli space $\mathcal{M}_m(x^-, x^+)$ is a smooth manifold.

For $m = 0$ the points x^- and x^+ belong to the same connected component of $\text{Crit}(\mathcal{A}^H)$ and the moduli space $\mathcal{M}_0(x^-, x^+)$ consists of the Morse flowlines. By assumption the pair (f, g) was chosen to be Morse-Smale, hence by standard Morse-theoretical arguments [20] we conclude that $\mathcal{M}_0(x^-, x^+)$ is a smooth manifold of dimension $\mu_\sigma(x^+) - \mu_\sigma(x^-) - 1$. As in this case x^- and q belong to the same connected component of $\text{Crit}(\mathcal{A}^H)$, so their Conley-Zehnder indexes are the same, which implies that the dimension of $\mathcal{M}_0(x^-, x^+)$ is in fact equal to $\mu(x^+) - \mu(x^-) - 1$.

For $m \geq 1$ whenever $\mathcal{M}_m(x^-, x^+) \neq \emptyset$ then the points x^- and x^+ belong to different critical components of $\text{Crit}(\mathcal{A}^{H+h})$. Following the arguments from Appendix of [14] and [11] section 2.4 we conclude that $\mathcal{M}_m(x^-, x^+)$ is a smooth manifold of dimension $\mu(x^+) -$

$\mu(x^-) - 1$ with boundary. By Theorem 4.2 if we sum up $\mathcal{M}_m(x^-, x^+)$ by all $m \geq 1$ as in (4.1), then the obtained set $\mathcal{M}(x^-, x^+)$ is precompact. In fact we can glue the manifolds with different number of cascades along the boundary (see Proposition 1b in [12], Corollary C.15 in [14] or Theorem 9.2.3 [2]) to equip the set $\mathcal{M}(x^-, x^+)$ with a smooth structure. As a result $\mathcal{M}(x^-, x^+)$ becomes a smooth, precompact manifold without boundary, of dimension $\mu(x^+) - \mu(x^-) - 1$. $\square$

6. DEFINITION OF RFH

In this section we will extend the definition given in [6] so that it also applies to a class of non-compact hypersurfaces.

Theorem 6.1. *Consider a smooth Hamiltonian H on an exact symplectic manifold (M, ω) , such that $\Sigma = H^{-1}(0)$ is a regular level set and a smooth hypersurface of exact contact type. Additionally, assume that H satisfies (PO) and (MB) and there exists $J \in \mathcal{J}(M, \omega)$, such that for every pair of connected submanifolds without boundary $\Lambda^-, \Lambda^+ \subseteq \text{Crit}(\mathcal{A}^H)$ the associated moduli space $\mathcal{M}_{J,H}(\Lambda^-, \Lambda^+)$ is a smooth manifold without boundary, whose image in $M \times \mathbb{R}$ is bounded. Then the associated Rabinowitz Floer homology*

$$RFH(H, J),$$

is well defined.

Proof. The proof follows the construction of Rabinowitz Floer homology defined in [6] by combining the results from previous sections into a generalization for the non-compact setting.

Fix a Morse-Smale pair (f, g) on $\text{Crit}(\mathcal{A}^H)$ and consider the chain complex $CF_*(\mathcal{A}^H, f)$ generated by the critical points of f , with coefficients in $\mathbb{Z}_2$ and grading given by the index μ . More precisely, $CF_k(\mathcal{A}^H, f)$ consists of formal sums of the form

$$p = \sum_{\substack{x \in \text{Crit}(f), \\ \mu(x)=k}} p_x x,$$

where the coefficients $p_x \in \mathbb{Z}_2$ satisfy the Novikov finiteness condition

$$\#\{x \in \text{Crit}(f) \mid \mu(x) = k, p_x \neq 0, \mathcal{A}^H(x) \geq a\} < \infty \quad \forall a \in \mathbb{R}. \quad (6.1)$$

By Proposition 5.2, for a given a pair of points $x, y \in \text{Crit}(f)$ with $\mu(x) - \mu(y) = 1$, the moduli space $\mathcal{M}(y, x)$ is a discrete set of points. On the other hand, by Theorem 4.2 $\mathcal{M}(y, x)$ is also compact, hence it is a finite set. Denote its cardinality by

$$n(y, x) := \#\mathcal{M}(y, x) \bmod 2 \in \mathbb{Z}_2 \quad (6.2)$$

and define the Floer boundary operator

$$\partial : CF_{*+1}(\mathcal{A}^H, f) \rightarrow CF_*(\mathcal{A}^H, f),$$

as the linear extension of

$$\partial x := \sum_{\substack{y \in \text{Crit}(f), \\ \mu(x) - \mu(y) = 1}} n(y, x) y.$$

In order for the boundary operator to be well defined, one has to check that for ∂x the finiteness condition (6.1) holds. In other words, we must show that for all $x \in \text{Crit}(f)$ and $a \in \mathbb{R}$

$$\#\{y \in \text{Crit}(f) \mid \mu(x) - \mu(y) = 1, n(y, x) \neq 0 \text{ \& } \mathcal{A}^H(y) \geq a\} < +\infty, \quad (6.3)$$

Set $b := \mathcal{A}^H(x)$. Note that $n(y, x) \neq 0$ implies $b \geq \mathcal{A}^H(y) \geq a$. Therefore, we can consider the following two cases:

- (1) When $\mathcal{A}^H(x) = \mathcal{A}^H(y) = 0$, then $n(y, x) \neq 0$ implies $y \in \text{Crit}(f) \cap W_f^u(x)$. Moreover, $W_f^u(x) \subseteq f^{-1}((-\infty, f(x)])$, which is compact due to coercivity of f on Σ , and thus $W_f^u(x) \cap \text{Crit}(f)$ is finite.
- (2) In the other case condition $n(y, x) \neq 0$ implies

$$y \in \text{Crit}(f) \cap \left((\mathcal{A}^H)^{-1}([a, b] \setminus \{0\}) \cup K(b) \right).$$

Observe that set $(\mathcal{A}^H)^{-1}([a, b] \setminus \{0\})$ is a finite union of nonzero connected components of $\text{Crit}(\mathcal{A}^H)$, which are all compact. Moreover, the set $K(b)$ is compact by Lemma 4.1. Thus their sum contains finitely many critical points of f and the condition (6.3) follows.

Now if we combine the results from Theorem 4.2 and Proposition 5.2 with the standard gluing argument in Floer theory (see for example Proposition 2d.1 in [12] or Theorem 9.2.2 in [2]), then we obtain that the operator ∂ defined above is indeed a boundary operator, i.e. $\partial^2 = 0$. As a result, we can define Rabinowitz Floer homology of the quadruple (H, J, f, g) by setting:

$$RFH_k(H, J, f, g) := \frac{\text{Im}(\partial_{k+1})}{\text{Ker}(\partial_k)}.$$

In fact, by Theorem 8 in [20], such defined homology is independent of the choice of the Morse-Smale pair (f, g) as long as f is coercive on $\text{Crit}(\mathcal{A}^H)$. Therefore, we can denote it by $RFH(H, J)$. $\square$

Just as in the compact case, the Rabinowitz Floer homology has the following property:

Corollary 6.2. *Whenever $\Sigma = H^{-1}(0)$ carries no closed characteristics, then $RFH(H)$ is well defined and isomorphic to the singular homology of Σ .*

In that case the assumptions of Theorem 6.1 are trivially satisfied and the corresponding Rabinowitz Floer homology is generated by the positive gradient flow of the coercive function f on Σ . Observe, that in that case, if we change the the sign of the gradient flow and change the order in the definition of the differential, we retrieve the Morse homology of Σ with respect to Σ with the signature grading. Indeed, the number of positive gradient flowlines flowing out of a critical point are equal to the number of negative gradient flowlines flowing in the same point. On the other hand the Morse homology of Σ with signature grading is

isomorphic to its Morse homology shifted by half of the dimension of Σ . Finally, the Morse homology of Σ with Morse grading is equal to its singular homology (see [20] for a proof of the last isomorphism in the non-compact case):

$$RFH(H, J) = RFH(H, J, f, g) \cong MH(f, g, \Sigma) \cong H(\Sigma).$$

This observation allows us to use the Rabinowitz Floer homology to detect periodic orbits. Indeed, if the Rabinowitz Floer homology of H is well defined and different from the singular homology of the hypersurface $\Sigma := H^{-1}(0)$, then Σ carries a closed characteristic.

The following corollary shows that Rabinowitz Floer homology is invariant under symplectomorphisms:

Proposition 6.3. *Let $\varphi : (M_1, \omega_1) \rightarrow (M_2, \omega_2)$ be a symplectomorphism of exact symplectic manifolds and suppose that Rabinowitz Floer homology is well defined for the pair $(H, J) \in C^\infty(M_2) \times \mathcal{J}(M_2, \omega_2)$. Then the Rabinowitz Floer homology is also well defined for the pair $\varphi^*(H, J) \in C^\infty(M_1) \times \mathcal{J}(M_1, \omega_1)$, $\varphi^*(H, J) := (H \circ \varphi, D\varphi^{-1} \circ J \circ D\varphi)$ and*

$$RFH(\varphi^*(H, J)) \cong RFH(H, J).$$

Proof. By assumption the pair (H, J) satisfies the assertions of Theorem 6.1. We will show that the pair $\varphi^*(H, J)$ also satisfies assertions of Theorem 6.1.

Observe that φ is a symplectomorphism, hence if λ_2 is a primitive of ω_2 then $\varphi^*\lambda_2$ is a primitive of ω_1 . As a result, the symplectomorphism φ preserves the contact type property of the hypersurface. Further on, the symplectomorphism φ induces a diffeomorphism of the loop spaces

$$\begin{aligned} \varphi_\# : C^\infty(\mathbb{R}/\mathbb{Z}; M_2) \times \mathbb{R} &\rightarrow C^\infty(\mathbb{R}/\mathbb{Z}; M_1) \times \mathbb{R}, \\ (v, \eta) &\mapsto (\varphi^{-1} \circ v, \eta). \end{aligned}$$

We will show that this diffeomorphism maps $\mathcal{A}^H$ to $\mathcal{A}^{H \circ \varphi}$. For $(v, \eta) \in C^\infty(\mathbb{R}/\mathbb{Z}; M_2) \times \mathbb{R}$ we get

$$\begin{aligned} \mathcal{A}^{H \circ \varphi} \circ \varphi_\#(v, \eta) &= \int \varphi^*\lambda_2(D\varphi^{-1}\partial_t v) - \eta \int H \circ \varphi(\varphi^{-1} \circ v) \\ &= \int \lambda_2(\partial_t v) - \eta \int H(v) = \mathcal{A}^H(v, \eta). \end{aligned}$$

As a result $d\mathcal{A}^{H \circ \varphi} = d\mathcal{A}^H \circ D(\varphi_\#)^{-1}$. Consequently, $(v, \eta) \in \text{Crit}(\mathcal{A}^H)$ if and only if $\varphi_\#(v, \eta) \in \text{Crit}(\mathcal{A}^{H \circ \varphi})$. In particular, for every $a, b \in \mathbb{R}$ there is a one-to-one correspondence between

$$\text{Crit}(\mathcal{A}^H) \cap (\mathcal{A}^H)^{-1}([a, b]) \quad \text{and} \quad \text{Crit}(\mathcal{A}^{H \circ \varphi}) \cap (\mathcal{A}^{H \circ \varphi})^{-1}([a, b]).$$

Since φ is a diffeomorphism, pre-images of compact sets are compact, and so we can conclude that the Hamiltonian $H \circ \varphi$ on (M_1, ω_1) satisfies (PO) if and only if H on (M_2, ω_2) satisfies (PO).

Now, we will show that $\varphi_{\#}$ preserves the Conley-Zehnder indices. Denote by ϕ^t the Hamiltonian flow of X_H on M_2 and let γ be a contractible, closed characteristic of ϕ^t on $\Sigma := H^{-1}(0)$ with period η . Let λ_2 be the primitive of ω_2 , such that $\xi = \text{Ker} \left(\lambda_2 \Big|_{T\Sigma} \right)$ is the contact structure on Σ and let $\Phi : [0, \eta] \times \mathbb{R}^{2n-2} \rightarrow \gamma^*(TM_2)$ be the trivialization of ξ along the closed characteristics γ . There is a splitting $T\Sigma = X_H \oplus \xi$, which is preserved by the flow ϕ_t . Then the Conley-Zehnder index of $(\gamma(\frac{t}{\eta}), \eta) \in \text{Crit}(\mathcal{A}^H)$ is equal to the Maslov index of the path of symplectic matrices $\Gamma_1 : [0, \eta] \rightarrow Sp(n-1)$ defined by

$$\Gamma_1(t) := \Phi^{-1}(t) \circ D\phi^t \circ \Phi(0).$$

Let us now construct the path of symplectic matrices corresponding to the closed characteristic $\varphi^{-1} \circ \gamma$ on $(\varphi)^{-1}(\Sigma)$. The Hamiltonian flow of $X_{H \circ \varphi}$ on M_1 is given by $\varphi^{-1} \circ \phi^t \circ \varphi$ and the contact structure on $\varphi^{-1}(\Sigma)$ is given by $D\phi^{-1}(\xi)$. Then $D\varphi^{-1} \circ \Phi$ is a trivialization of $D\phi^{-1}(\xi)$ along $\varphi^{-1} \circ \gamma$. We can construct the corresponding path of symplectic matrices $\Gamma_2 : [0, \eta] \rightarrow Sp(n-1)$ as before

$$\Gamma_2(t) := (D\varphi^{-1} \circ \Phi(t))^{-1} \circ D(\varphi^{-1} \circ \phi^t \circ \varphi) \circ D\varphi^{-1} \circ \Phi(0) = \Gamma_1(t).$$

It turns out that the paths of symplectic matrices corresponding to γ and to $\varphi^{-1} \circ \gamma$ are the same, hence the corresponding Conley-Zehnder indices are the same.

Now, we will show that the map $\varphi_{\#}$ preserves the Morse-Bott property of the Rabinowitz action functional. Using (2.1) we can define a metric g_J on $C^\infty(\mathbb{R}/\mathbb{Z}; M_2) \times \mathbb{R}$ associated to the almost complex structure J and a metric g_{φ^*J} on $C^\infty(\mathbb{R}/\mathbb{Z}; M_1) \times \mathbb{R}$ associated to the almost complex structure $\varphi^*J = D\varphi^{-1} \circ J \circ D\varphi$. Observe that the push-forward of g_J to $C^\infty(\mathbb{R}/\mathbb{Z}; M_1) \times \mathbb{R}$ via $\varphi_{\#}$ is equal to g_{φ^*J}

$$(\varphi_{\#})^* g_J = g_{\varphi^*J}.$$

As a result

$$\begin{aligned} \nabla_{g_{\varphi^*J}} \mathcal{A}^{H \circ \varphi}(\varphi_{\#}(v, \eta)) &= D\varphi_{\#}(\nabla_{g_J} \mathcal{A}^H(v, \eta)), \\ \nabla_{g_{\varphi^*J}}^2 \mathcal{A}^{H \circ \varphi}(\varphi_{\#}(v, \eta)) &= D\varphi_{\#} \circ \nabla_{g_J}^2 \mathcal{A}^H \circ D(\varphi_{\#})^{-1} \end{aligned} \quad (6.4)$$

Consequently, if $\mathcal{A}^H$ is Morse-Bott then $\mathcal{A}^{H \circ \varphi}$ is Morse-Bott, as for $(v, \eta) \in \text{Crit}(\mathcal{A}^{H \circ \varphi})$ we have

$$\begin{aligned} \text{Ker} \left(\nabla_{g_{\varphi^*J}}^2 \mathcal{A}^{H \circ \varphi}(v, \eta) \right) &= \text{Ker} \left(D\varphi_{\#} \circ \nabla_{g_J}^2 \mathcal{A}^H \circ D(\varphi_{\#})^{-1} \right) \\ &= \text{Ker} \left(\nabla_{g_J}^2 \mathcal{A}^H \circ D(\varphi_{\#})^{-1} \right) \\ &= \text{Ker} \left(d\mathcal{A}^H \circ D(\varphi_{\#})^{-1} \right) \\ &= \text{Ker} (d_{(v, \eta)} \mathcal{A}^{H \circ \varphi}) \\ &= T_{(v, \eta)} \mathcal{A}^{H \circ \varphi}. \end{aligned}$$

Finally, we show that there is a one-to-one correspondence between the Floer trajectories of the pair (H, J) and the pair $\varphi^*(H, J)$. In other words, for every two connected components

$\Lambda^-, \Lambda^+ \subseteq \text{Crit}(\mathcal{A}^H)$ we have

$$u \in \mathcal{M}_{J,H}(\Lambda^-, \Lambda^+) \iff \varphi_{\#} \circ u \in \mathcal{M}_{\varphi^*(J,H)}(\varphi_{\#}(\Lambda^-), \varphi_{\#}(\Lambda^+)). \quad (6.5)$$

Indeed, if $u \in \mathcal{M}_{J,H}(\Lambda^-, \Lambda^+)$ then by (6.4) we have

$$0 = D\varphi_{\#}(\partial_s u + \nabla_{g_J} \mathcal{A}^H(u)) = \partial_s(\varphi_{\#} \circ u) + \nabla_{g_{\varphi^*J}} \mathcal{A}^{H \circ \varphi}(\varphi_{\#} \circ u).$$

As φ is a diffeomorphism, the pre-images of compact sets are compact, and thus we can conclude that $\mathcal{M}_{\varphi^*(J,H)}(\varphi_{\#}(\Lambda^-), \varphi_{\#}(\Lambda^+))$ are bounded smooth manifolds. We can conclude that the pair $\varphi^*(H, J)$ satisfies assertion of Theorem 6.1 and thus $RFH_*(\varphi^*(H, J))$ is well defined. On the other hand, there is a one-to-one correspondence between $\text{Crit}(\mathcal{A}^H)$ and $\text{Crit}(\mathcal{A}^{H \circ \varphi})$, which preserves the Conley-Zehnder indices and by (6.5) there is a one-to-one correspondence between the Floer trajectories of the pair (H, J) and the pair $\varphi^*(H, J)$. As a result, the corresponding Rabinowitz Floer homologies are isomorphic. $\square$

7. INVARIANCE OF RFH

The aim of this section is to show that Rabinowitz Floer homology is independent of the choice of almost complex structure and invariant under small compactly supported homotopies of the Hamiltonian. In order to prove this, we first need to introduce the notion of perturbed flow lines with cascades. Let $\Gamma = \{(H_s, J_s)\}_{s \in \mathbb{R}}$ be a homotopy of Hamiltonians and almost-complex structures, constant outside $[0, 1]$: then for any pair $(p, q) \in \text{Crit}(f_0) \times \text{Crit}(f_1)$ we consider the space of perturbed flow lines with cascades $\mathcal{M}^{\Gamma}(p, q)$. This is obtained as follows: we consider flow lines which consist of exactly one perturbed Floer trajectory, i.e. solution of $\partial_s u = \nabla_{J_s} \mathcal{A}^{H_s}$ with bounded energy, while all the other trajectories are Floer trajectories of $\mathcal{A}^{H_0}$ or $\mathcal{A}^{H_1}$. Then we quotient the space of such lines by the natural $\mathbb{R}$ -action, in the same way as we did for the unperturbed trajectories in Section 4.

Proving independence of Rabinowitz Floer homology from the choice of almost complex structure and invariance under compactly supported perturbations requires uniform L^∞ bounds on perturbed Floer trajectories associated to different families of homotopies. In the compact case those uniform bounds are obtained by a standard isolating neighborhood argument, similar to Theorem 3 from [12]. Unfortunately, in the case of non-compact hypersurfaces the standard techniques are not directly applicable. In this section we will assume that all the necessary bounds hold and show how to prove invariance of Rabinowitz Floer homology consequently. Later we will apply this result to the case of tentacular Hamiltonians a class for which we have proven uniform bounds directly in [19].

Note that contrary to the unperturbed case, the action may not be monotonically increasing along the perturbed Floer trajectories. To deal with this phenomenon, we introduce the Novikov finiteness condition, which ensures that the action cannot decrease indefinitely along a perturbed Floer trajectory.

Definition 7.1. Let $\Gamma = \{(H_s, J_s)\}_{s \in \mathbb{R}}$ be a smooth homotopy of Hamiltonians and almost complex structures constant outside $[0, 1]$. For a pair $a, b \in \mathbb{R}$ denote

$$A(\Gamma, a, b) := \inf \left\{ A \in (-\infty, b] \left| \begin{array}{l} \exists \Lambda_i \subseteq \text{Crit}(\mathcal{A}^{H_i}) \quad i = 0, 1, \\ \mathcal{A}^{H_0}(\Lambda_0) \geq a, \mathcal{A}^{H_1}(\Lambda_1) = A, \\ \mathcal{M}^\Gamma(\Lambda_0, \Lambda_1) \neq \emptyset \end{array} \right. \right\}, \quad (7.1)$$

$$B(\Gamma, a, b) := \sup \left\{ B \in [a, +\infty) \left| \begin{array}{l} \exists \Lambda_i \subseteq \text{Crit}(\mathcal{A}^{H_i}) \quad i = 0, 1, \\ \mathcal{A}^{H_0}(\Lambda_0) = B, \mathcal{A}^{H_1}(\Lambda_1) \leq b, \\ \mathcal{M}^\Gamma(\Lambda_0, \Lambda_1) \neq \emptyset \end{array} \right. \right\} \quad (7.2)$$

We say that the homotopy Γ satisfies the *Novikov finiteness condition* if for each pair $(a, b) \in \text{CritVal}(\mathcal{A}^{H_0}) \times \text{CritVal}(\mathcal{A}^{H_1})$ the corresponding $A(\Gamma, a, b)$ and $B(\Gamma, a, b)$ are finite.

The first step to construct an isomorphism between $RFH(H_0, J_0)$ and $RFH(H_1, J_1)$ is to show that for any pair $(p, q) \in \text{Crit}(f_0) \times \text{Crit}(f_1)$ the corresponding $\mathcal{M}^\Gamma(p, q)$ is a smooth manifold without boundary, which can be compactified by moduli spaces of lower index. That will allow us to define a homomorphism between $RFH(H_1, J_1)$ and $RFH(H_0, J_0)$. To equip $\mathcal{M}^\Gamma(p, q)$ with a manifold structure we follow the argument as in Theorem 5.1, but to achieve the transversality, rather than perturbing the almost complex structures, we perturb the homotopy in its Hamiltonian component. Therefore one needs uniform bounds on the moduli spaces not only for a fixed homotopy, but also for families of homotopies. Since obtaining those bounds is a substantial difference with respect to the non-compact, we will show how to do it in Lemma 7.2.

We introduce below some notation which is necessary to formulate the lemma. For a compact subset $N \subseteq H^{-1}(0)$ we define

$$\mathcal{C}(\mathcal{A}^H, N) := \left\{ x \in \text{Crit}(\mathcal{A}^H) \mid 0 < |\mathcal{A}^H(x)| \text{ or } x \in N \times \{0\} \right\} \quad (7.3)$$

Let (H_0, J_0) and (H_1, J_1) be pairs of Hamiltonians and compatible almost complex structures satisfying the assumptions of Theorem 6.1. Assume that there exists a compact subset $K \subseteq M$, such that $H_1 - H_0 \in C_0^\infty(K)$. Let $\Gamma = \{(H_s, J_s)\}_{s \in \mathbb{R}}$ be a smooth homotopy satisfying (2.5). For $h \in C_0^\infty([0, 1] \times K)$, we define the “homotopy of homotopies”:

$$\Gamma(h) := \{(H_s + h_s, J_s)\}_{s \in \mathbb{R}} \quad (7.4)$$

Lemma 7.2. *Suppose there exists an open neighborhood $\mathcal{O}(\Gamma)$ of 0 in $C_0^\infty([0, 1] \times K)$, such that for every $h \in \mathcal{O}(\Gamma)$ and every pair of critical values $a \in \text{CritVal}(\mathcal{A}^{H_0})$ and $b \in \text{CritVal}(\mathcal{A}^{H_1})$ the following holds:*

(i)

$$\sup_{h \in \mathcal{O}(\Gamma)} B(\Gamma(h), a, b) < +\infty \quad \text{and} \quad \inf_{h \in \mathcal{O}(\Gamma)} A(\Gamma(h), a, b) > -\infty;$$

(ii) *given a compact subset $N \subseteq H_1^{-1}(0)$ and a pair of connected components*

$$(\Lambda_0, \Lambda_1) \subseteq \text{Crit}(\mathcal{A}^{H_0}) \times \mathcal{C}(\mathcal{A}^{H_1}, N),$$

such that $a \leq \mathcal{A}^{H_0}(\Lambda_0)$ and $\mathcal{A}^{H_1}(\Lambda_1) \leq b$, the moduli space $\mathcal{M}^{\Gamma(h)}(\Lambda_0, \Lambda_1)$ is contained in a compact set of $M \times \mathbb{R}$ and admits a uniform bound on the energy, that

is

$$\sup_{u \in \mathcal{M}^{\Gamma(h)}(\Lambda_0, \Lambda_1)} \int_{\mathbb{R}} \|\partial_s u\|^2 ds < +\infty,$$

which only depend on a, b and N .

Then for a generic choice of $h \in \mathcal{O}(\Gamma)$, there exists a homomorphism

$$\Psi^{\Gamma(h)} : RFH(H_1, J_1) \rightarrow RFH(H_0, J_0).$$

Proof. In this proof we will show how to obtain uniform bounds on the moduli spaces corresponding to families of homotopies and later just refer to standard Floer theoretical arguments for the actual definition of the homomorphisms.

Let us fix $h \in \mathcal{O}(\bar{\Gamma})$ and let $\Gamma(h)$ be the corresponding homotopy defined as in (7.4). We will start with proving compactness. Fix two Morse-Smale pairs (f_0, g_0) and (f_1, g_1) on $\text{Crit}(\mathcal{A}^{H_0})$ and $\text{Crit}(\mathcal{A}^{H_1})$, respectively, such that f_0 and f_1 are coercive. For a fixed pair $(p, q) \in \text{Crit}(f_0) \times \text{Crit}(f_1)$ we will show that there exists a compact subset of $M \times \mathbb{R}$ that contains all the broken trajectories with cascades between p and q .

Since H_0 and H_1 satisfy the assumptions of Lemma 3.6, $\text{Crit}(H_0)$ and $\text{Crit}(H_1)$ are closed and discrete. Let $a := \mathcal{A}^{H_0}(p)$, $b := \mathcal{A}^{H_1}(q)$. By assumption, for a fixed $h \in \mathcal{O}(\bar{\Gamma})$ the corresponding values $\alpha := A(\Gamma(h), a, b)$ and $\beta := B(\Gamma(h), a, b)$ are finite. As a result, the sets

$$[a, \beta] \cap \text{Crit}(\mathcal{A}^{H_0}), \quad [\alpha, b] \cap \text{Crit}(\mathcal{A}^{H_1})$$

are finite.

We are going to consider different spaces of trajectories and argue that they all have bounded image in $M \times \mathbb{R}$. The first two spaces consist of unperturbed Floer trajectories between connected components of the critical set of a fixed Hamiltonian:

$$\begin{aligned} \mathfrak{M}_0(a, b) &:= \bigcup_{\substack{\Lambda^\pm \subseteq \text{Crit}(\mathcal{A}^{H_0}) \\ \mathcal{A}^{H_0}(\Lambda^\pm) \in [a, \beta]}} \mathcal{M}(\Lambda^-, \Lambda^+), \\ \mathfrak{M}_1(a, b) &:= \bigcup_{\substack{\Lambda^\pm \subseteq \text{Crit}(\mathcal{A}^{H_1}) \\ \mathcal{A}^{H_1}(\Lambda^\pm) \in [\alpha, b]}} \mathcal{M}(\Lambda^-, \Lambda^+) \end{aligned}$$

Since the pairs (H_0, J_0) and (H_1, J_1) satisfy the assumptions of Theorem 6.1, the images of $\mathfrak{M}_0(a, b)$ and $\mathfrak{M}_1(a, b)$ are bounded in $M \times \mathbb{R}$ (finite unions of bounded sets).

The third space consists of perturbed trajectories connecting components of the critical set of $\mathcal{A}^{H_0}$ with components of the critical set of $\mathcal{A}^{H_1}$:

$$\mathfrak{M}_2(a, b) := \bigcup_{\substack{\Lambda_0 \subseteq \text{Crit}(\mathcal{A}^{H_0}) \\ \mathcal{A}^{H_0}(\Lambda_0) \in [a, \beta]}} \bigcup_{\substack{\Lambda_1 \subseteq \mathcal{C}(\mathcal{A}^{H_1}, N) \\ \mathcal{A}^{H_1}(\Lambda_1) \in [\alpha, b]}} \mathcal{M}_{\Gamma(h)}(\Lambda_0, \Lambda_1).$$

Note that for a compact $N \subseteq (H_1)^{-1}(0)$, the set $\mathfrak{M}_2(a, b)$ is a finite union of sets whose image in $M \times \mathbb{R}$ is by assumption (ii) bounded, so the image of $\mathfrak{M}_2(a, b)$ itself is also bounded.

Now we need to distinguish three different cases to show that the cascades between p and q are passing through $\mathfrak{M}_2(a, b)$. First of all, whenever $0 \notin [\alpha, b]$, then by the definition of α (7.1) there are no cascades passing through the non-compact component of $\text{Crit}(\mathcal{A}^{H_1})$ and hence we can take $N = \emptyset$. If $0 \in [\alpha, b]$, we set $N = K(b)$ where $K(b)$ is the compact set defined in (4.3), since all the cascades passing through the non-compact component of $\text{Crit}(\mathcal{A}^{H_1})$ are in fact passing through $K(b)$. The last case we need to examine is $\mathcal{A}^{H_0}(q) = b = 0$: then we set $N = f_1^{-1}((-\infty, f_1(q)])$ as all the trajectories passing through the non-compact component of $\text{Crit}(\mathcal{A}^{H_1})$ are confined in this case to $f_1^{-1}((-\infty, f_1(q)])$, which is also a compact subset of $(H_1)^{-1}(0)$ due to the coercivity of f_1 .

Let

$$\{(\{u_j^i\}_{j=1}^m, \{t_j^i\}_{j=1}^{m-1})\}_{i=1}^k$$

be a broken trajectory with cascades between p and q . Then for all $i = 1, \dots, k$, $j = 1, \dots, m$

$$u_j^i \in \mathfrak{M}_0(a, b) \cup \mathfrak{M}_1(a, b) \cup \mathfrak{M}_2(a, b),$$

hence there exists a compact subset of $M \times \mathbb{R}$ that contains Floer components of all the broken trajectories with cascades between p and q . In fact, by assumption (i) and (ii) the bounds are uniform and do not depend on the choice of $h \in \mathcal{O}(\Gamma)$.

Recall the evaluation maps (4.2) and define

$$K_{\Gamma(h)}(a, b) := (ev^{-1}(\mathfrak{M}_2(a, b)) \cap ((H_0)^{-1}(0) \times \{0\})). \quad (7.5)$$

Then due to boundedness of $\mathfrak{M}_2(a, b)$ in $M \times \mathbb{R}$ and compactness argument as in Lemma 4.1, we infer that $K_{\Gamma(h)}(a, b)$ is a compact subset of $(H_0)^{-1}(0)$. Uniform bounds on the Floer components together with the fact that both f_0 and f_1 are coercive, imply compactness of the Morse components of all the broken trajectories with cascades between p and q .

Having obtained uniform bounds, we can now apply the usual Floer techniques to our setting: it follows from Theorem 11.1.6 in [2] that for a generic choice of $h \in \mathcal{O}(\Gamma)$ and a pair $(p, q) \in \text{Crit}(f_0) \times \text{Crit}(f_1)$, all the moduli spaces $\mathcal{M}^{\Gamma(h)}(p, q)$ have the structure of a smooth manifold without boundary, of dimension $\mu(p) - \mu(q)$. That allows us to define a homomorphism between the chain complexes $CF(\mathcal{A}^{H_1}, f_1)$ and $CF(\mathcal{A}^{H_0}, f_0)$, by counting perturbed flow lines with cascades between points of equal indices. In order for this map to be well defined, one has to make sure that for all $q \in \text{Crit}(f_1)$ and $a \in \mathbb{R}$

$$\#\{p \in \text{Crit}(f_0) \mid \mu(y) = \mu(x), \mathcal{M}_{\Gamma(h)}(p, q) \neq \emptyset \text{ \& } \mathcal{A}^H(p) \geq a\} < +\infty.$$

With $b = \mathcal{A}^{H_1}(x)$, observe that $\mathcal{M}_{\Gamma(h)}(p, q) \neq \emptyset$ implies that $\mathcal{A}^{H_0}(y) \in [a, \beta]$.

Recall that by Lemma 3.6 the number of connected components of $\text{Crit}(\mathcal{A}^{H_0}) \cap (\mathcal{A}^{H_0})^{-1}([a, \beta] \setminus \{0\})$ is finite and each of them is compact, hence

$$\#\{p \in \text{Crit}(f_0) \mid \mathcal{A}^{H_0}(p) \in [a, \beta] \setminus \{0\}\} < +\infty.$$

On the other hand, if $\mathcal{A}^{H_0}(y) = 0$, then

$$y \in K(\beta) \cup f_0^{-1}((-\infty, \max_{K^{\Gamma(h)}(a, b)} f_0]),$$

with the sets in the union above defined in (4.3) and (7.5), respectively. The first set is compact by Lemma 4.1, as is the second one by virtue of a similar argument and coercivity of f_0 . As compact sets they contain only finitely many critical points of f_0 . This concludes the proof that the map between the chain complexes $CF(\mathcal{A}^{H_1}, f_1)$ and $CF(\mathcal{A}^{H_0}, f_0)$ is well defined. Moreover, for any pair $(p, q) \in \text{Crit}(f_0) \times \text{Crit}(f_1)$ the associated moduli space can be compactified (Theorem 4.2 and Theorem 11.1.16 in [2]) by moduli spaces of lower index. This enables us to show that the associated map of chain complexes commutes with the differentials, hence it descends to the desired homomorphism $\Phi^{\Gamma(h)}$ on the homology level. $\square$

Constructing the above homomorphism is only the first step in the proof of invariance. To prove that it is indeed an isomorphism one has to obtain similar bounds for Γ^{-1} and its concatenations

$$\Gamma \# \Gamma^{-1} := \{(H_{1-|2s-1|}, J_{1-|2s-1|})\}_{s \in \mathbb{R}} \quad (7.6)$$

$$\Gamma^{-1} \# \Gamma := \{(H_{|2s-1|}, J_{|2s-1|})\}_{s \in \mathbb{R}}. \quad (7.7)$$

Note that if $\Gamma, \Gamma^{-1}, \Gamma \# \Gamma^{-1}$ and $\Gamma^{-1} \# \Gamma$ satisfy the assumptions of Lemma 7.2, then there exists a generic h , such that $\Gamma(h), \Gamma(h)^{-1}, \Gamma(h) \# \Gamma(h)^{-1}$ and $\Gamma(h)^{-1} \# \Gamma(h)$ are regular (as intersections of generic sets are generic). In that case, by Lemma 7.2, the corresponding homomorphisms $\Phi^{\Gamma(h)}, \Phi^{\Gamma(h)^{-1}}, \Phi^{\Gamma(h) \# \Gamma(h)^{-1}}$ and $\Phi^{\Gamma(h)^{-1} \# \Gamma(h)}$ are well defined.

The next step would be to ensure that for a fixed generic $\bar{h}$ the following λ -dependent families of homotopies $\{\bar{\Gamma}^\lambda(h)\}_{\lambda \in [0,1]}$ and $\{\tilde{\Gamma}^\lambda(h)\}_{\lambda \in [0,1]}$ families of homotopies

$$\bar{\Gamma}^\lambda(h) := \{(\bar{H}_{\lambda(1-|2s-1|)} + \bar{h}_{\lambda(1-|2s-1|)} + h_s^\lambda, \bar{J}_{\lambda(1-|2s-1|)})\}_{s \in \mathbb{R}}, \quad (7.8)$$

$$\tilde{\Gamma}^\lambda(h) := \{(\bar{H}_{|2\lambda s-1|} + \bar{h}_{|2\lambda s-1|} + h_s^\lambda, \bar{J}_{|2\lambda s-1|})\}_{s \in \mathbb{R}}, \quad (7.9)$$

satisfy assumptions of Lemma 7.2 uniformly in $\lambda \in [0, 1]$ and $h \in \mathcal{O}^\lambda(\bar{\Gamma}, \bar{h}) \subseteq C_0^\infty([0, 1]^2 \times K)$. Observe that those homotopies of homotopies connect $\Gamma(\bar{h}) \# \Gamma(\bar{h})^{-1}$ with a constant homotopy $(\bar{H}_0, \bar{J}_0)$ and $\Gamma(\bar{h})^{-1} \# \Gamma(\bar{h})$ with a constant homotopy $(\bar{H}_1, \bar{J}_1)$, respectively. If they satisfy assumptions of Lemma 7.2 uniformly, then by Theorem 11.3.11 and Proposition 11.2.8 in [2] we can infer that $\Phi^{\Gamma(\bar{h}) \# \Gamma(\bar{h})^{-1}}$ and $\Phi^{\Gamma(\bar{h})^{-1} \# \Gamma(\bar{h})}$ are in fact isomorphisms on the homology level.

The last step would be to ensure that the following R -dependent families of homotopies $\{\bar{\Gamma}^R(h)\}_{R \geq 0}$ and $\{\tilde{\Gamma}^R(h)\}_{R \geq 0}$ defined

$$\bar{\Gamma}^R(h) := \left\{ \left(\bar{H}_{(1+e^{-R})(R+\frac{1}{2}-|s-\frac{1}{2}|)} + \bar{h}_{(1+e^{-R})(R+\frac{1}{2}-|s-\frac{1}{2}|)} + h_s^R, \bar{J}_{(1+e^{-R})(R+\frac{1}{2}-|s-\frac{1}{2}|)} \right) \right\}_{s \in \mathbb{R}},$$

$$\tilde{\Gamma}^R(h) := \left\{ \left(\bar{H}_{(1+e^{-R})(|s-\frac{1}{2}|-R)} + \bar{h}_{(1+e^{-R})(|s-\frac{1}{2}|-R)} + h_s^R, \bar{J}_{(1+e^{-R})(|s-\frac{1}{2}|-R)} \right) \right\}_{s \in \mathbb{R}}$$

satisfy assumptions of Lemma 7.2 uniformly in $R \geq 0$ and $h \in \mathcal{O}^R(\bar{\Gamma}, \bar{h}) \subseteq C_c^\infty([0, +\infty) \times [0, 1] \times K)$. Observe that for $R = 0$, $\bar{\Gamma}^0(h) = \Gamma(\bar{h}) \# \Gamma(\bar{h})^{-1}$ and $\tilde{\Gamma}^0(h) = \Gamma(\bar{h})^{-1} \# \Gamma(\bar{h})$. If $\{\bar{\Gamma}^R(h)\}_{R \geq 0}$ and $\{\tilde{\Gamma}^R(h)\}_{R \geq 0}$ satisfy assumptions of Lemma 7.2 uniformly, then by Proposition 11.2.9 in [2] we can infer that

$$\Phi^{\Gamma(\bar{h})^{-1}} \circ \Phi^{\Gamma(\bar{h})} = \Phi^{\Gamma(\bar{h}) \# \Gamma(\bar{h})^{-1}} \quad \text{and} \quad \Phi^{\Gamma(\bar{h})} \circ \Phi^{\Gamma(\bar{h})^{-1}} = \Phi^{\Gamma(\bar{h})^{-1} \# \Gamma(\bar{h})}.$$

In the end we can conclude that $\Phi^{\Gamma(\bar{h})} \circ \Phi^{\Gamma(\bar{h})^{-1}}$ and $\Phi^{\Gamma(\bar{h})^{-1}} \circ \Phi^{\Gamma(\bar{h})}$ are isomorphisms, so $\Phi^{\Gamma(\bar{h})}$ is an isomorphism between $RFH(H_1, J_1)$ and $RFH(H_0, J_0)$.

8. TENTACULAR RFH

To prove Theorem 1.4, we will show that the conditions in the definition of tentacular Hamiltonians guarantee the appropriate bounds for the Rabinowitz Floer equations based on the estimates in [19]. First we verify Hypotheses (H1)-(H3) from [19] for admissible Hamiltonians and then apply Theorem 1. therein to obtain uniform bounds on the Floer trajectories necessary to construct the Rabinowitz Floer homology. Afterwards, we show that hypothesis (h4) for strongly tentacular Hamiltonians implies uniform continuity of (PO+). We will combine it with the result from Theorem 3.4, which ensures genericity of the Morse-Bott property in the affine space of compactly supported perturbations under the assumption of the uniform continuity of (PO+). This will allow us to construct the Rabinowitz Floer homology on a generic subset of strongly tentacular Hamiltonians. Finally, we will argue how to obtain uniform estimates for homotopies of homotopies in order to prove invariance of Rabinowitz Floer homology of the choice of the almost complex structure and under compactly supported perturbations of the Hamiltonians. That will allow us to extend the definition of Rabinowitz Floer homology to the whole set of strongly tentacular Hamiltonians.

8.1. Strongly tentacular Hamiltonians. Let us verify Hypotheses (H1)-(H3) from [19] for admissible Hamiltonians. Observe that Hypothesis (H1) and (H2) are identical to (h1) and (h2). In the following lemmas we will show how Hypothesis (H3) follows from (h1) and (h3).

Lemma 8.1. *For a Hamiltonian H satisfying Axiom (h1) and (h2) we have*

$$|\nabla H(x)| \geq c, \text{ and } \|D^2 H(x)\| \leq c|\nabla H(x)|, \quad (8.1)$$

for some $c > 0$ as $|x| \rightarrow \infty$.

Proof. From the conditions on $X^\dagger \in \mathcal{L}(\mathbb{R}^{2n})$ and $H \in \mathcal{H}(\mathbb{R}^{2n})$ we have that $|X^\dagger(x)| \leq c + c|x|$, $\|D^2 H(x)\| \leq c$ and $|\nabla H(x)| \leq c + c|x|$. Axiom (h1) implies that

$$|\nabla H(x)||X^\dagger(x)| \geq \langle \nabla H(x), X^\dagger(x) \rangle \geq -c' + c|x|^2,$$

and therefore

$$|\nabla H(x)| \geq \frac{-c' + c|x|^2}{|X^\dagger(x)|} \geq \frac{-c' + c|x|^2}{c + c|x|} \geq c|x| - c.$$

which implies the above estimates. $\square$

Lemma 8.2. *For $X^\dagger \in \mathcal{L}(\mathbb{R}^{2n})$ satisfying Axiom (h1) and (h2) we have that*

$$|X^\dagger(x)| \geq c, \text{ and } \|DX^\dagger(x)\| \leq c|X^\dagger(x)| \quad (8.2)$$

for some $c > 0$ as $|x| \rightarrow \infty$.

Proof. By the same token as before we have

$$|X^\dagger(x)| \geq \frac{-c' + c|x|^2}{|\nabla H(x)|} \geq \frac{-c' + c|x|^2}{c + c|x|} \geq c|x| - c,$$

which implies the above estimates. $\square$

Lemma 8.3. *For an admissible Hamiltonian there exists constants $\delta > 0$ and $c_\delta > 0$ and a Liouville vector field $X \in \mathcal{L}(\mathbb{R}^{2n})$ such that $dH(X)(x) \geq c_\delta$ for all $x \in H^{-1}(-\delta, \delta)$.*

Proof. Choose $0 < \epsilon < 1$ and consider the Liouville vector field $X_\epsilon = (1 - \epsilon)X^\dagger + \epsilon X^\dagger$. From Axiom (h1) we have

$$\begin{aligned} dH(X_\epsilon)(x) &\geq (1 - \epsilon)dH(X^\dagger)(x) + \epsilon dH(X^\dagger)(x) \\ &\geq (1 - \epsilon)dH(X^\dagger)(x) - \epsilon c' + \epsilon c|x|^2. \end{aligned}$$

By contact type Axiom (h3) we have that $dH(X^\dagger)(x) \geq \varepsilon' > 0$ for all $x \in \Sigma \cap \{x \in \mathbb{R}^{2n} : |x| \leq c\}$, and thus for small enough ε the above inequality yields

$$dH(X_\epsilon)(x) \geq c_* + c_*|x|^2, \quad c_* > 0,$$

for all $x \in \Sigma$ — a uniform contact type condition. Lemma 8.2 applies to X_ϵ since the above holds in a tubular neighborhood of the ball of radius 2.

Consider the normalizing gradient flow ψ^σ defined by the equation $\dot{x} = \frac{\nabla H(x)}{\|\nabla H(x)\|^2}$. This provides local coordinates $(x, \sigma) \in \Sigma \times (-\delta, \delta)$, for some $\delta > 0$, via $H(\psi^\sigma(x)) = \sigma$. By the growth properties on ∇H and X_ϵ and the uniform contact type condition above we have that

$$c(x) := \frac{\langle \nabla H(x), X_\epsilon(x) \rangle}{|\nabla H(x)||X_\epsilon(x)|} = \frac{dH(X_\epsilon)(x)}{|\nabla H(x)||X_\epsilon(x)|} \geq \frac{c_* + c_*|x|^2}{(c + c|x|)^2} \geq c_* > 0,$$

for some $c > 0$ and for all $x \in \Sigma$. In order to extend the estimate to a neighborhood of Σ we estimate the σ -derivative of $c(\psi^\sigma(x))$ using Lemma 8.1 and 8.2:

$$\begin{aligned} \left| \frac{d}{ds} c(\psi^\sigma(x)) \right| &= \left| \left\langle \frac{d}{ds} \left(\frac{\nabla H(x)}{|\nabla H(x)|} \right), \frac{X_\epsilon}{|X_\epsilon|} \right\rangle + \left\langle \frac{\nabla H(x)}{|\nabla H(x)|}, \frac{d}{ds} \left(\frac{X_\epsilon}{|X_\epsilon|} \right) \right\rangle \right| \\ &\leq \left| \left\langle \frac{D^2 H(x) \dot{x}}{|\nabla H(x)|} - \frac{\langle \nabla H(x), D^2 H(x) \dot{x} \rangle \nabla H(x)}{|\nabla H(x)|^3}, \frac{X_\epsilon}{|X_\epsilon|} \right\rangle \right| \\ &\quad + \left| \left\langle \frac{\nabla H(x)}{|\nabla H(x)|}, \frac{DX_\epsilon \dot{x}}{|X_\epsilon|} - \frac{\langle X_\epsilon, DX_\epsilon \dot{x} \rangle X_\epsilon}{|X_\epsilon|^3} \right\rangle \right| \leq c|\dot{x}| \leq c, \end{aligned}$$

uniformly for all $x \in \Sigma$. This implies that there exists a $\delta > 0$ such that $c(\psi^\sigma(x)) \geq \frac{1}{2}c_* > 0$, for all $|\sigma| < \delta$, which proves that $c(x) \geq \frac{1}{2}c_*$ on $H^{-1}(-\delta, \delta)$. Consequently, $dH(X_\epsilon)(x) \geq \frac{1}{2}c_*(c_* + c_*|x|^2) \geq c_\delta > 0$, which completes the proof. We denote the vector field X_ϵ by X . $\square$

Remark 8.4. We have thus concluded that Axioms (h1)-(h3) imply Hypothesis (H3) in [19]. Notice that if we were to consider only asymptotically regular vector fields, then the two sets of assumptions (h1)-(h3) and (H1)-(H3) would in fact be equivalent.

In the following lemma we will show yet another property of admissible Hamiltonians: that the length of the non-degenerate closed characteristics is linearly bounded by action.

Lemma 8.5. *Let H be an admissible Hamiltonian. Then there exists a constant $\bar{c} > 0$ and an open subset $B(H) \subseteq C_c^\infty(\mathbb{R}^{2n})$, such that for all $h \in B(H)$ and all $(v, \eta) \in \text{Crit}(\mathcal{A}^{H+h})$, $\eta \neq 0$*

$$l(v) = \int |\partial_t v| dt \leq \bar{c} |\mathcal{A}^{H+h}(v, \eta)|.$$

Proof. By axiom (h1) $dH(X^\dagger)$ is quadratic and by (h2) X_H is linear. Therefore, there exist $c, c' > 0$, such that for all $x \in \mathbb{R}^{2n}$

$$|X_H(x)| \leq c(dH(X^\dagger)(x) + c'). \quad (8.3)$$

On the other hand, by Lemma 8.3 there exists a Liouville vector field $X^\dagger$, such that $dH(X^\dagger)$ is bounded away from 0 in the neighborhood of $H^{-1}(0)$ therefore

$$\inf_{H^{-1}(0)} dH(X^\dagger) \geq \frac{1}{c}. \quad (8.4)$$

Fixing c to satisfy both (8.3) and (8.4) we define a subset of $C_c^\infty(\mathbb{R}^{2n})$ in the following way:

$$B(H) := \left\{ h \in C_c^\infty(\mathbb{R}^{2n}) \left| \begin{array}{l} |X_{H+h}(x)| < 2c(d(H+h)(X^\dagger)(x) + 2c') \quad \forall x \in \mathbb{R}^{2n} \\ d(H+h)(X^\dagger)(x) > \frac{1}{2c} \quad \forall x \in (H+h)^{-1}(0) \end{array} \right. \right\}$$

As both conditions defining $B(H)$ are open, hence $B(H)$ itself is open in $C_c^\infty(\mathbb{R}^{2n})$. Moreover, by (8.3) and (8.4) we can see that $0 \in B(H)$ thus $B(H)$ is an open neighborhood of 0 in $C_c^\infty(\mathbb{R}^{2n})$.

Fix $h \in B(K_a)$ and take $(v, \eta) \in \text{Crit}(\mathcal{A}^{H+h})$, with $\eta \neq 0$. As $h \in B(K_a)$ we see that the period of the loop is bounded by the action:

$$|\mathcal{A}^{H+h}(v, \eta)| \geq |\mathcal{A}^{H+h}(v, \eta)| = |\eta| \int d(H+h)(X^\dagger) > \frac{|\eta|}{2c}.$$

Let us now calculate the length of the loop v . Using conditions defining $B(H)$ we get a uniform bound by the action:

$$\begin{aligned} l(v) &= \int |\partial_t v| = |\eta| \int |X_{H+h}(v)| \leq 2c|\eta| \int (d(H+h)(X^\dagger) + 2c') \\ &= 2c |\mathcal{A}^{H+h}(v, \eta) + \eta 2c'| \leq 2c(|\mathcal{A}^{H+h}(v, \eta)| + 2|\eta|c') \\ &< 2c |\mathcal{A}^{H+h}(v, \eta)| (1 + 4c'c) = \bar{c} |\mathcal{A}^{H+h}(v, \eta)|, \end{aligned}$$

with $\bar{c} := 2c(1 + 4c'c)$. □

In the following lemma we will show that the last property, (h4) implies the uniform continuity of (PO).

Lemma 8.6. *A tentacular Hamiltonian satisfies the axiom of uniform continuity of (PO).*

Proof. We will start by constructing a sequence of compact sets $K_a \subseteq \mathbb{R}^{2n}$ and the corresponding sequence of open subsets $B(K_a) \subseteq C_0^\infty(K_a)$ and later we show that they satisfy the hypothesis of uniform continuity of (PO).

By assumption (h4) we are given a function $F : \mathcal{N}(\Sigma) \rightarrow \mathbb{R}$, where $\mathcal{N}(\Sigma)$ is a neighborhood of $\Sigma := H^{-1}(0)$. Define

$$A := \{x \in \mathcal{N}(\Sigma) \mid \{H, F\} \neq 0\}, \quad B := \{x \in \mathcal{N}(\Sigma) \mid \{H, \{H, F\}\} > 0\}.$$

Then by assumption (h4) there exist $r > 0$, such that for an open ball $B(r)$ (with center in the origin) we have $\Sigma \setminus B(r) \subseteq A \cup B$.

Let $B(H) \subseteq C_c^\infty(\mathbb{R}^{2n})$ be the open neighborhood of 0 from Lemma 8.5. We define a subset $\tilde{B}(H) \subseteq B(H)$ in the following way: $h \in \tilde{B}(H)$ if it satisfies the following conditions:

- (a) $(H + h)^{-1}(0) \setminus B(r) \subseteq A \cup B$;
- (b) for all $x \in (H + h)^{-1}(0) \setminus (B(r) \cup A)$

$$|\{H, \{h, F\}\}(x) + \{h, \{H, F\}\}(x) + \{h, \{h, F\}\}(x)| < \{H, \{H, F\}\}(x);$$

- (c) for all $x \in (H + h)^{-1}(0) \setminus (B(r) \cup B)$

$$|\{h, F\}(x)| < |\{H, F\}(x)|.$$

First observe that $\tilde{B}(H) \neq \emptyset$ as by axiom (h4) we know that $0 \in \tilde{B}(H)$. Moreover, since the sets A and B are open and so are the above conditions, it follows that $\tilde{B}(H)$ is an open neighborhood of 0 in $C_c^\infty(\mathbb{R}^{2n})$. Now for every $a \in \mathbb{N}$, let K_a be a closed ball of radius $r + 2\bar{c}a$, where $\bar{c}$ is as in Lemma 8.5

$$K_a := B_0(r + 2da), \quad B(K_a) := C_0^\infty(K_a) \cap \tilde{B}(H).$$

Naturally, $\{K_a\}_{a \in \mathbb{N}}$ is an exhaustion of $\mathbb{R}^{2n}$ by compact sets. We will show that $\{B(K_a)\}_{a \in \mathbb{N}}$ satisfies the hypothesis of uniform continuity of (PO). Fix $a \in \mathbb{N}$, $h \in B(K_a)$ and take $(v, \eta) \in \text{Crit}(\mathcal{A}^{H+h})$, with $0 < |\mathcal{A}^{H+h}(v, \eta)| \leq a$. By Lemma 8.5 we know that the length of the loop is uniformly bounded by the action $l(v) \leq \bar{c}a$. Let $\bar{v} := \int v(t)dt$ be the average then $|v(t) - \bar{v}| \leq l(v) \leq \bar{c}a$ and thus $v \subseteq B_{\bar{v}}(\bar{c}a)$. We will show now that $\bar{v}$ satisfies the bound: $|\bar{v}| \leq \bar{c}a + r$. Suppose not, then it would follow that $|v(t)| \geq ||\bar{v}| - |v(t) - \bar{v}|| > r$, hence by (a) we have

$$v(S^1) \subseteq (H + h)^{-1}(0) \setminus B(r) \subseteq A \cup B. \quad (8.5)$$

The function $F \circ v$ obtains its maximum at the point $t_0 \in S^1$, where

$$F \circ v(t_0) = \max_{S^1} F \circ v, \quad \frac{d}{dt} F \circ v(t_0) = 0 \quad \text{and} \quad \frac{d^2}{dt^2} F \circ v(t_0) \leq 0.$$

Since $(v, \eta) \in \text{Crit}(\mathcal{A}^{H+h})$, we have

$$0 = \frac{d}{dt} F \circ v(t_0) = dF(\partial_t v(t_0)) = \eta dF(X_{H+h}(v(t_0))) = \eta \{H + h, F\}(v(t_0)), \quad (8.6)$$

$$0 \geq \frac{d^2}{dt^2} F \circ v(t_0) = \eta^2 d(\{H + h, F\})(X_{H+h}(v(t_0))) = \eta^2 \{H + h, \{H + h, F\}\}(v(t_0)). \quad (8.7)$$

By (8.5) we know that $v(t_0) \in A \cup B$. Therefore, we can consider the two following cases:

(1) $v(t_0) \notin A$.

Then $v(t_0) \in (H+h)^{-1}(0) \setminus (B(r) \cup A)$ and we have that (b) contradicts with (8.7) in the following way:

$$\begin{aligned} 0 &\geq \frac{1}{\eta^2} \frac{d^2}{dt^2} F \circ v(t_0) = \{H+h, \{H+h, F\}\}(v(t_0)) \\ &= \{H, \{h, F\}\}(v(t_0)) + \{h, \{H, F\}\}(v(t_0)) \\ &\quad + \{h, \{h, F\}\}(v(t_0)) + \{H, \{H, F\}\}(v(t_0)) > 0. \end{aligned}$$

(2) $v(t_0) \notin B$.

Then $v(t_0) \in (H+h)^{-1}(0) \setminus (B(r) \cup B)$ and we have that (c) contradicts with (8.6):

$$\begin{aligned} 0 &= \left| \frac{1}{\eta} \frac{d}{dt} F \circ v(t_0) \right| = |\{H+h, F\}(v(t_0))| \\ &\geq |\{H, F\}(v(t_0))| - |\{h, F\}(v(t_0))| > 0. \end{aligned}$$

Summarizing, we can conclude that for $h \in B(K_a)$, if (v, η) is a non-degenerate critical point of $\mathcal{A}^{H+h}$ with action $0 < |\mathcal{A}^{H+h}(v, \eta)| \leq a$, then $v(S^1) \subseteq K_a$. In other words, (PO) is uniformly continuous at H . $\square$

However, to construct Rabinowitz Floer homology, we need stronger condition, namely uniform continuity of (PO+). In the following lemma we will show how using a coercive function in Axiom (h4) assures uniform continuity of (PO+). As a result, (PO+) is uniformly continuous for the strongly tentacular Hamiltonians. In fact we can state the result in a greater generality, for a Hamiltonian on any exact symplectic manifold, not necessarily on $\mathbb{R}^{2n}$.

Lemma 8.7. *Let H, F be smooth Hamiltonians on a symplectic manifold (M, ω) . If F is coercive and the set*

$$K_0 := \left\{ x \in H^{-1}(0) \mid \{H, \{H, F\}\}(x) \leq 0 \quad \text{and} \quad \{H, F\}(x) = 0 \right\},$$

is compact, then (PO+) is uniformly continuous at H .

Proof. For $n = 0, 1, 2, \dots$ let us denote

$$K_n := F^{-1}((-\infty, n + \sup_{K_0} F]).$$

By assumption F is coercive so every K_n is compact and we have an exhaustion of M by compact sets

$$M = \bigcup_{n \in \mathbb{N}} K_n.$$

We will show that the family $\{K_n\}_{n \in \mathbb{N}}$ satisfies the uniform continuity of (PO+) hypothesis. More precisely, we will show that for every $n = 1, 2, \dots$, whenever $h \in C_0^\infty(K_n)$ then

$$(v, \eta) \in \text{Crit}(\mathcal{A}^{H+h}) \setminus (\mathcal{A}^{H+h})^{-1}(0) \implies v(t) \in K_n \quad \forall t \in S^1.$$

Fix $h \in C_0^\infty(K_n)$, take $(v, \eta) \in \text{Crit}(\mathcal{A}^{H+h}) \setminus (\mathcal{A}^{H+h})^{-1}(0)$ and calculate the maximum of F over v . There exists a $t_0 \in S^1$, such that

$$F \circ v(t_0) = \max_{S^1} F \circ v, \quad \frac{d}{dt} F \circ v(t_0) = 0 \quad \& \quad \frac{d^2}{dt^2} F \circ v(t_0) \leq 0.$$

Now suppose, that $v(S^1) \setminus K_n \neq \emptyset$. Naturally, then $v(t_0) \notin K_n \supseteq \text{supp}(h)$. Since $(v, \eta) \in \text{Crit}(\mathcal{A}^{H+h})$, we would have

$$\begin{aligned} 0 &= \frac{d}{dt} F \circ v(t_0) = dF(\partial_t v(t_0)) = \eta dF(X_H(v(t_0))) = \eta \{H, F\}(v(t_0)), \\ 0 &\geq \frac{d^2}{dt^2} F \circ v(t_0) = \eta^2 d(\{H, F\})(X_H(v(t_0))) = \eta^2 \{H, \{H, F\}\}(v(t_0)). \end{aligned}$$

But this would imply that $v(t_0) \in K_0 \subseteq K_n$, which leads us to contradiction and concludes the proof. $\square$

8.2. Estimates for Floer trajectories. In this subsection we will recall Theorem 1. from [19], which ensures uniform estimates for tentacular Hamiltonians.

Let us consider the following setting: Let J_0 be the standard almost complex structure on $\mathbb{R}^{2n}$. Fix an admissible Hamiltonian $H : \mathbb{R}^{2n} \rightarrow \mathbb{R}$; a compact set $K \subseteq \mathbb{R}^{2n}$, $K \neq \emptyset$; an open, precompact subset $\mathcal{V} \subseteq \mathbb{R}^{2n}$ and a constant $\eta > 0$. Let $\mathcal{O}(H) \subseteq C_0^\infty(K)$ be the open subset associated to H and K by Lemma 2.1 [19] and let $\tilde{c}, \varepsilon_0 > 0$ be the constants associated to H and K by Lemma 3.1 [19].

Theorem 8.8. *Let $\Gamma := \{(H_s, J_s)\}_{s \in \mathbb{R}}$ be a homotopy as described in (2.5) with*

$$\begin{aligned} H_s &\in H + \mathcal{O}(H), \quad J_s \in \mathcal{J}^\infty(\mathbb{R}^{2n}, \omega_0, \mathcal{V} \times ((-\infty, \eta) \cup (\eta, \infty))), \\ \left(\tilde{c} + \frac{1}{\varepsilon_0} \|J\|_{L^\infty}^{\frac{3}{2}} \right) \|\partial_s H_s\|_{L^\infty} &< \frac{1}{8}. \end{aligned} \tag{8.8}$$

If we fix $a, b \in \mathbb{R}$ and a compact subset $N \subseteq H_1^{-1}(0)$, then for each pair (Λ_0, Λ_1) of connected components

$$\begin{aligned} \Lambda_0 &\subseteq \text{Crit}(\mathcal{A}^{H_0}) \cap (\mathcal{A}^{H_0})^{-1}([a, \infty)), \\ \Lambda_1 &\subseteq \mathcal{C}(\mathcal{A}^{H_1}, N) \cap (\mathcal{A}^{H_1})^{-1}((-\infty, b]), \end{aligned}$$

the image in $\mathbb{R}^{2n+1}$ of the associated moduli space $\mathcal{M}^\Gamma(\Lambda_0, \Lambda_1)$ is uniformly bounded.

Note that the theorem presented above varies slightly from Theorem 1. in [19], as we consider N to be a compact subset of $H_1^{-1}(0)$ rather than $H_0^{-1}(0)$. This change is due to the fact that we dropped the assumption on bounded topology of the hypersurface and thus have to consider the Morse function used in the construction of the cascades to have compact sublevel sets rather than compact superlevel sets as it was in [21]. Yet, the change does not influence the arguments in the proof - we just assume that the Floer trajectories end in a compact subset of $\mathbb{R}^{2n}$ rather than start in it. Thus the theorem holds true also in this setting.

8.3. RFH for tentacular Hamiltonians. The goal of this subsection is to prove Theorem 1.4. We do it first by proving in Lemma 8.9 that Rabinowitz Floer homology is well defined on a dense subset of pairs of the strongly tentacular Hamiltonians and the almost complex structures. Secondly, in Lemma 8.10 we show that the Rabinowitz Floer homology on this set is in fact independent of the choice of the almost complex structure. Finally, in Lemma 8.12 we show that the Rabinowitz Floer homology is invariant under small enough compactly supported perturbations of the Hamiltonians, allowing us to define the Rabinowitz Floer homology on the whole subset of strongly tentacular Hamiltonians.

Let J_0 be the standard almost complex structure on $\mathbb{R}^{2n}$. Denote

$$\begin{aligned} \mathcal{J}_\eta^\mathcal{V} &:= \mathcal{J}^\infty(\mathbb{R}^{2n}, \omega_0, \mathcal{V} \times ((-\infty, \eta) \cup (\eta, \infty))), \\ \mathcal{J}_\star &:= \bigcup_{\eta > 0} \bigcup_{\mathcal{V} \subseteq \mathbb{R}^{2n}} \mathcal{J}_\eta^\mathcal{V}, \end{aligned}$$

where the union is taken over all $\eta > 0$ and all open, pre-compact subsets $\mathcal{V} \subseteq \mathbb{R}^{2n}$. This $\mathcal{J}_\star$ is a contractible subspace of $\mathcal{J}^\infty(\mathbb{R}^{2n}, \omega_0)$ with the limit topology. Recall that by $\mathcal{H}(\mathbb{R}^{2n})$ we denote the set of strongly tentacular Hamiltonians.

Lemma 8.9. *There exists a dense subset of $\mathcal{H}(\mathbb{R}^{2n}) \times \mathcal{J}_\star$ on which Rabinowitz Floer homology is well defined.*

Proof. First observe that $\mathcal{H}(\mathbb{R}^{2n})$ is open under compact perturbations. In other words for every $H \in \mathcal{H}(\mathbb{R}^{2n})$ there exists an open subset $B(H) \subseteq C_c^\infty(\mathbb{R}^{2n})$, such that $H + B(H) \subseteq \mathcal{H}(\mathbb{R}^{2n})$. By Lemma 8.7 we know that $(\text{PO}+)$ is uniformly continuous at H , hence by Theorem 3.4 for $H \in \mathcal{H}(\mathbb{R}^{2n})$ there exists a generic subset $A(H) \subseteq B(H)$, such that for every $h \in A(H)$ the corresponding $\mathcal{A}^{H+h}$ satisfies (MB) and its set of critical values is discrete. Denote

$$\mathcal{H}^{reg}(\mathbb{R}^{2n}) := \bigcup_{H \in \mathcal{H}(\mathbb{R}^{2n})} (H + A(H)).$$

Now fix $H \in \mathcal{H}^{reg}(\mathbb{R}^{2n})$. By uniform continuity of $(\text{PO}+)$ at H all the nondegenerate periodic orbits of X_H on $H^{-1}(0)$ are uniformly bounded. In particular, there exists an open, precompact set $\mathcal{V} \subseteq \mathbb{R}^{2n}$ containing all of them. By Lemma 3.5 the infimum of η parameter over $\text{Crit}(\mathcal{A}^H) \setminus (H^{-1}(0) \times \{0\})$ is bounded away from 0. In other words, for $\eta > 0$ small enough all the non-degenerate periodic orbits are contained in $\mathcal{V} \times ((-\infty, \eta) \cup (\eta, \infty))$. On the other hand, we can apply Theorem 8.8 to a “constant” homotopy $(H_s, J_s) \equiv (H, J)$, $\forall s \in \mathbb{R}$ with $J \in \mathcal{J}_\eta^\mathcal{V}$ obtaining uniform bounds on the moduli spaces $\mathcal{M}_{J,H}(\Lambda^-, \Lambda^+)$ for any pair of connected components $\Lambda^-, \Lambda^+ \subseteq \text{Crit}(\mathcal{A}^H)$. In particular, the estimates on the moduli spaces do depend continuously on the choice of the almost complex structure. That allows us to apply Theorem 5.1 and Theorem 6.1 to infer that for a generic subset of $\mathcal{J}_\eta^\mathcal{V}$ the corresponding $\text{RFH}(H, J)$ is well defined. As the result holds for any big enough $\mathcal{V} \subseteq \mathbb{R}^{2n}$ and small enough $\eta > 0$, we can conclude that the Rabinowitz Floer homology is well defined for a generic subset of $\mathcal{H}(\mathbb{R}^{2n}) \times \mathcal{J}_\star$. $\square$

Lemma 8.10. *For any $H \in \mathcal{H}^{reg}(\mathbb{R}^{2n})$ the associated Rabinowitz Floer homology is independent of the choice of the almost complex structure.*

Proof. Fix $H \in \mathcal{H}^{reg}(\mathbb{R}^{2n})$ and let $J_0, J_1 \in \mathcal{J}_\star$ be two almost complex structures for which $RFH(J_0, H)$ and $RFH(J_1, H)$ are well defined. There exists a homotopy $\bar{\Gamma} := \{(H, J_s)\}_{s \in \mathbb{R}}$ between (J_0, H) and $RFH(J_1, H)$ satisfying (2.5) and with $J_s \in \mathcal{J}_\eta^\mathcal{V}$ for some small enough $\eta > 0$ and big enough, open, pre-compact $\mathcal{V} \subseteq \mathbb{R}^{2n}$. Fix a compact set $K \subseteq \mathbb{R}^{2n}$, $K \neq \emptyset$ and let $\mathcal{O}(H) \subseteq C_0^\infty(K)$ be the open subset associated to H and K by Lemma 2.1 [19] along with $\tilde{c}, \varepsilon_0 > 0$ being the constants associated to H and K by Lemma 3.1 [19]. Denote

$$\mathcal{O}(J_1, J_2) := \left\{ h \in C_0^\infty([0, 1] \times K) \left| \begin{array}{l} \|\partial_s h\|_\infty < \frac{1}{16} \left(\tilde{c} + \frac{1}{\varepsilon_0} \|J_s\|_\infty^{\frac{3}{2}} \right)^{-1} \\ \forall s \in [0, 1] \quad h_s \in \mathcal{O}(H) \end{array} \right. \right\}$$

Then $\mathcal{O}(J_1, J_2)$ is a neighborhood of 0 in $C_0^\infty([0, 1] \times K)$. Moreover, for every $h \in \mathcal{O}(J_1, J_2)$ the corresponding homotopy $\Gamma(h)$ defined

$$\Gamma(h) := \{(H + h_s, J_s)\}_{s \in \mathbb{R}}$$

satisfies inequality (8.8) and the Hamiltonian part of the homotopy lies in $H + \mathcal{O}(H)$. In other words, for every $h \in \mathcal{O}(J_1, J_2)$ the corresponding homotopy $\Gamma(h)$ satisfies the assumptions of Theorem 8.8 and as a result one obtains uniform $L^\infty \times \mathbb{R}$ bounds on the corresponding moduli spaces and uniform bounds on the energy as shown in Proposition 3.3 [19]. Moreover, using the linearity condition between the action and the η , as stated in Lemma 2.1 [19], together with inequality (8.8), we can directly apply Corollary 3.8 from [6] to show that every $\Gamma(h)$ satisfies the Novikov conditions. In fact the estimates do not depend on the chosen perturbation, but can be obtained uniformly over the whole set $\mathcal{O}(J_1, J_2)$. In consequence, the homotopy $\{H, J_s\}_{s \in \mathbb{R}}$ and the set $\mathcal{O}(J_1, J_2)$ satisfy the assumptions of Lemma 7.2 and we can deduce that for a generic choice of $h \in \mathcal{O}(J_1, J_2)$ there exists a homomorphism

$$\Psi^{\Gamma(h)} : RFH(H, J_2) \rightarrow RFH(H, J_1).$$

Note that for every $h \in \mathcal{O}(J_1, J_2)$ the corresponding homotopies $\Gamma(h)^{-1}$, $\Gamma(h) \# \Gamma(h)^{-1}$ and $\Gamma(h)^{-1} \# \Gamma(h)$ as in (7.6) and (7.7) also satisfy inequality (8.8) and their Hamiltonian parts lie in $H + \mathcal{O}(H)$. By the same argument as the one above, we can conclude that for a generic choice of $h \in \mathcal{O}(J_1, J_2)$ all four homomorphisms, namely $\Psi^{\Gamma(h)}$, $\Psi^{\Gamma(h)^{-1}}$, $\Psi^{\Gamma(h) \# \Gamma(h)^{-1}}$ and $\Psi^{\Gamma(h)^{-1} \# \Gamma(h)}$ are well defined. The set $\mathcal{O}(J_1, J_2)$ is convex and open, hence for every fixed, generic $\bar{h} \in \mathcal{O}(J_1, J_2)$ the set

$$\mathcal{O}^\lambda(\bar{\Gamma}, \bar{h}) := \left\{ h \in C_0^\infty([0, 1]^2 \times K) \mid 2\lambda\bar{h} + h^\lambda \in 2\mathcal{O}(J_1, J_2) \right\},$$

is an open subset of $C_0^\infty([0, 1]^2 \times K)$. Moreover, for every $h \in \mathcal{O}^\lambda(\bar{\Gamma}, \bar{h})$ the corresponding homotopies $\bar{\Gamma}^\lambda(h)$ and $\tilde{\Gamma}^\lambda(h)$ also satisfy inequality (8.8) and their Hamiltonian parts lie in $H + \mathcal{O}(H)$. Therefore, the corresponding moduli spaces are uniformly bounded by Theorem 8.8 and the homotopies satisfy assumptions of Lemma 7.2 uniformly, which proves that $\Psi^{\Gamma(\bar{h}) \# \Gamma(\bar{h})^{-1}}$ and $\Psi^{\Gamma(\bar{h})^{-1} \# \Gamma(\bar{h})}$ are isomorphisms on the homology level. Using similar arguments one can prove uniform bounds for the R -parametric families of homotopies, which assures that the homomorphism of a concatenations $\Gamma(\bar{h}) \# \Gamma(\bar{h})^{-1}$ and $\Gamma(\bar{h})^{-1} \# \Gamma(\bar{h})$

are the composition of the homomorphisms $\Psi^{\Gamma(\bar{h})}$ and $\Psi^{\Gamma(\bar{h})^{-1}}$. This proves that $\Psi^{\Gamma(\bar{h})}$ is an isomorphism, which concludes the proof. $\square$

Remark 8.11. In view of Lemma 8.10, from now on we will omit the almost complex structure from the notation of the Rabinowitz Floer homology of $H \in \mathcal{H}^{reg}(\mathbb{R}^{2n})$.

Lemma 8.12. *For any $H \in \mathcal{H}(\mathbb{R}^{2n})$ there exists an open neighborhood $\tilde{\mathcal{O}}(H) \subseteq C_c^\infty(\mathbb{R}^{2n})$, such that for every pair $h_0, h_1 \in \tilde{\mathcal{O}}(H)$ with $H + h_0, H + h_1 \in \mathcal{H}^{reg}(\mathbb{R}^{2n})$*

$$RFH(H + h_0) \cong RFH(H + h_1).$$

Proof. Let $B(H) \subseteq C_c^\infty(\mathbb{R}^{2n})$ be the open subset and let $\{K_m\}_{m \in \mathbb{N}}$ be the sequence of compact subsets in $\mathbb{R}^{2n}$ guaranteed by the uniform continuity of (PO+) at H . Fix one of the compact subsets $K := K_m$ and let $\mathcal{O}(H) \subseteq C_0^\infty(K)$ be the convex, open subset from Lemma 2.1 [19] along with $\tilde{c}, \varepsilon_0 > 0$ being the constants associated to H and K by Lemma 3.1 [19]. Denote

$$\tilde{\mathcal{O}}(H) := \left\{ h \in \mathcal{O}(H) \cap B(H) \mid \|h\|_\infty < 2^{-7} \left(\tilde{c} + \frac{\sqrt{8}}{\varepsilon_0} \right)^{-1} \right\}.$$

Fix $h_0, h_1 \in \tilde{\mathcal{O}}(H)$. By uniform continuity of (PO+) at H and Lemma 3.5 there exists an open, precompact subset $\mathcal{V} \subseteq \mathbb{R}^{2n}$ and a constant $\mathfrak{y} > 0$, such that all nondegenerate critical points of $\mathcal{A}^{H+h_0}$ and $\mathcal{A}^{H+h_1}$ lie in $\mathcal{V} \times ((-\infty, \mathfrak{y}) \cup (\mathfrak{y}, \infty))$. By Theorem 5.1 the sets of regular almost complex structures corresponding to $H + h_0$ and $H + h_1$, respectively, are dense in $\mathcal{J}_\mathfrak{y}^\mathcal{V}$. Moreover, this set is contractible. Therefore, in this set there exist two almost complex structures J_0 and J_1 regular in the sense of Theorem 5.1 for $H + h_0$ and $H + h_1$ respectively, which are connected by a homotopy $\{J_s\}_{s \in \mathbb{R}}$ with $J_s \in \mathcal{J}_\mathfrak{y}^\mathcal{V}$ and

$$s \leq 0 \quad J_s := J_1, \quad s \geq 1 \quad J_s := J_2 \quad \& \quad \|J_s\|_\infty < 2.$$

Now if we take a smooth function $\beta \in C^\infty(\mathbb{R})$ with

$$s \leq 0 \quad \beta(s) := 0, \quad s \geq 1 \quad \beta(s) := 1 \quad \& \quad \|\beta'\|_\infty < 2,$$

then the homotopy

$$\bar{\Gamma} := \{(H + h_0(1 - \beta(s)) + h_1\beta(s)), J_s\}_{s \in \mathbb{R}}$$

connects $(H + h_0, J_0)$ with $(H + h_1, J_1)$. Moreover, $\bar{\Gamma}$ satisfies (8.8) and due to convexity of $\mathcal{O}(H)$ the Hamiltonian part of $\bar{\Gamma}$ lies in $H + \mathcal{O}(H)$. This puts us in the setting of Theorem 8.8, which gives us uniform $L^\infty \times \mathbb{R}$ bounds on the corresponding moduli spaces and uniform bounds on the energy as shown in Proposition 3.3 [19]. Using similar arguments as in the proof of Lemma 8.10, we can show that the uniform bounds hold also for perturbations of $\bar{\Gamma}$, their inverses and concatenations and homotopies of homotopies. Having uniform bounds on the families of homotopies using the standard, Floer-theoretical techniques presented in section 7 we conclude that $RFH(H + h_0)$ and $RFH(H + h_1)$ are indeed isomorphic. $\square$

We conclude by proving the main theorem of this article:

Proof of Theorem 1.4: By Lemma 8.9 and Lemma 8.10 there exists a dense subset of the strongly tentacular Hamiltonians for which the Rabinowitz Floer homology is well defined. We denote it by $\mathcal{H}^{reg}(\mathbb{R}^{2n})$. Let H be a strongly tentacular Hamiltonian and let $\tilde{O}(H) \subseteq C_c^\infty(\mathbb{R}^{2n})$ be the open neighborhood of 0 from Lemma 8.12. For $h \in \tilde{O}(H)$ and $H + h \in \mathcal{H}^{reg}(\mathbb{R}^{2n})$ we define

$$RFH(H) := RFH(H + h).$$

This definition is independent on the choice of the perturbation $h \in \tilde{O}(H)$, as by Lemma 8.12 for any pair $h_1, h_2 \in \tilde{O}(H)$, such that $H + h_1, H + h_2 \in \mathcal{H}^{reg}(\mathbb{R}^{2n})$ the corresponding Rabinowitz Floer homologies are isomorphic. $\square$

Remark 8.13. It follows from the above lemma that for any one-parameter family of tentacular Hamiltonians $\{H_s\}$ in the affine space of compactly supported perturbations of a given Hamiltonian H , the Rabinowitz Floer homology is constant along $\{H_s\}$.

8.4. Invariance under symplectomorphisms. The goal of this section is to prove Corollary 1.5 and to determine which subgroup of the group of symplectomorphisms leaves the set of strongly tentacular Hamiltonians invariant.

Corollary 1.5. *For every strongly tentacular Hamiltonian H and every symplectomorphism φ , the Rabinowitz Floer homology of $H \circ \varphi$ is well-defined and isomorphic to the Rabinowitz Floer homology of H .*

Proof. Let φ be a diffeomorphism of $\mathbb{R}^{2n}$. In particular, φ takes compact sets to compact sets. We define a map

$$\tilde{\varphi} : C_c^\infty(\mathbb{R}^{2n}) \rightarrow C_c^\infty(\mathbb{R}^{2n}), \quad f \mapsto f \circ \varphi^{-1}.$$

The map $\tilde{\varphi}$ is a diffeomorphism of $C_c^\infty(\mathbb{R}^{2n})$, so it maps open sets to open sets and dense sets to dense sets.

Let H be a strongly tentacular Hamiltonian and let φ be a symplectomorphism of $\mathbb{R}^{2n}$. Let $\tilde{O}(H) \subseteq C_c^\infty(\mathbb{R}^{2n})$ be the open neighborhood of 0 associated to H by Lemma 8.12. Then $\tilde{O}(H \circ \varphi^{-1}) := \tilde{\varphi}(\tilde{O}(H))$ is an open subset of 0 in $C_c^\infty(\mathbb{R}^{2n})$. On the other hand, by Lemma 8.9 and Lemma 8.10 there exists a dense subset of the strongly tentacular Hamiltonians, denoted by $\mathcal{H}^{reg}(\mathbb{R}^{2n})$, for which the Rabinowitz Floer homology is well defined. Define

$$\tilde{O}^{reg}(H \circ \varphi^{-1}) := \left\{ h \circ \varphi^{-1} \mid h \in \tilde{O}(H), H + h \in \mathcal{H}^{reg}(\mathbb{R}^{2n}) \right\}.$$

In view of Lemma 8.9 and Lemma 8.10 the set $\tilde{\varphi}^{-1}(\tilde{O}^{reg}(H \circ \varphi^{-1}))$ is dense in $\tilde{O}(H)$. Consequently, the set $\tilde{O}^{reg}(H \circ \varphi^{-1})$ is dense in $\tilde{O}(H \circ \varphi^{-1})$. By Proposition 6.3 for every $h \in \tilde{O}^{reg}(H \circ \varphi^{-1})$ the corresponding Rabinowitz Floer homology $RFH_*(H \circ \varphi^{-1} + h)$ is well defined. What is left to show is that for all $h \in \tilde{O}^{reg}(H \circ \varphi^{-1})$ the corresponding

$RFH(H \circ \varphi^{-1} + h)$ is independent of the chosen h . Let us take $h_1, h_2 \in \tilde{O}^{reg}(H \circ \varphi^{-1})$. Then we have the following sequence of isomorphisms:

$$RFH(H \circ \varphi^{-1} + h_1) \cong RFH(H + h_1 \circ \varphi) \cong RFH(H + h_2 \circ \varphi) \cong RFH(H \circ \varphi^{-1} + h_2),$$

where the first and last isomorphism is guaranteed by Proposition 6.3 and the middle isomorphism is guaranteed by Lemma 8.12. As a result, we can define

$$RFH(H \circ \varphi^{-1}) := RFH(H \circ \varphi^{-1} + h),$$

where $h \in \tilde{O}^{reg}(H \circ \varphi^{-1})$. $\square$

Even though the Rabinowitz Floer homology of the image of a strongly tentacular Hamiltonian under a symplectomorphism is well-defined, this image will not, in general, be a strongly tentacular Hamiltonian. The next lemma describes which elements of $Symp(\mathbb{R}^{2n})$ preserve the set of tentacular Hamiltonians.

Lemma 8.14. *The set of (strongly) tentacular Hamiltonians is invariant under the action of the group*

$$Symp^{ten}(\mathbb{R}^{2n}) := \left\{ \varphi \in Symp(\mathbb{R}^{2n}) \left| \begin{array}{l} \sup \|D^k \varphi(x)\| \cdot |x|^{k-1} < +\infty, \\ \sup \|D^k \varphi^{-1}(x)\| \cdot |x|^{k-1} < +\infty, \end{array} \quad k = 1, 2, 3 \right. \right\}.$$

Proof. We will show that for every (strongly) tentacular Hamiltonian H and any $\varphi \in Symp^{ten}(\mathbb{R}^{2n})$, the composition $H \circ \varphi$ is also a (strongly) tentacular Hamiltonian. Since φ is a symplectomorphism it preserves the symplectic properties. On the other hand, the bounds on the derivatives of φ and φ^{-1} assure that the analytic properties of the Hamiltonian are preserved.

Observe, that the bounds on the derivatives of φ and φ^{-1} imply

$$|\varphi(x) - \varphi(0)| \leq |x| \cdot \sup \|D\varphi\|, \quad (8.9)$$

$$|x - \varphi^{-1}(0)| \leq |\varphi(x)| \cdot \sup \|D\varphi^{-1}\|. \quad (8.10)$$

Since φ is a symplectomorphism it sends Liouville vector fields to Liouville vector fields. Moreover, if $X \in \mathcal{L}(\mathbb{R}^{2n})$ and $\varphi \in Symp^{ten}(\mathbb{R}^{2n})$, then

$$\begin{aligned} \|D(D\varphi^{-1}X)(x)\| &\leq \|D^2\varphi^{-1}(x)\| \cdot \|X(\varphi^{-1}(x))\| + \|DX(\varphi^{-1}(x))\| \cdot \|D\varphi^{-1}(x)\|^2 \\ &\leq c\|D^2\varphi^{-1}(x)\|(|\varphi^{-1}(x)| + 1) + c' \\ &\leq c\|D^2\varphi^{-1}(x)\|(|x| \sup \|D\varphi^{-1}\| + 1) + c' < +\infty, \end{aligned}$$

and $D\varphi^{-1}X \in \mathcal{L}(\mathbb{R}^{2n})$. As a result, $d(H \circ \varphi)(D\varphi^{-1}X^\dagger)(x) = dH(X^\dagger)(\varphi(x)) > 0$ for all $x \in (H \circ \varphi)^{-1}(0) = \varphi^{-1}(H^{-1}(0))$. In other words, $H \circ \varphi$ satisfies condition (h3). Similarly, we can infer that

$$d(H \circ \varphi)(D\varphi^{-1}X^\dagger)(x) = dH(X^\dagger)(\varphi(x)) \geq c|\varphi(x)|^2 - c',$$

which combined with (8.10) assure that $H \circ \varphi$ satisfies (h1).

Observe that whenever φ is a symplectomorphism, then $\{H \circ \varphi, F \circ \varphi\}(x) = \{H, F\}(\varphi(x))$. Pre-images of compact sets under diffeomorphisms are compact, in particular, the set

$$\begin{aligned} & \varphi^{-1} \left(\left\{ x \in H^{-1}(0) \mid \{H, F\}(x) = 0 \text{ or } \{H, \{H, F\}\}(x) \leq 0 \right\} \right) \\ &= \left\{ x \in (H \circ \varphi)^{-1}(0) \mid \{H, F\}(\varphi(x)) = 0 \text{ or } \{H, \{H, F\}\}(\varphi(x)) \leq 0 \right\} \end{aligned}$$

is compact. Moreover, if F is a coercive function, then $F \circ \varphi$ is also a coercive function. Hence $H \circ \varphi$ satisfies (h4) with $F \circ \varphi$.

Consider $x \in \mathbb{R}^{2n}$, such that $|\varphi(x)| > 1$ and $|x| > 1$. Using (8.9) and (8.10) we estimate

$$\begin{aligned} & \|D^3(H \circ \varphi)(x)\| \cdot |x| \\ &= \|D^3H(\varphi(x))(D\varphi(x))^3 + 3D^2H(\varphi(x))D^2\varphi(x)D\varphi(x) + DH(\varphi(x))D^3\varphi(x)\| \cdot |x| \\ &\leq c \left(\|D^3H(\varphi(x))\| \cdot |\varphi(x)| \frac{|x|}{|\varphi(x)|} + \|D^2\varphi(x)\| \cdot |x| + (|\varphi(x)| + 1)\|D^3\varphi(x)\| \cdot |x| \right) \\ &\leq c \left(\frac{|x|}{|\varphi(x)|} + |x| (\|D^2\varphi(x)\| + \|D^3\varphi(x)\|) + \frac{|\varphi(x)|}{|x|} \|D^3\varphi(x)\| \cdot |x|^2 \right) \\ &\leq c (\sup \|D\varphi^{-1}\| + |\varphi^{-1}(0)| + 1 + \sup \|D\varphi\| + |\varphi(0)|) < +\infty. \end{aligned}$$

The function $\|D^3(H \circ \varphi)(x)\| \cdot |x|$ obtains its maximum on the compact set

$$\{x \in \mathbb{R}^{2n} \mid |\varphi(x)| \leq 1 \text{ or } |x| \leq 1\},$$

hence we can infer that $H \circ \varphi$ satisfies (h2). $\square$

9. SYMPLECTIC HYPERBOLOIDS

In this last section, we use Hörmander's symplectic classification of quadratic forms [16] to introduce the notion of *symplectic hyperboloids* and to show that a large number of quadratic Hamiltonians are strongly tentacular.

Suppose H is a non-degenerate quadratic Hamiltonian on $(\mathbb{R}^{2n}, \omega_0)$. Then there exist an affine symplectic change of coordinates such that H can be written in the form

$$H(x) := Q(x, x) - c, \tag{9.1}$$

where Q is a non-degenerate quadratic form and $c \in \mathbb{R}$ is a constant. In particular, for $c > 0$ the hypersurface $\Sigma := H^{-1}(0)$ is diffeomorphic to $S^{k-1} \times \mathbb{R}^l$, where (k, l) is the signature of Q . In general, a $2n$ -dimensional hyperboloid in $\mathbb{R}^{2n}$ can be described as $H^{-1}((-\infty, 0))$ for a quadratic Hamiltonian of the form (9.1) with $c = 1$ and $k, l \geq 1$ (in other words, the quadratic form Q is indefinite). We would like to investigate the equivalence classes of hyperboloids under the action of the linear symplectic group $Sp(\mathbb{R}^{2n})$ and find a unique quadratic form corresponding to each equivalence class of this action.

According to Hörmander's classification, $\mathbb{R}^{2n}$ splits into a direct sum of subspaces, orthogonal with respect to Q and ω , uniquely determined by the Jordan decomposition of J_0Q , of one of the following types:

- (a) $S = T^*\mathbb{R}^m$ and there exists a constant $\lambda > 0$ such that

$$Q(q, p) = 2\lambda \sum_{j=1}^m q_j p_j + 2 \sum_{j=1}^{m-1} q_{j+1} p_j.$$

Then the Jordan decomposition of J_0Q has one $m \times m$ box for each of the eigenvalues λ and $-\lambda$. The signature of Q is (m, m) .

(b) $S = T^*\mathbb{R}^{2m}$ and there exist constants $\lambda_1, \lambda_2 > 0$ such that

$$Q(q, p) = 2 \left(\sum_{j=1}^{2m-2} q_j p_{j+2} + \lambda_1 \sum_{j=1}^{2m} q_j p_j + \lambda_2 \sum_{j=1}^m (q_{2j} p_{2j-1} - q_{2j-1} p_{2j}) \right).$$

The Jordan decomposition of J_0Q has one $m \times m$ box for each of the eigenvalues $\pm \lambda_1 \pm i\lambda_2$. The signature of Q is $(2m, 2m)$.

(c) $S = T^*\mathbb{R}^m$ and there exist constants $\mu > 0$, $\gamma = \pm 1$ such that

$$Q(q, p) = \gamma \left(\mu \sum_{j=1}^m q_j q_{m+1-j} - \sum_{j=2}^m q_j q_{m+2-j} + \mu \sum_{j=1}^m p_j p_{m+1-j} - \sum_{j=1}^{m-1} p_j p_{m-j} \right).$$

The Jordan decomposition of J_0Q has one $m \times m$ box for each of the eigenvalues $\pm i\mu$. The signature of Q is (m, m) if m is even and $(m + \gamma, m - \gamma)$ when m is odd.

The number of spaces of each type in this decomposition is uniquely determined.

According to Hörmander classification every equivalence class of the action of the linear symplectic group acting on the set of hyperboloids admits a quadratic representant Q in the above form, that is, consisting of blocks of types (a), (b) or (c), which is unique up to a permutation of these blocks. In analogy with the case of positive definite quadratic forms and symplectic ellipsoids, we would like to call *symplectic hyperboloid* a sublevel set of such an indefinite quadratic representant:

$$W = \{x \in \mathbb{R}^{2n} : Q(x, x) < 1\}.$$

Our hope is that Rabinowitz Floer homology can eventually be applied to obtain a classification of symplectic hyperboloids, similar to the classification of symplectic ellipsoids obtained as an application of symplectic homology in [13].

Example 9.1. Consider the Hamiltonian

$$H = \frac{1}{2} (p_1^2 + \dots + p_n^2 + q_1^2 + \dots + q_k^2 - q_{k+1}^2 - \dots - q_n^2) - 1.$$

In this case we get a splitting of $\mathbb{R}^{2n}$ into n two-dimensional subspaces, orthogonal with respect to both the quadratic and the symplectic form:

$$S_i = \{(p, q) : p_j = q_j = 0 \text{ for all } j \neq i\}.$$

These subspaces are of type (c) with $\gamma = 1$ for $i = 1, \dots, k$ and they are of type (a) for $i = k + 1, \dots, n$.

Remark 9.2. Note that in the example above the corresponding quadratic representant is not diagonal, whereas the Hamiltonian H obviously is. The criterium for symplectic diagonalisability is presented in [8], where it is shown that a diagonalizable matrix A is symplectically diagonalizable if and only if it satisfies $AJ_0A^TJ_0 = J_0A^TJ_0A$.

The next proposition can be applied in order to determine whether a given quadratic Hamiltonian is strongly tentacular, just on the basis of the orthogonal decomposition and the type of subspaces in Hörmander's classification. For instance, one immediate consequence will be that the Hamiltonian in the example above is in fact strongly tentacular, for any $1 \leq k \leq n$. In turn, this implies that the Rabinowitz Floer Homology can be defined.

Proposition 9.3. *Consider $(\mathbb{R}^{2n}, \omega_0)$ and a Hamiltonian H defined by a non-degenerate quadratic form Q . Decompose $(\mathbb{R}^{2n}, \omega_0)$ into symplectic subspaces S_i which are orthogonal with respect to ω_0 and Q and let Q_i denote the restriction of Q to S_i . Suppose the pair (S_i, Q_i) belongs to one of the following types in Hörmander's classification:*

- (1) type (a) with $m = 1$ or $m > 1$ and $\lambda > 2$;
- (2) type (b) with $m = 1$ or $m > 1$ and $\lambda_1 > 2$;
- (3) type (c) with $m = 1$ and $\gamma = 1$.

Then H is strongly tentacular.

Proof. Step 1: We will start by proving that set of strongly tentacular Hamiltonians is invariant under linear symplectic changes of coordinates. Suppose H is strongly tentacular and $\phi : \mathbb{R}^{2n} \rightarrow \mathbb{R}^{2n}$ is a linear symplectic transformation. Observe that if X is an asymptotically regular Liouville vector field, then $\phi^* X$ is too, since $\phi^* \omega_0 = \omega_0$. As a result, axioms (h1)–(h3) are satisfied for $H \circ \phi$. Analogously, if F is a coercive function, then $F \circ \phi$ is also coercive, since the sublevel sets $(F \circ \phi)^{-1}((-\infty, a]) = \phi^{-1}(F^{-1}((-\infty, a]))$ are compact for all $a \in \mathbb{R}$. Moreover, $\{H \circ \phi, F \circ \phi\}(x) = \{H, F\}(\phi(x))$ and $\{H \circ \phi, \{H \circ \phi, F \circ \phi\}\}(x) = \{H, \{H, F\}\}(\phi(x))$, hence $H \circ \phi$ satisfies also axiom (h4).

Step 2: It follows from step 1 that, without loss of generality, we can assume $H(x) := \frac{1}{2}Q(x, x) - c$. We prove that if a Hamiltonian of this form satisfies the assumptions of the proposition, then it satisfies hypotheses (h1) and (h3).

By assumption we can split the Hamiltonian $H(x) = \frac{1}{2} \sum_i Q_i(x_i, x_i) - c$, where $x_i \in S_i$ and (S_i, Q_i) is of type (1), (2) or (3). Suppose that for every i the Hamiltonian $H_i(x_i) := \frac{1}{2}Q_i(x_i, x_i) - c$ satisfies the hypotheses $dH_i(X^i)(x_i) \geq c_i \|x_i\|^2$ for $x_i \in S_i$, with X^1 an asymptotically regular Liouville vector field and a constant $c_i > 0$. Then $X^\dagger := \sum_i X^i(x_i)$ is an asymptotically regular Liouville vector field on $T^*\mathbb{R}^m$, such that $dH_x(X^\dagger) = \sum dH_i(X^i)(x_i) \geq \min_i c_i \|x\|^2$. In other words, $H = \sum_i H_i$ satisfies hypotheses (h1) and (h3).

Therefore it suffices to prove hypotheses (h1) and (h3) for Hamiltonians H_i in each of the three distinct cases (1), (2) and (3). Our proof of existence will be in fact a constructive proof. In the standard coordinates on $(T^*\mathbb{R}^m, \omega_0)$, for each $\alpha \in \mathbb{R}$ we define the vector field

$$X^\alpha(q, p) = \sum_{i=1}^m \left(\left(\frac{1}{2}p_i + \alpha q_i \right) \partial_{p_i} + \left(\frac{1}{2}q_i + \alpha p_i \right) \partial_{q_i} \right).$$

A straightforward calculation shows that X^α is an asymptotically regular Liouville vector field for every $\alpha \in \mathbb{R}$.

Case (1): Let $S_i = T^*\mathbb{R}^m$ and Q_i be of class (a) with coefficient $\lambda > 0$. Then

$$dH_i(X^\alpha)(q, p) = \lambda \sum_{j=1}^m (\alpha q_j^2 + q_j p_j + \alpha p_j^2) + \sum_{j=1}^{m-1} (p_j q_{j+1} + \alpha p_j p_{j+1} + \alpha q_j q_{j+1}) \geq c \sum_{j=1}^n (q_j^2 + p_j^2),$$

$$\text{with } c_i := \begin{cases} \lambda(\alpha - \frac{1}{2}), & m = 1, \\ \alpha(\lambda - 1) - \frac{1}{2}(\lambda + 1), & m > 1. \end{cases}$$

The last inequality is obtained by square completion. We can see that for $m = 1$, the constant c_i is positive provided $\alpha > \frac{1}{2}$, and for $m > 1$ the constant c_i is positive, provided $\lambda > 1$ and $\alpha > \frac{\lambda+1}{2(\lambda-1)}$. Therefore, in both cases, H_i satisfies hypotheses (h1) and (h3) on S_i .

Case (2): Let $S_i = T^*\mathbb{R}^{2m}$ and Q_i be of class (b) with coefficient $\lambda_1, \lambda_2 > 0$. In this case, if we calculate $dH_i(X^\alpha)$, we obtain:

$$dH_i(X^\alpha)(q, p) = \lambda_1 \sum_{j=1}^{2m} (\alpha q_j^2 + q_j p_j + \alpha p_j^2) + \lambda_2 \sum_{j=1}^n (p_{2j-1} q_{2j} + q_{2j-1} p_{2j})$$

$$+ \sum_{j=1}^{2m-2} (p_{j+2} q_j + \alpha(p_{j+2} p_j + q_j q_{j+2})) \geq c \sum_{j=1}^{2m} (q_j^2 + p_j^2),$$

$$\text{with } c_i := \begin{cases} \alpha \lambda_1 - \frac{1}{2}(\lambda_1 + \lambda_2), & m = 1, \\ \alpha(\lambda_1 - 1) - \frac{1}{2}(\lambda_1 + \lambda_2 + 1), & m > 1. \end{cases}$$

We can see that for $m = 1$, the constant c_i is positive provided $\alpha > \frac{\lambda_1 + \lambda_2}{2\lambda_1}$ and for $m > 1$ the constant c_i is positive provided $\lambda_1 > 1$ and $\alpha > \frac{\lambda_1 + \lambda_2 + 1}{2(\lambda_1 - 1)}$, which in turn shows that H_i satisfies hypotheses (h1) and (h3).

Case (3): Let $S_i = T^*\mathbb{R}$ and Q_i be of class (c) with coefficients $\mu > 0$ and $\gamma = 1$. Then the associated Hamiltonian H_i is of the form $H_i(q_1, p_1) = \frac{\mu}{2}(p_1^2 + q_1^2)$. A straightforward calculation shows that $dH_i(X^0) = \mu(p_1^2 + q_1^2)$, which proves the claim in this last case.

Step 3: In this last step we will prove that the Hamiltonian H satisfies hypothesis (h4).

For every i , let $F_i(x_i) := \frac{1}{2}\|x_i\|^2$ and assume that for some constants $c_i, c'_i > 0$ we have

$$\forall i \in I \quad \{H_i, \{H_i, F_i\}\}(x_i) \geq c_i \|x_i\|^2 \quad \text{and} \quad |H_i(x_i)| \leq c'_i \|x_i\|^2, \quad (9.2)$$

$$\forall i \in I' \quad \{H_i, \{H_i, F_i\}\}(x_i) = 0 \quad \text{and} \quad H_i(x_i) \geq c_i \|x_i\|^2. \quad (9.3)$$

Now we take $\varepsilon > 0$ and with $F(x) := \frac{1}{2}\|x\|^2$ we calculate

$$\begin{aligned} \{H, \{H, F\}\}(x) + \varepsilon H(x) &= \sum_{i \in I \cup I'} (\{H_i, \{H_i, F_i\}\}(x_i) + \varepsilon H_i(x_i)) - \varepsilon c \\ &\geq \sum_{i \in I} (c_i - \varepsilon c'_i) \|x_i\|^2 + \varepsilon \sum_{i \in I} c_i \|x_i\|^2 - \varepsilon c \end{aligned}$$

In particular, if we take ε small enough then we get the estimate

$$\{H, \{H, F\}\}(x) + \varepsilon H(x) \geq \varepsilon' \|x\|^2 - \varepsilon c,$$

which brings us to the conclusion, that the function above is positive outside a compact set. As a result, for $x \in H^{-1}(0)$ the function $\{H, \{H, F\}\}(x)$ is positive outside a compact set, proving hypothesis (h4). Therefore, to prove hypothesis (h4) for H it suffices to show that each H_i of type (1), (2) or (3) satisfies either (9.2) or (9.3).

Let us introduce the following notation: Let J_0 be the standard almost complex structure on $T^*\mathbb{R}^m$ and let A be a real, symmetric, $2m \times 2m$ matrix, such that $H(x) = \frac{1}{2}Q(x, x) - c = \frac{1}{2}\langle x, Ax \rangle - c$. Then for the radial function F the corresponding Poisson brackets are given by

$$\begin{aligned}\{H, F\}(x) &:= \langle x, \frac{1}{2}(J_0 A + (J_0 A)^T)x \rangle, \\ \{H, \{H, F\}\}(x) &:= \langle x, (A^2 + \frac{1}{2}((J_0 A)^2 + ((J_0 A)^2)^T))x \rangle.\end{aligned}$$

In particular, $\{H, \{H, F\}\}(x) > 0$ for all $x \neq 0$, whenever the matrix $A^2 + \frac{1}{2}((J_0 A)^2 + ((J_0 A)^2)^T)$ is positive definite. We define $B := A^2 + \frac{1}{2}((J_0 A)^2 + ((J_0 A)^2)^T)$.

Case (1): Let A_i be the symmetric matrix corresponding to Q_i . Then $B_i = \{b_{l,k}\}_{l,k=1}^{2m}$ has entries

$$b_{l,k} := \begin{cases} 2\lambda^2 & l = k = 1 \quad \text{or} \quad l = k = 2m, \\ 1 + 2\lambda^2 & l = k = 2, \dots, 2m-1, \\ 2\lambda & k = l + 2 \quad \text{or} \quad l = k + 2, \\ \frac{1}{2} & k = l + 4 \quad \text{or} \quad l = k + 4, \\ 0 & \text{otherwise.} \end{cases}$$

In particular, for $m = 1$ the matrix B is diagonal with $2\lambda^2$ on the diagonal, hence always positive definite. On the other hand, for $m > 1$ the matrix B_i is diagonally dominant for $\lambda > 2$, hence by a theorem of Levy-Desplanques (cf. [17] and [9]), B_i is in this case positive definite. We can conclude that in this case (9.2) is satisfied.

Case (2): Let $S_i = T^*\mathbb{R}^{2m}$, let Q_i be of class (b) with coefficient $\lambda_1, \lambda_2 > 0$ and let A_i be the matrix corresponding to Q_i . Then for $m = 1$ the corresponding matrix B_i is diagonal with $2\lambda_1^2$ on the diagonal, hence always positive definite. For $m > 1$ the corresponding $B_i = \{b_{l,k}\}_{l,k=1}^{2m}$ has entries

$$b_{l,k} := \begin{cases} 2\lambda_1^2 & k = l = 2, 4, 4m-3, 4m-1, \\ 1 + 2\lambda_1^2 & k = l = 1, 3, 5, \dots, 4m-4, 4m-2, 4m, \\ 2\lambda_1 & k = l + 4 \quad \text{or} \quad l = k + 4, \\ \frac{1}{2} & k = l + 8 \quad \text{or} \quad l = k + 8, \\ 0 & \text{otherwise.} \end{cases}$$

In particular, if $\lambda_1 > 2$, then B_i is diagonally dominant, hence by the same theorem of Levy-Desplanques, B_i is in fact positive definite. We can conclude that in this case (9.2) is satisfied.

Case (3): Let $S_i = T^*\mathbb{R}$ and Q_i be of class (c) with coefficients $\mu > 0$ and $\gamma = 1$. Then the associated Hamiltonian H_i is of the form $H_i(q_1, p_1) = \frac{\mu}{2}(p_1^2 + q_1^2)$, hence it is always positive accept at 0. On the other hand the associated matrix $B_i = 0$. We can conclude that in this case (9.3) is satisfied. $\square$

Remark 9.4. It follows from Proposition 9.3 that Rabinowitz Floer homology is well-defined for the Hamiltonians corresponding to one of the three types in the statement of the proposition. Moreover, in view of Corollary 1.5 and Lemma 8.14, it will also be well-defined for the composition of such Hamiltonians with any symplectomorphism. Finally, by invariance the homology will also be well-defined for any sufficiently small compactly supported perturbation of any of the previous Hamiltonians.

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