Stationary solutions of a fourth order nonlinear diffusion equation

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1. Introduction

In this paper we shall study stationary solutions of the Extended Fisher-Kolmogorov (EFK) equation

$$\frac{\partial u}{\partial t} = -\gamma \frac{\partial^4 u}{\partial x^4} + \frac{\partial^2 u}{\partial x^2} + u - u^3 \qquad \text{on} \quad \mathbf{R} \times (0, \infty), \tag{1.1}$$

in which γ is a positive parameter. This equation is a higher order generalization of the classical Fisher-Kolmogorov (FK) equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + u - u^3 \qquad \text{on} \quad \mathbf{R} \times (0, \infty),$$

which has been studied in detail as a model equation for phase transitions and pattern formation in bistable systems [AW,FMcL,KPP]. Recently, the EFK-equation has been proposed as a model equation for comparable, but more complex, situations, such as phase transitions near a Lifschitz point [DS,CER,HLS,Z].

In this paper we study two types of stationary solutions of equations (1.1): *periodic solutions* and *transition layers*, also known as *domain walls* or *Kinks*.

Let $u(x)$ be a stationary periodic solution of (1.1). Then without loss of generality we may set $u(0) = 0$ and denote its smallest positive zero by $x = L$. We shall study those periodic solutions which are antisymmetric with respect to $x = 0$ and $x = L$. Such solutions can be constructed by piecing together solutions of the two-point boundary value problem

$$\text{(I)} \begin{cases} -\gamma u^{iv} + u'' + u - u^3 = 0 \quad \text{for} \quad 0 < x < L, & (1.2\text{a}) \\ u(0) = 0, \quad u''(0) = 0 \quad \text{and} \quad u(L) = 0, \quad u''(L) = 0. & (1.2\text{b}) \end{cases}$$

This is a nonlinear eigenvalue problem with eigenvalues γ and L. We shall show that if

$$L \leq L_1 = \pi \sqrt{\frac{2\gamma}{\sqrt{1 + 4\gamma} - 1}}$$

then Problem (I) has the trivial solution only, whilst if $L > L_1$ then Problem (I) has nontrivial solutions.

Theorem A. *Let $\gamma > 0$.*
(a) *If $L \leq L_1$, then Problem (I) has the trivial solution only.*
(b) *If $nL_1 \leq L < (n+1)L_1$ $(n \geq 1)$, then Problem (I) has n solutions $u_1,\ u_2, \ldots, u_n$; for each k, u_k has $k - 1$ zeros on $(0, L)$.*

Solutions which only differ by a factor -1 *are counted as one.*

In this theorem we characterise solutions by means of the period $2L$, There is, however, another parameter which can be used for this purpose. If we multiply (1.2a) by $-2u'$ and integrate the resulting equation we find the first integral

$$2\gamma u'u''' - \gamma(u'')^2 - (u')^2 + \frac{1}{2}(1-u^2)^2 = \text{constant} \stackrel{\text{def}}{=} \frac{\mu}{2}. \tag{1.4}$$

A characterisation of periodic solutions by means of the parameter μ is given in [PT2].

Kinks are solutions of equation (1.2a) which are defined on the whole real line and converge to $u = \pm 1$ at $x \to \pm\infty$. This leads to the two-point boundary value problem problem

$$\text{(II)} \begin{cases} -\gamma u^{iv} + u'' + u - u^3 = 0 \quad \text{for} \quad -\infty < x < \infty, & (1.4\text{a}) \\ u(-\infty) = -1, \quad u(0) = 0 \quad \text{and} \quad u(+\infty) = 1. & (1.4\text{b}) \end{cases}$$

The condition that $u(x)$ vanishes at the origin involves no loss of generality, since the equation is autonomous. It is well known that the FK-equation has the unique Kink

$$u(x) = \tanh\Big(\frac{x}{\sqrt{2}}\Big),$$

which is monotone and satisfies the symmetry condition

$$u(-x) = -u(x) \quad \text{for all} \quad x \in \mathbf{R}. \tag{1.5}$$

Recently, in [PT1], it has been shown that if $\gamma \leq \frac{1}{8}$, then the EFK-equation continues to have a unique monotone Kink which satisfies (1.5).

In this paper we shall show that Kinks, which satisify (1.5), exist for *all* $\gamma > 0$.

Theorem B. *Let* $\gamma > 0$. *Then Problem (II) has an odd solution* $u \in C^\infty(\mathbf{R})$ *which has the properties:*

(a) $\quad u \leq 0$ *on* $(-\infty, 0)$ *and* $u \geq 0$ *on* $(0, \infty)$,

(b) $\quad 1 + u \in H^2(-\infty, 0)$ *and* $1 - u \in H^2(0, \infty)$.

It is easily seen from a local analysis near $u = +1$ or $u = -1$, that if $\gamma > \frac{1}{8}$, then Problem (II) cannot have monotone solutions.

In contrast to [PT1,2], where shooting methods were used, we shall here use variational methods.

As we shall see, many of the arguments developed here are also applicable to the 6^{th} order equation

$$\frac{\partial u}{\partial t} = \frac{\partial^6 u}{\partial x^6} + A\frac{\partial^4 u}{\partial x^4} + B\frac{\partial^2 u}{\partial x^2} + u - u^3, \tag{1.6}$$

discussed in [GJ] when the positive constants A and B satisfy the inequality

$$A^2 < 4B.$$

This inequality was also found to be critical in [GJ].

The plan of the paper is the following. In Section 2 we shall discuss the existence of periodic solutions and their qualitative properties, and in Section 3 we discuss Kinks.

2. A nonlinear eigenvalue problem

To cast Problem (I) into a variational form we introduce the functional

$$\phi(u) = \int_0^L \left\{ \frac{\gamma}{2}(u'')^2 + \frac{1}{2}(u')^2 - \frac{1}{2}u^2 + \frac{1}{4}u^4 \right\}, \qquad \gamma > 0 \tag{2.1}$$

on the set of functions $X = H^2(0,L) \cap H_0^1(0,L)$. Plainly

$$\phi(u) > -\frac{1}{4}L \quad \text{for all} \quad u \in X, \tag{2.2}$$

so that the minimization problem, to find a function $u \in X$ so that

$$\phi(u) = \inf\{\phi(v) : v \in X\}, \tag{2.3}$$

makes sense.

Lemma 2.1. *Suppose u is a minimizer of (2.3). Then u is a classical solution of Problem (I).*

Proof. Define

$$F(t) = \phi(u + tv),$$

where v is an arbitrary function in X. Then, since F has a minimun at $t = 0$, and F is clearly differentiable near the origin, it follows that $F'(0) = 0$, or

$$\int_0^L \{\gamma u''v'' + u'v' - uv + u^3 v\}\, dx = 0. \tag{2.4}$$

This means that u is a weak solution of equation (1.2a). From standard regularity theory we deduce that $u \in C^4([0,L])$ and because $u \in X$, we know that $u(0) = 0$ and $u(L) = 0$. Therefore, it remains to show that u'' vanishes on the boundary.

Choose $v \in C^2([0,L])$ so that $v(0) = 0$ and $v(L) = 0$. Then $v \in X$ and we may substitute it into (2.4). When we integrate by parts and use the differential equation we obtain

$$\gamma u''(L)v'(L) - \gamma u''(0)v'(0) = 0.$$

Because $v'(L)$ and $v'(0)$ are arbitrary, it follows that we must have $u''(L) = 0$ and $u''(0) = 0$.

Lemma 2.2. *There exists a function $u \in X$ which satisfies (2.3).*

Proof. Write

$$M = \inf\{\phi(v) : v \in X\}$$

and let u_k be a minimizing sequence of (2.3), so that

$$\phi(u_k) \to M \quad \text{as} \quad k \to \infty.$$

Then,

$$\int_0^L \left\{\frac{\gamma}{2}(u_k'')^2 + \frac{1}{2}(u_k')^2 + \frac{1}{4}(1-u_k^2)^2\right\} \leq C$$

so that

$$\int_0^L (u_k'')^2 \leq C, \quad \int_0^L (u_k')^2 \leq C$$

and by Poincaré's inequality

$$\|u_k\|_{H^2(0,L)} \leq C.$$

Here C is a constant, which may be different at different occasions, which does not depend on k. Because X is a Hilbert space we can extract a subsequence, which we denote again by $\{u_k\}$, and a function u such that as $k \to \infty$,

$$\begin{aligned} u_k &\rightharpoonup u \quad &&\text{in} \quad H^2 \\ u_k &\to u \quad &&\text{in} \quad H_0^1 \\ u_k &\to u \quad &&\text{in} \quad L^2 \\ u_k &\to u \quad &&\text{in} \quad L^4. \end{aligned}$$

Hence, if we write

$$\phi(u) = \phi_1(u) + \phi_2(u),$$

where

$$\phi_1(u) = \frac{\gamma}{2}\int_0^L (u'')^2 \quad \text{and} \quad \phi_2(u) = \int_0^L \left\{\frac{1}{2}(u')^2 - \frac{1}{2}u^2 + \frac{1}{4}u^4\right\},$$

then

$$\phi_2(u_k) \to \phi_2(u) \quad \text{as} \quad k \to \infty. \tag{2.5}$$

As to ϕ_1, we note that

$$\phi_1(u_k) \geq \gamma \int_0^L u_k'' u'' - \phi_1(u).$$

Hence, by the weak convergence of u_k in H^2, and the strong convergence in H^1 it follows that

$$\liminf_{k\to\infty} \phi_1(u_k) \geq \phi_1(u). \tag{2.6}$$

Combining (2.5) and (2.6) we find that

$$\phi(u) \le \liminf_{k\to\infty} \phi(u_k) = M.$$

Because $M = \inf \phi$ we conclude that

$$\phi(u) = M$$

which we set out to prove.

In the remainder of this section we investigate the influence of the length L of the interval, or the period $2L$ of the periodic solution on the solution set of Problem (I). Thus we shall set

$$\phi(u) = \phi(u; L), \quad \text{and} \quad X = X(L)$$

and

$$M(L) = \min\{\phi(u; L) : u \in X(L)\}. \tag{2.7}$$

We need the following Poincaré type inequality.

Lemma 2.3. *We have*

$$\int_0^L u^2 \le \frac{L^4}{\pi^4} \int_0^L (u'')^2 \qquad \textit{for any} \quad u \in X(L). \tag{2.8}$$

Proof. By Poincaré's inequality

$$\int_0^L u^2 \le \kappa \int_0^L (u')^2, \qquad \text{where} \quad \kappa = \frac{L^2}{\pi^2}.$$

But since $u \in X(L)$,

$$\int_0^L (u')^2 = -\int_0^L uu''$$

so that

$$\int_0^L (u')^2 \le \Big(\int_0^L u^2\Big)^{1/2} \Big(\int_0^L (u'')^2\Big)^{1/2}$$

and hence

$$\int_0^L u^2 \le \kappa \Big(\int_0^L u^2\Big)^{1/2} \Big(\int_0^L (u'')^2\Big)^{1/2}$$

which yields

$$\int_0^L u^2 \le \kappa^2 \int_0^L (u'')^2.$$

The inequality (2.8) enables us to prove the nonexistence of nontrivial solutions for small values of L. Let

$$L_1 = \pi \sqrt{\frac{2\gamma}{\sqrt{4\gamma+1} - 1}}.$$

Lemma 2.4. *If $L \leq L_1$. Then Problem (I) has the trivial solution only.*

Proof. Suppose that u is a nontrivial solution. Then,

$$\int_0^L \{\gamma(u'')^2 + (u')^2 - u^2\} = -\frac{1}{2}\int_0^L u^4 < 0.$$

However, when we multiply the differential equation by u and integrate over $(0, L)$, we obtain by means of Lemma 2.3,

$$\int_0^L \{\gamma(u'')^2 + (u')^2 - u^2\} \geq \left\{\gamma\left(\frac{\pi}{L}\right)^4 + \left(\frac{\pi}{L}\right)^2 - 1\right\}\int_0^L u^2. \tag{2.9}$$

An easy computation shows that the right hand side of (2.9) becomes nonnegative when $L \leq L_1$, so that we then have a contradiction.

In Lemma 2.4 we have shown that

$$M(L) = 0 \qquad \text{if} \quad L \leq L_1.$$

In the next lemma we show that this is no longer so when $L > L_1$.

Lemma 2.5. *If $L > L_1$, then $M(L) < 0$.*

Proof. Consider the trial function

$$w(x) = \varepsilon \sin\left(\frac{\pi x}{L}\right)$$

in which ε is a positive number, to be chosen later. Then

$$\phi(w; L) = \frac{1}{4}L\varepsilon^2\left\{\gamma\left(\frac{\pi}{L}\right)^4 + \left(\frac{\pi}{L}\right)^2 - 1 + \frac{3}{8}\varepsilon^2\right\}.$$

If $L > L_1$, then

$$\delta = \gamma\left(\frac{\pi}{L}\right)^4 + \left(\frac{\pi}{L}\right)^2 - 1 < 0$$

and

$$\phi(w; L) = \frac{1}{4}L\varepsilon^2\left(-|\delta| + \frac{3}{8}\varepsilon^2\right).$$

Thus, if we choose $\varepsilon < 2\sqrt{2|\delta|/3}$, then $\phi(w; L) < 0$, which implies that $M(L) < 0$.

Lemma 2.6. *If $L > L_1$, then Problem (I) has a nontrivial solution u, which is a minimizer of ϕ. Moreover, this solution has no zeros on $(0, L)$.*

Proof. That there exists a nontrivial solution follows from Lemma 2.5, which implies that the minimizer of (2.3) cannot be $u = 0$.

Suppose u is a global minimizer of $\phi(\cdot;L)$ which has a zero at a point $\xi \in (0,L)$:

$$\phi(u;L) = M(L). \tag{2.10}$$

Define

$$y = \frac{L}{\xi}x = kx \quad \text{and} \quad v_1(y) = u(x), \quad 0 \le x \le \xi.$$

Then

$$\begin{aligned} \phi(v_1;L) &= k\int_0^\xi \left\{\frac{\gamma}{2k^4}(u'')^2 + \frac{1}{2k^2}(u')^2 - \frac{1}{2}u^2 + \frac{1}{4}u^4\right\}dx, \\ &< k\int_0^\xi \left\{\frac{\gamma}{2}(u'')^2 + \frac{1}{2}(u')^2 - \frac{1}{2}u^2 + \frac{1}{4}u^4\right\}dx, \end{aligned} \tag{2.11}$$

because $k > 1$. Next, define

$$y = \frac{L}{L-\xi}(x-\xi) = \ell(x-\xi) \quad \text{and} \quad v_2(y) = u(x), \quad \xi \le x \le L.$$

Then

$$\begin{aligned} \phi(v_2;L) &= \ell\int_\xi^L \left\{\frac{\gamma}{2\ell^4}(u'')^2 + \frac{1}{2\ell^2}(u')^2 - \frac{1}{2}u^2 + \frac{1}{4}u^4\right\}dx, \\ &< \ell\int_\xi^L \left\{\frac{\gamma}{2}(u'')^2 + \frac{1}{2}(u')^2 - \frac{1}{2}u^2 + \frac{1}{4}u^4\right\}dx, \end{aligned} \tag{2.12}$$

since $\ell > 1$. Thus, putting (2.10), (2.11) and (2.12) together we find that

$$M(L) > \frac{\xi}{L}\phi(v_1;L) + \frac{L-\xi}{L}\phi(v_2;L). \tag{2.13}$$

Because $v_i \in X(L)$ $(i = 1,2)$ it follows that

$$\phi(v_i;L) \ge M(L), \quad i = 1,2,$$

so that (2.13) yields

$$M(L) > \frac{\xi}{L}M(L) + \frac{L-\xi}{L}M(L) = M(L).$$

This is impossible and so we have a contradiction.

Remark. Lemma 2.6 can also be proved by means of the observation, made in [MS], that the operator

$$L \stackrel{\text{def}}{=} \left(\frac{d^4}{dx^4} - \frac{d^2}{dx^2}\right)^{-1}$$

with the same boundary conditions, maps the positive cone into itself.

Next suppose $L > 2L_1$. Then we can construct two solutions of Problem (I): u_1, which is a global minimizer of $\phi(u; L)$ and u_2, which is a composition of a global minimizer $\hat{u}$ of $\phi(u, \frac{1}{2}L)$ and a translate of $\hat{u}$ over $\frac{1}{2}L$:

$$u_2(x) = \begin{cases} \hat{u}(x) & \text{if } \ 0 \le x \le \dfrac{1}{2}L \\ -\,\hat{u}(x - \dfrac{1}{2}L) & \text{if } \ \dfrac{1}{2}L \le x \le L. \end{cases}$$

By the symmetry of ϕ with respect to u we have

$$\phi(u_2; L) = 2\phi(\hat{u}; \frac{1}{2}L). \tag{2.14}$$

On the other hand, if we proceed as in the proof of Lemma 2.6, stretch the variable x and define

$$y = 2x \quad \text{and} \quad v(y) = \hat{u}(x), \quad 0 \le x \le \frac{1}{2}L.$$

we find that $v \in X(L)$, whilst

$$\phi(v; L) = 2\int_0^{L/2} \left\{ \frac{\gamma}{32}(\hat{u}'')^2 + \frac{1}{8}(\hat{u}')^2 - \frac{1}{2}\hat{u}^2 + \frac{1}{4}\hat{u}^4 \right\} dx < 2\phi(\hat{u}; \frac{1}{2}L).$$

Hence, by (2.14),

$$\phi(v; L) < \phi(u_2; L). \tag{2.15}$$

Therefore, u_2 cannot be a global minimizer of $\phi(\cdot; L)$ and so $u_2 \ne u_1$.

Remark. The above argument also yields an interesting property of $M(R)$. Namely, from (2.14) and (2.15) we deduce that

$$M(L) < 2M(\frac{1}{2}L) \quad \text{if} \quad L > 2L_1. \tag{2.16}$$

Thus we have found that if $L > 2L_1$, we can construct two different solutions – modulo a factor -1 – of Problem (I), of which one has no interior zero, and the other has precisely one zero.

If $L > 3L_1$ we can use the same argument to construct three different solutions of which one has no zeros, one has one zero and one has two zeros.

Continuing this process we obtain the following family of solutions.

Theorem 2.7. *Let $\gamma > 0$.*
(a) *If $L \le L_1$, then Problem (I) has the trivial solution only.*

(b) *If* $nL_1 < L \leq (n+1)L_1$ $(n \geq 1)$, *then Problem (I) has* n *different solutions* $u_1, u_2, \ldots, u_n$. *For each* k, u_k *has* $k-1$ *zeros on* $(0, L)$. *Here in the counting of solutions a factor* -1 *is ignored.*

Remark. The existence of multiple solutions can also be proved by means of an abstract critical point theorem (Theorem 4.1 in Chapter II of [C]) which links the number of solutions of Problem (I) to the Morse index of the zero solution of the linear operator

$$\mathcal{L}u = \gamma u^{iv} - u'' - u.$$

In order for this theorem to be applicable we need to verify that ϕ is bounded from below on $H^2 \cap H_0^1(0, L)$ by a constant a and that there exists an n-dimensional subspace X_n, for some $1 \leq n < \infty$, such that ϕ is bounded above by a constant $b > a$ on $X_n \cap S_\rho$, for some $\rho > 0$. Other conditions are that ϕ is even, $\phi(0) = 0$ and ϕ satisfies the (PS)-condition. It may be obvious that the latter conditions are indeed satisfied.

As for the geometric conditions we proceed as follows. Choose $a = -\frac{1}{4}L$ and $b = 0$ and recall that

$$\phi(u) > -\frac{1}{4}L \quad \text{for all} \quad u \in H^2 \cap H_0^1(0, L).$$

Next we introduce the orthonormal family of functions

$$\psi_k(x) = \sqrt{\frac{2}{L}} \sin\Big(\frac{k\pi x}{L}\Big)$$

in $L^2(0, L)$ and the associated linear space

$$X_n = \operatorname{span}\{\psi_1, \ldots, \psi_n\}, \quad n \geq 1.$$

Any $w \in X_n$ can be written uniquely as

$$w = \sum_{k=1}^{n} w_k \psi_k,$$

and its norm in $L^2(0, L)$ is given by

$$\|w\|_{X_n}^2 = \sum_{k=1}^{n} w_k^2.$$

Observe that

$$\begin{aligned} \phi(w; L) &= \sum_{k=1}^{n} \Big\{ \frac{\gamma}{2}\Big(\frac{k\pi}{L}\Big)^4 + \frac{1}{2}\Big(\frac{k\pi}{L}\Big)^2 - \frac{1}{2} \Big\} w_k^2 + \frac{1}{4}\int_0^L w^4 \\ &\leq \sum_{k=1}^{n} \Big\{ \frac{\gamma}{2}\Big(\frac{k\pi}{L}\Big)^4 + \frac{1}{2}\Big(\frac{k\pi}{L}\Big)^2 - \frac{1}{2} \Big\} w_k^2 + C(n)\|w\|_{X_n}^4. \end{aligned} \tag{2.17}$$

For $k \le n < \frac{L}{L_1}$ there exists a constant $\delta > 0$ such that

$$\frac{\gamma}{2}\Big(\frac{k\pi}{L}\Big)^4 + \frac{1}{2}\Big(\frac{k\pi}{L}\Big)^2 - \frac{1}{2} \le -\delta,$$

so that (2.17) yields

$$\phi(w;L) \le -\delta \|w\|^2_{X_n} + C(n)\|w\|^2_{X_n}.$$

Thus, for some $\rho > 0$ small enough, we have

$$\phi(w;L) < 0 \quad \text{if} \quad \|w\|_{X_n} < \rho.$$

From Theorem 4.1 in [C] we now deduce that ϕ has at least $2n$ nontrivial geometrically distinct critical points, with critical values between $-\frac{1}{4}L$ and 0.

Remark. To study stationary periodic solutions of the 6^{th} order equation (1.6) we can argue analogously and analyze the functional

$$\phi(u) = \int_0^L \Big\{\frac{1}{2}(u''')^2 - \frac{A}{2}(u'')^2 + \frac{B}{2}(u')^2 - \frac{1}{2}u^2 + \frac{1}{4}u^4\Big\}$$

on the space $Y(L) = \{u \in H^3(0,L) : u(0) = 0,\ u(L) = 0,\ u''(0) = 0,\ u''(L) = 0\}$. One finds by means of Young's inequality that if $A^2 < 4B$, then there exists an $\varepsilon > 0$ such that

$$A\int_0^L (u'')^2 \le (1-\varepsilon)\int_0^L (u''')^2 + B\int_0^L (u')^2.$$

This yields

$$\int_0^L \Big\{\frac{1}{2}(u''')^2 - \frac{A}{2}(u'')^2 + \frac{B}{2}(u')^2 - \frac{1}{2}u^2 + \frac{1}{4}u^4\Big\} \ge \frac{\varepsilon}{2}\int_0^L (u''')^2 - \frac{1}{4}L \ge -\frac{1}{4}L.$$

Thus, if $A^2 < 4B$ the minimization problem

$$\phi(u) = \inf\{\phi(v) : v \in Y\}$$

is well posed, and we can proceed as with the 4^{th} order equation (1.1).

3. Kinks

In this section we are interested in finding solutions of the equation

$$-\gamma u^{iv} + u'' + u - u^3 = 0 \qquad \text{on} \quad \mathbf{R} \tag{3.1}$$

which connect the equilibrium solutions $u = \pm 1$ at infinity:

$$\lim_{x\to-\infty} u(x) = -1, \quad \lim_{x\to\infty} u(x) = 1. \tag{3.2}$$

We refer to such solutions as *Kinks*. Their existence was established in [PT1] when $0 < \gamma \le \frac{1}{8}$. Here we shall show that they exist for *every* $\gamma > 0$.

The solutions that were constructed for $\gamma \le \frac{1}{8}$ were monotone and had the symmetry property

$$u(-x) = -u(x) \quad \text{for} \quad -\infty < x < \infty. \tag{3.3}$$

Kinks which satisfy (3.3) will be called *symmetric*.

We shall focuss on symmetric kinks, so that we seek solutions of (3.1) on the half line, which satisfy two conditions at the origin, dictated by the symmetry condition (3.3), and tend to 1 at infinity:

$$u(0) = 0, \quad u''(0) = 0 \quad \text{and} \quad \lim_{x\to\infty} u(x) = 1. \tag{3.4}$$

Defining $u(x)$ for negative values of x through (3.3) we then obtain a symmetric kink.

As was shown in [PT1] by linearizing equation (3.1) around $u = 1$, kinks cannot be monotone on $\mathbf{R}$ when $\gamma > \frac{1}{8}$.

For technical reasons it is convenient first to consider the approximate problem

$$(\mathrm{S}_a^+)\begin{cases} -\gamma u^{iv} + u'' + (u-a)(1-u^2) = 0, \quad x > 0, & (3.5\text{a}) \\ u(0) = 0, \quad u''(0) = 0, \quad \lim_{x\to\infty} u(x) = 1, & (3.5\text{b}) \end{cases}$$

in which $a < 0$, and prove the existence of a solution u_a of Problem (S_a^+) for every $-3 < a < 0$. Then we pass to the limit as $a \to 0$ and show that

$$u_a(x) \to \overline{u}(x) \quad \text{as} \quad a \to 0,$$

and that the limit function $\overline{u}$ satisfies (3.1) on $\mathbf{R}^+$, as well as the boundary conditions (3.4).

Remark. In anology with Problem (S_a^+) one can also study solutions of equation (3.5a) on $(-\infty, 0]$ with the same conditions at the origin, but tending to -1 as $x \to -\infty$, and with $0 < a < 3$. We shall refer to this problem as Problem (S_a^-).

In view of the symmetry properties of equation (3.1) and the boundary conditions (3.2) the composite function

$$u(x) = \begin{cases} \overline{u}(x) & \text{for} \quad x \ge 0 \\ -\overline{u}(-x) & \text{for} \quad x \le 0 \end{cases}$$

is then a kink. Let us now continue by studying Problem (S_a^+).

Theorem 3.1. *Let $\gamma > 0$ and $-3 < a < 0$. Then Problem (S_a^+) possesses a solution $u_a \in C^\infty([0,\infty))$, which is positive on $\mathbf{R}^+$.*

In order to prove this theorem we use a variational approach. It will be convenient to change variables and set $v = 1 - u$. This yields upon substitution in (3.1), the problem

$$\begin{cases} -\gamma v^{iv} + v'' - v(1-a-v)(2-v) = 0, \quad x > 0, & (3.6\text{a}) \\ v(0) = 1, \quad v''(0) = 0, \quad \lim_{x\to\infty} v(x) = 0. & (3.6\text{b}) \end{cases}$$

Equation (3.6a) is the Euler-Lagrange equation of the functional

$$I_a(v) = \frac{\gamma}{2} \int_{\mathbf{R}^+} (v'')^2 + \frac{1}{2} \int_{\mathbf{R}^+} (v')^2 + \frac{1}{4} \int_{\mathbf{R}^+} v^2 g(v, a), \tag{3.7}$$

in which

$$g(v, a) = 4(1-a) - 4\frac{3-a}{3}v + v^2. \tag{3.8}$$

The functional analytic setting. We introduce the affine space $X = \chi + H^2 \cap H_0^1(\mathbf{R}^+)$ in which $\chi \in C^\infty([0,\infty))$ has the properties

$$\chi(x) = 1 \quad \text{for} \quad 0 \le x \le 1 \quad \text{and} \quad \chi(x) = 0 \quad \text{for} \quad x \ge 2.$$

That is, $v \in X$ if and only if $v - \chi \in H^2 \cap H_0^1(\mathbf{R}^+)$. Plainly, X is a closed and therefore weakly closed subset of $H^2(\mathbf{R}^+)$. On X we define the metric

$$d(u, v) = \int_{\mathbf{R}^+} |u'' - v''|^2 + \int_{\mathbf{R}^+} |u - v|^2.$$

From the Sobolev embedding theorem we know that $X \hookrightarrow L^\infty(\mathbf{R}^+)$ and by definition $X \hookrightarrow L^2(\mathbf{R}^+)$. Therefore the interpolation-inequality

$$\|u\|_{L^p} \le \|u\|_{L^2}^{\alpha} \|u\|_{L^\infty}^{1-\alpha}$$

with $p \ge 2$ and $\alpha = 2/p$, yields the embedding $X \in L^p(\mathbf{R}^+)$ for $p \ge 2$. It is straightforward now to show that I_a is a C^1-functional on X [S].

Lemma 3.2. *For every $-3 \le a \le 0$, the functional I_a is sequentially weakly lower-semicontinuous in X.*

Proof. We first make the following observation. We have

$$g(v, a) = \left(v - 2\frac{3-a}{3}\right)^2 - \frac{4}{9}(a^2 + 3a) \ge 0, \tag{3.9}$$

whenever $-3 \le a \le 0$ and so the functional

$$G_a(v) = \frac{1}{4} \int_{\mathbf{R}^+} v^2 g(v, a) \tag{3.10}$$

is non-negative on X whenever $-3 \le a \le 0$.

Let $v \in X$ and let $\{v_n\}$ be a sequence in X which converges weakly in $H^2(\mathbf{R}^+)$ to v. Again by the Sobolev embeddings for $H^2(\mathbf{R}^+)$ we know that weakly convergent sequences in H^2 have convergent subsequences in $C([0, R])$ for every $0 < R < \infty$. This shows that $\{v_n\}$ converges pointwise on $\mathbf{R}^+$ along a subsequence, which we denote again by $\{v_n\}$. Therefore

$$\lim_{n\to\infty} v_n^2 g(v_n, a) = v^2 g(v, a) \quad \text{pointwise on} \quad \mathbf{R}^+.$$

Combining this with Fatou's Lemma we obtain the inequality

$$\liminf_{n\to\infty} \int_{\mathbf{R}+} v_n^2 g(v_n,a) \geq \int_{\mathbf{R}+} v^2 g(v,a),$$

which proves the lower-semicontinuity of G_a. Since the first two terms in I_a are seminorms, the verification that I_a is weakly lower-semicontinuous on X is complete.

Lemma 3.3. *Let $-3 < a < 0$ and let $\{v_n\}$ be a minimizing sequence of I_a. Then there exists a constant $C(a)$, such that*

$$\int_{\mathbf{R}+} (v_n'')^2 + \int_{\mathbf{R}+} (v_n')^2 + \int_{\mathbf{R}+} v_n^2 + \int_{\mathbf{R}+} |v_n|^3 + \int_{\mathbf{R}+} v_n^4 < C(a).$$

Proof. Define the auxiliary functional

$$J(v) = \frac{\gamma}{2}\int_{\mathbf{R}+} (v'')^2 + \frac{1}{2}\int_{\mathbf{R}+} (v')^2 + 4\int_{\mathbf{R}+} v^2 + 2\int_{\mathbf{R}+} |v|^3 + \frac{1}{4}\int_{\mathbf{R}+} v^4, \tag{3.11}$$

Then an easy computation shows that for every $a \in (-3,0)$

$$I_a(v) \leq J(v) \quad \text{when} \quad v \in X. \tag{3.12}$$

Let $\{v_n\}$ be a minimizing sequence for I_a, such that

$$I_a(v_n) \leq \inf I_a + \frac{1}{n} \quad \text{for every} \quad n \geq 1.$$

Such a sequence exists for every $a \in (-3,0)$. It follows from (3.12) that

$$\inf I_a \leq \min J \quad \text{for every} \quad a \in (-3,0),$$

and thus

$$I_a(v_n) \leq \min J + \frac{1}{n} \leq \min J + 1 \stackrel{\text{def}}{=} c^* \tag{3.13}$$

for every $a \in (-3,0)$ and every $n \geq 1$.

We use this uniform upper bound to estimate v_n in L^2 and L^4. From the inequality

$$\int_{\mathbf{R}+} v^3 \leq \frac{\varepsilon}{2}\int_{\mathbf{R}+} v^2 + \frac{1}{2\varepsilon}\int_{\mathbf{R}+} v^4$$

with

$$\varepsilon = 3\frac{1-a}{3-a} + \frac{3-a}{3}.$$

we conclude that

$$\int_{\mathbf{R}+} v^2 g(v,a) < \frac{|a|(3+a)}{3(3-a)}\Big(\int_{\mathbf{R}+} v_n^2 + \int_{\mathbf{R}+} v_n^4\Big),$$

Therefore, by (3.13),

$$\int_{\mathbf{R}^+} v_n^2 + \int_{\mathbf{R}^+} v_n^4 \leq \frac{288c^*}{|a|(3+a)}.$$

It follows that for any $-3 < a < 0$ the sequence $\{v_n\}$ is bounded in $H^2(\mathbf{R}^+)$, and hence in X. Thus, I_a is coercive for every $a \in (-3, 0)$.

Remark. Here we use critically the fact that $-3 < a < 0$, to obtain bounds for $\|v_n\|_{L^2}$ and $\|v_n\|_{L^4}$.

We now turn to the minimization problem: find a function $v \in X$ such that

$$K_a = \inf\{I_a(v) : v \in X\}. \tag{3.14}$$

Lemma 3.4. *Let $-3 < a < 0$. Then the infimum K_a of I_a over X is attained.*

Proof. If $\{v_n\} \subset X$ is a minimizing sequence of I_a, it follows from Lemma 3.3 that $\|v_n\|_{H^2} \leq C(a)$ and thus that $\{v_n\}$ has a weakly convergent subsequence $\{v_{n_k}\}$ in $H^2(\mathbf{R}^+)$, with limit v. Since X is weakly closed in $H^2(\mathbf{R}^+)$ the weak limit of $\{v_{n_k}\}$ is also an element of X. Using the weak lower-semicontinuity of I_a we conclude that

$$K_a \leq I_a(v) \leq \liminf_{k\to\infty} I_a(v_{n_k}) = K_a,$$

which proves that the infimum is indeed attainded by v and therefore that v is a minimizer of I_a.

Because $v \in X$ and $v(0) = 1$, it follows that v is non-trivial and that it is a weak solution of Problem (S_a^+). From the regularity theory for ODE's it then follows that v is also a C^∞-solution.

Lemma 3.5. *Let $-3 < a < 0$. Then I_a has a minimizer which is positive on $\mathbf{R}^+$.*

Proof. Suppose v is a minimizer of (3.14) and $u = 1 - v$. Then $1 - u \in X$ and so $\lim_{x\to\infty} u(x) = 1$. Therefore,

$$y = \inf\{x > 0 : u > 0 \text{ on } (x, \infty)\} < \infty.$$

Remembering from (3.9) that $g(v, a) \geq 0$ when $a \in [-3, 0]$ we find that

$$K_a = I_a(v) \geq \frac{\gamma}{2}\int_y^\infty (v'')^2 + \frac{1}{2}\int_y^\infty (v')^2 + \frac{1}{4}\int_y^\infty v^2 g(v, a) = I_a(v(\cdot + y)).$$

However, $v(\cdot + y) \in X$. Therefore

$$K_a \geq I_a(v(\cdot + y) \geq K_a,$$

and thus $v(\cdot + y)$ is a minimizer which is positive on $(0, \infty)$.

This completes the proof of Theorem 3.1.

For Problem (S_a^-) we can mimic the proof of Theorem 3.1 and so obtain a similar result for the parameter range $0 < a < 3$.

Our next step consists of proving that the solutions u_a of Problem (S_a^+) obtained in Theorem 3.1 converge along a subsequence to a solution of Problem (S_0^+) as a tends to zero. This is the content of the following lemma.

Let

$$\mathcal{A} = \{u_a : -3 < a < 0\}$$

be the family of *positive* minimizers of the functionals I_a.

Lemma 3.6. *The family $\mathcal{A}$ contains a sequence which converges uniformly on compact intervals to a solution $u \in C^\infty([0,\infty))$ of Problem (S_0^+), which has the properties*

$$u \geq 0 \quad \text{on} \quad [0,\infty) \quad \text{and} \quad 1 - u \in H^2(0,\infty).$$

Proof. The proof of this lemma relies on the fact that we can find an a-priori estimate for solutions v_a in $H^2(0,\infty)$ which is uniform with respect to $a \in (-3,0)$. As we observed in (3.12),

$$I_a(v) \leq J(v) \quad \text{for every} \quad v \in X,$$

where J was defined in (3.11). Thus, take any $v^* \in X$. Then

$$K_a = \min I_a = I_a(v_a) \leq I_a(v^*) \leq J(v^*) \stackrel{\text{def}}{=} M. \tag{3.15}$$

Plainly, M does not depend on a. Since we have chosen u_a from $\mathcal{A}$, we know that $u_a > 0$ and hence that $v_a < 1$. Because $\partial g/\partial v < 0$ if $v \leq 1$ and $a < 0$, this implies that

$$g(v_a, a) \geq g(1,a) = -\frac{8}{3}a + 1 > 1 \quad \text{if} \quad a < 0.$$

Therefore

$$\int_{\mathbf{R}^+} v_a^2 \leq \int_{\mathbf{R}^+} v_a^2 g(v_a, a) \leq 4M$$

for all $a \in (-3,0)$. This proves that

$$\|v_a\|_{H^2(0,\infty)} \leq \sqrt{2M(3+\gamma^{-1})}. \tag{3.16}$$

Thus, there exists a constant C such that

$$\|v_a\|_{L^\infty(0,\infty)} \leq C, \tag{3.17}$$

for every $a \in (-3,0)$.

The differential equation (3.6a) for v_a expresses v_a^{iv} in terms of v_a'' and v_a, and so (3.16) and (3.17) yield an estimate for v_a in $H^4(0,\infty)$, which is uniform in $a \in (-3,0)$.

Continuing this bootstrap argument we find that for any $k \geq 1$, there exists a constant C, which does not depend on a, such that

$$\|v_a\|_{H^{2k}(0,\infty)} \leq C.$$

By the Sobolev embedding theorem this means that for any $k \geq 1$,

$$\|v_a\|_{C^{2k-1}([0,\infty))} \leq C,$$

where C is again a constant which does not depend on a. Thus, by compactness there exists a sequence $\{v_{a_n}\}$ such that $a_n \to 0$ as $n \to \infty$, which converges to a function $v \in C^\infty([0,\infty))$ in $C^\ell([0,R])$ for every $\ell \geq 0$ and any $R > 0$.

By passing to the limit in Problem (3.6) we find that v satisfies the differential equation (3.6a) with $a = 0$ and that

$$v(0) = 1, \quad v''(0) = 0 \quad \text{and} \quad v(x) \leq 1 \quad \text{for} \quad 0 \leq x < \infty. \tag{3.18}$$

The upper bound follows from Lemma 3.5.

In addition, because $g(v,a) \geq 0$ by (3.9), we find from (3.14) and (3.15) that for every $R > 0$

$$\frac{\gamma}{2}\int_0^R (v_a'')^2 + \frac{1}{2}\int_0^R (v_a')^2 + \frac{1}{4}\int_0^R v_a^2 g(v_a,a) < M$$

along the sequence $\{v_{a_n}\}$. Hence, in the limit as $n \to \infty$ we obtain

$$\frac{\gamma}{2}\int_0^R (v'')^2 + \frac{1}{2}\int_0^R (v')^2 + \frac{1}{4}\int_0^R v^2(2-v)^2 \leq M. \tag{3.19}$$

If we now let $R \to \infty$ in (3.19) we arrive at the upper bound for I_0:

$$I_0(v) \leq M. \tag{3.20}$$

Since $2 - v \geq 1$, this inequality implies in particular that

$$\int_0^\infty v^2 \leq 4M \tag{3.21}$$

and so, that

$$v \in H^2(0,\infty). \tag{3.22}$$

It follows immediately from (3.22) that the condition at infinity is satisfied:

$$\lim_{x\to\infty} v(x) = 0.$$

Because equation (3.1) is invariant under reflection with respect to the origin, we obtain a kink by extending the solution obtained in Lemma 3.6 symmetrically with respect to the origin.

Theorem 3.7. *Let $\gamma > 0$. Then Problem (I) has an odd solution $u \in C^{\infty}(\mathbf{R})$, such that*

$$u \leq 0 \quad on \quad (-\infty, 0), \qquad u \geq 0 \quad on \quad (0, \infty) \quad and \quad 1 - u \in H^2(0, \infty).$$

Remark. The methods that were used in this section to establish the existence of kinks also apply to equation (1.5), provided the positive coefficients A and B satisfy the inequality $A^2 < 4B$. This condition is used to show that the associated functional is coercive and weakly lower-semicontinuous.

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