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On Some Quasilinear Systems

by

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Introduction

In this paper we will be interested in systems of quasilinear equations whose action-functional is strongly indefinite. The general problem is the following

$$(P) \qquad \begin{cases} -\Delta_p u = F_u(u,v) & \text{in} \quad \Omega \\ \Delta_q v = F_v(u,v), & \text{in} \quad \Omega \\ u = v = 0 \quad \text{on} \quad \partial\Omega \end{cases}$$

where $\Omega \subset \mathbf{R}^N$ is a bounded domain with smooth boundary, $N > 2$, $1 < p, q < N$ and F is a $\mathcal{C}^1$-function. On the function F we impose the following

Coupling Condition

(C): the equations $F_u(0,v) = 0$ and $F_v(u,0) = 0$ have only finitely many solutions.

We shall see later on that due to this coupling condition it is impossible for the problem (P) to have solutions of the form $(u,0)$ or $(0,v)$.

Our goal is to give two methods for finding solutions of Problem (P). Section 1 will be devoted to study problem (P) by means of the reduction to a semilinear system following the ideas by [8], [9]. Here the method consist of a convenient splitting and using a duality argument (see [7] and [4]). The remaining sections will be dealing with a more general case using a Galerkin type argument in combination with finite dimensional linking theorem. (See [1],[14]). The Galerkin approach was previously used by others authors ([3], [11], [14]).

§1.- The spliting method

Due to technical obstructions we study a particular case where we assume $\frac{2N}{N+2} < p < 2$, $q = 2$. Notice that the equations of problem (P) are the Euler Lagrange equations of the functional

$$J_p(u,v) = \frac{1}{p}\int_\Omega |\nabla u|^p dx - \frac{1}{2}\int_\Omega |\nabla u|^2 dx - \int_\Omega F(u,v)dx.$$

The functional J_p is defined on the Banach space $W_0^{1,p}(\Omega) \times W_0^{1,2}(\Omega)$ and clearly J_p it is strongly indefinite, which makes it difficult to apply standard critical point theory for finding solutions of Problem (P). We shall, however, rewrite the equations of problem (P) in such a way that the problem becomes semilinear and stated in a Hilbert space, which simplifies matters. The systems however keeps it strongly indefinite structure. Problem (P) can be equivalently formulated as follows. Define

$$w = |\nabla u|^{p-2}\nabla u,$$

consequently

$$\nabla u = |w|^{\frac{2-p}{p-1}} w.$$

Then the problem (P) becomes

$$
\left\{ \begin{array}{ll} -\operatorname{div} w = & F_u(u,v) \\ \Delta v = & F_v(u,v) \\ \nabla u = & |w|^{\frac{2-p}{p-1}} w \end{array} \right. \tag{1.1}
$$

In order to find weak solutions of Problem (P) we shall study (1.1), since these two formulations are equivalent. This spliting of the p-laplacian allows us to consider the linear differential operator

$$
\mathcal{A} = \begin{pmatrix} 0 & 0 & -\operatorname{div} \\ 0 & \Delta & 0 \\ \nabla & 0 & 0 \end{pmatrix} \tag{1.2}
$$

It is very important to observe that $\operatorname{Ker}(\mathcal{A}) = N(\mathcal{A})$ is infinite dimensional.

THE FUNCTIONAL ANALYTIC SETTING. The operator $\mathcal{A}$ will be considered in the following functional analytic context (see Clément-van der Vorst [8] for notation)

(1) $\mathcal{A} : D(\mathcal{A}) \subset H \longrightarrow H$ where $H = L^2(\Omega) \times L^2(\Omega) \times (L^2(\Omega))^N$ and
$D(\mathcal{A}) = H_0(\Omega)^1 \times H^2 \cap H_0^1(\Omega) \times H_{div}(\Omega)$
and the domain of the divergence is
$H_{div}(\Omega) = \{v \in (L^2(\Omega))^N | v = v_1 + v_2,\ \operatorname{div} v_1 = 0,\ v_2 = \nabla u,\ u \in H^2 \cap H_0^1(\Omega)\}$.
The operator $\mathcal{A}$ is selfadjoint.

(2) With the scalar product in the Hilbert space H and the operator $\mathcal{A}$ we have the following strongly indefinite quadratic form

$$
\frac{1}{2}(\mathcal{A}\phi, \phi)_H = \int_\Omega w \nabla u dx - \frac{1}{2} \int_\Omega |\nabla v|^2 dx \equiv Q(\phi) \tag{1.3}
$$

Here $\phi = (u, v, w)^t$.

Performing integration by parts we have continuous extensions of Q to:

(1) $E_0 = L^2(\Omega) \times H_0^1(\Omega) \times H_{\text{div}}(\Omega)$,
(2) $E_1 = H_0^1(\Omega) \times H_0^1(\Omega) \times (L^2(\Omega))^N$.

Then by interpolation between E_0 and E_1 (see [8]) we obtain continuous extensions of Q to the 1-parameter family of spaces;

$$
E_\alpha = [E_0, E_1]_\alpha = \Theta_1^\alpha(\Omega) \times H_0^1(\Omega) \times \Theta_3^{1-\alpha}(\Omega),
$$

where

$$
\Theta_1^\alpha(\Omega) = D(\Delta^{a/2}) = \{u \in L^2(\Omega);\ (-\Delta)^{\alpha/2} u \in L^2(\Omega)\} = \left\{ \begin{array}{ll} H^\alpha(\Omega), & 0 \le \alpha < 1/2, \\ H_{00}^\alpha(\Omega), & \alpha = 1/2, \\ H^\alpha(\Omega), & 1/2 < \alpha \le 1, \end{array} \right.
$$

and

$$
\Theta_3^{1-\alpha}(\Omega) = \{v \in (L^2(\Omega))^N;\ v = v_1 + v_2,\ \operatorname{div} v_1 = 0,\ v_2 = \nabla u,\ u \in H^{2-\alpha} \cap H_0^1(\Omega)\}.
$$

On $\Theta_1^\alpha(\Omega)$ and $\Theta_3^{1-\alpha}(\Omega)$ we have the usual equivalent norms $\|\cdot\|_{\Theta_1^\alpha}$ and $\|\cdot\|_{\Theta_3^{1-\alpha}}$ and therefore $\|\cdot\|^2_{E_\alpha} = \|\cdot\|^2_{\Theta_1^\alpha} + \|\cdot\|^2_{H_0^1} + \|\cdot\|^2_{\Theta_3^{1-\alpha}}$.

As in [8] we can associate with $\mathcal{A}$ a selfadjoint partial isometry L, such that $\frac{1}{2} < \mathcal{A}\phi, \phi >= \frac{1}{2}(L\phi, \phi)_{E_\alpha}$.

We have the decomposition of E_α $E_\alpha = E^+ \oplus E^- \oplus E^0$, where E^+, E^- and E^0 the eigenspaces associated with the eigenvalues $\lambda = 1$, $\lambda = -1$ and $\lambda = 0$ respectively, of the operator L. Let us consider the operator

$$\partial_\nabla = \begin{pmatrix} 0 & -\mathrm{div} \\ \nabla & 0 \end{pmatrix} \tag{1.4}$$

on $\Theta_1^\alpha(\Omega) \times \Theta_3^{1-\alpha}(\Omega)$, with associated operator L_α (see [8]).

Associated with the eigenvalues $\lambda = 1, -1, 0$ of L_α, we have the eigenspaces E_r^+, E_r^- and $E_r^0 = N(div)$. Using the latter we can chacterize the spaces E^+, E^- and E^0;

(1) $E^+ = \{\phi \in E_\alpha;\ (u, w) \in E_r^+,\ v = 0\}$
(2) $E^- = \{\phi \in E_\alpha;\ (u, w) \in E_r^-,\ v \in H_0^1(\Omega)\}$
(3) $E_0 = \{\phi \in E_\alpha;\ (u, w) \in E_r^0,\ v = 0\}$.

With this decomposition of E_α we have
i) $\mathcal{Q} > 0$ in E^+
ii) $\mathcal{Q} < 0$ in E^-
iii) $\mathcal{Q} = 0$ in E^0.

Sobolev embeddings. In the last section we saw that the quadratic form Q is well-defined on the 1-parameter family of interpolation spaces E_α, $\alpha \in [0, 1]$. If we include the nonlinear part of Problem (P), we need embeddings of E_α in proper L^p-spaces, the so-called Sobolev embeddings. Using the characterization of E_α in terms of Sobolev spaces, we have from the Sobolev embeddings;

$$\begin{cases} E^+ \oplus E^- \hookrightarrow L^{r_1+1} \times L^{r_2+1} \times L^{r_3+1} \equiv X \\ r_1 + 1 < \dfrac{2N}{N-2\alpha}, \quad r_2 + 1 < \dfrac{2N}{N-2}, \quad r_3 + 1 < \dfrac{2N}{N-2+2\alpha}. \end{cases}$$

For r_1 and r_3 this yields the relation

$$\frac{1}{r_1+1} + \frac{1}{r_3+1} > \frac{N-1}{N}.$$

Furthermore $E_0 \subset (L^2)^N$ and (see e.g. [8])
i) $E_0 \cap X$ is closed in X
ii) $E^+ \oplus E^- \oplus (E_0 \cap X)$ is dense in X
iii) $(E_0 \cap X)$ is dense in E_0

Formulation of the Hamiltonian. System (1.1) has of course a variational structure and the equations of System (1.1) are the Euler-Lagrange equations of the function

$$\mathcal{L}(\phi) = \frac{1}{2} < \mathcal{A}\phi, \phi > -\mathcal{H}(\phi) = Q(\phi) - \mathcal{H}(\phi),$$

where Q is defined above and $\mathcal{H}$ is given by

$$\mathcal{H}(\phi) = \frac{p-1}{p} \int_\Omega |w|^{\frac{p}{p-1}} dx + \int_\Omega F(u,v) dx$$

We assume that F is a smooth function, which is stricly convex and meets the following growth conditions

(1) F_u, F_v are strictly monotone with respect to the preorder in $\mathbf{R}^2$.
(2) $c_1|u|^{r_1} \leq |F_u(u,v)| \leq c_2|u|^{r_1} + \sum |u|^{a-1}|v|^b$
(3) $c_3|v|^{r_3} \leq |F_v(u,v)| \leq c_4|v|^{r_3} + \sum |u|^a|v|^{b-1}$
(4) $uF_u + vF_v - \gamma F \geq A(|u|^{r_1} + |v|^{r_3}) - B$ for $\gamma > 2$

with

$$\frac{a}{r_1+1} + \frac{b}{r_3+1} < 1$$

and r_1, r_2, r_3 subcritical. On the other hand we need that for fixed p the component in w in $\mathcal{H}$ must satisfy

$$\frac{p}{p-1} < \frac{2N}{N-2+2\alpha} \qquad \text{for some } 0 < \alpha < 1, \tag{1.5}$$

which can indeed be met when $\frac{2N}{N+2} < p < 2$.

In this way we obtain that $\mathcal{H}$ is well defined and strictly convex on E_α for some appropriate $\alpha \in (0,1)$, depending on the growth of the function $F(u,v)$. We have that

$$\mathcal{L}(\phi) = \mathcal{Q}(\phi) - \mathcal{H}(\phi)$$

is a proper C^1-functional on E_α. We are now in a position to give an existence result using an abstract critical point theorem due to Benci & Fortunato [4], which is based on a dual method.(See [7]).

Theorem 1.1. *Suppose p verifies $\frac{2N}{N+2} < p < 2$ and F satisfies the growth conditions mentioned above. Then Problem (P) has at least one nontrivial solution. Moreover if $F(u,v)$ is even Problem (P) has infintely many non-trivial solutions.*

Proof of Theorem 1.1.. The Sobolev embeddings and the properties of the Hamiltonian allow us to consider the idea of combining duality with critical points theory due to Clarke and Ekeland [7]. See also Benci Fortunato [4], where this method is developped. For sake of completness we include the idea of the proof. More precisely consider $\mathcal{H}^*$ the Legendre transform of $\mathcal{H}$ and $\mathcal{K}$ compact extension of the inverse of $\mathcal{A}$, i.e.

$$\mathcal{K} : E_\alpha^* \overset{i^*}{\hookrightarrow} R(\mathcal{A})^* \overset{\text{Riesz}}{\equiv} R(\mathcal{A}) \overset{(\mathcal{A})^{-1}}{\longrightarrow} R(\mathcal{A}) \overset{i}{\hookrightarrow} E^\alpha$$

Then the new fuctional is defined for $\psi \in W = \overline{(E^+)^*} \oplus \overline{(E^-)^*}$

$$l(\phi) = \mathcal{H}^*(\psi) - \frac{1}{2}\langle w, \mathcal{K}(w)\rangle$$

We can apply the Mountain Pass Theorem in [2] and we get a critical point $w \in W$. This critical point verifies

$$\mathcal{K}(w) = \mathcal{P} d\mathcal{H}^*(w)$$

where the last equality is in $R(\mathcal{A})$ and $\mathcal{P}$ is the projection. But then there exists a $w_0 \in \overline{(E_0 \cap \mathcal{X})}$ such that $w_0 = \mathcal{K}(w) = d\mathcal{H}^*(w)$. Call $\phi = w_0 + \mathcal{K}(w)$, then we get $\mathcal{A}(\phi) = d\mathcal{H}(\phi)$ and this equivalent to the system (1.1). For the rest of details see [4], [5] and [8].

Remark 1.2. The case $q \neq 2$ and $2 > p \geq \frac{2N}{N+2}$ can be handled in a similar way.

The case $q \neq 2$ and $1 < p \leq \frac{2N}{N-2}$ or $p \geq 2$ can also be considered. For example by making an additional splitting for the v-equation (one obtains a semilinear system of four equations - ∂_T-system -). The proper functional analytic setting in a Banach space can be found in [9]. A slightly different system was considered there but the main ideas remain the same. The method for finding critical points seems more delicate in this case and the duality method does not seem straightforward since the Hamiltonian will no longer be convex. It remains an open question how to find critical point for this semilinear system.

§2.- The Galerkin approach

In this section we consider the following particular system, that is we take F to be $F(u,v) = G(u,v) - \frac{1}{r+1}|u|^{r+1} - \frac{1}{s+1}|v|^{s+1}$. Without using the splitting as described in Section 1, we search for solutions of Problem (P) by means of a Galerkin type method. The problem is,

$$\left\{ \begin{array}{ll} -\Delta_p u = |u|^{r-1}u + G_u(u,v) & \text{in} \quad \Omega \\ \Delta_q v = |v|^{s-1}v + G_v(u,v), & \text{in} \quad \Omega \\ u = v = 0 \qquad \text{on} \quad \partial\Omega & \end{array} \right. \tag{2.1}$$

The numbers r and s satisfy $p \leq r+1 < \frac{pN}{N-p}$ and $q \leq s+1 < \frac{qN}{N-q}$. On G we impose the following conditions:
(g1) : $G \in \mathcal{C}^1$, $G_u(0,0) = 0 = G_v(0,0)$ and $G \geq 0$,
(g2) $|\nabla G(u,v)| \leq c_1 + c_2|u|^d + c_3|v|^d$, where $\max(p,q) < d+1 \leq \min(r+1, s+1)$,
(g3) $|G_u(u,0)| = o(|u|^{p-1})$, as $|u| \to 0$,
(g4) $0 < G(u,v) < \frac{1}{p}uG_u + \frac{1}{q}vG_v$ for $|u|, |v| \geq \eta > 0$.
(g5) *Coupling condition:* The equations $G_u(0,v) = 0$ and $G_v(u,0) = 0$ posses only finitely many solutions.

We have

Theorem 2.1. *Let Ω, p, q, r and s as before and suppose that G satisfies the conditions (g1)-(g5). Then problem (2.1) has at least one weak solution $(u,v) \in W_0^{1,p}(\Omega) \times W_0^{1,q}(\Omega)$ with nontrivial components.*

The Lagrangian associated with problem (2.1) is given by
(2.2)

$$J(u,v) = \frac{1}{p}\int_\Omega |\nabla u|^p dx - \frac{1}{q}\int_\Omega |\nabla u|^q dx - \int_\Omega G(u,v)dx - \frac{1}{r+1}\int_\Omega |u|^{r+1} - \frac{1}{s+1}\int_\Omega |v|^{s+1}$$

By conditions (g1) and (g2) J is a $\mathcal{C}^1$-functional on the Banach space

$$\mathbf{E} = W_0^{1,p}(\Omega) \times W_0^{1,q}(\Omega).$$

Weak solutions must be understood as critical points of this functional.

With the hypothesis (g1)-(g5) we do not pursue full generality, since one can allow for G to satisfy less restrictives growth conditions. For instance (g2) can be generalized. A good example for G can be a function $G(u,v) \equiv h(u+v)$ with $h(t)$ satisfying similar conditions. The strategy of the proof is as follows:

1) Consider a family of projected problems on a finite dimensional family of subspaces and we look for positive critical values by means of the Linking Theorem ([1], [14]).
2) We prove a compactness property which is stronger than the usual Palais-Smale condition: the $(PS)^*$-condition.
3) Using the latter condition we pass to the limit and we find a postive critical value of J.

§3.- The proof of theorem 2.1

The space $\mathbf{E}$ is a separable Banach space and therefore there exists a filtering of finite dimensional subspaces such that

1) $E_n = E_n^u \times E_n^v$
2) $E_n \subset E_{n+1}$
3) $\overline{\cup E_n} = \mathbf{E}$. The spaces E_n^u and E_n^v are considered to be the same.

The restiction of J to E_n is denoted by J_n.

Definition 3.1. *We say that the functional J satisfies the $(PS)^*$-condition if for any sequence $\{z_n\} \subset \mathbf{E}$, with $z_n \in E_n$, such that $|J(z_n)| \leq C$ and $J_n'(z_n) \to 0$, implies that there exists a convergent subsequence and the limit is a critical point for J.*

Lemma 3.2. *Let Ω, p, q, r and s as before and suppose that G satisfies (g1), (g2) and (g4). Then the functional defined by (2.1) satisfies condition $(PS)^*$.*

Proof of Lemma 3.2. First we observe the following:

Let $\tilde{z}_n$ be a vector in $\mathbf{E}$ such that $P_n\tilde{z}_n = z_n$ is the vector z_n seen as element of $\mathbf{E}$. Then

$$J_n(z_n) = J(P_n\tilde{z}_n).$$

Now

$$< J_n'(z_n), \phi_n > = < J'(P_n\tilde{z}_n), P_n\tilde{\phi}_n >, \quad \forall \phi_n \in E_n$$

and thus if we write $P_n\tilde{z}_n = z_n$ and $P_n\tilde{\phi}_n = \phi_n$ as a vector in $\mathbf{E}$ we have

$$< J_n'(z_n), \phi_n > = < J'(z_n), \phi_n >$$

Suppose $\{z_n\} \subset \mathbf{E}$ is a sequence as indicated in Definition 3.1. Then

$$
\begin{aligned}
&C+\varepsilon||z_n||_{\mathbf{E}} \geq J(z_n) - < J'(z_n), (\frac{1}{p}u_n, \frac{1}{q}v_n) >= \\
&\Big(\frac{1}{p} - \frac{1}{r+1}\Big)\int\limits_\Omega |u_n|^{r+1} + \Big(\frac{1}{q} - \frac{1}{s+1}\Big)\int\limits_\Omega |v_n|^{s+1} + \\
&\frac{1}{p}\int\limits_\Omega u_n G_u(u_n,v_n) + \frac{1}{q}\int\limits_\Omega v_n G_v(u_n,v_n) - \int\limits_\Omega G(u_n,v_n) \geq \\
&\Big(\frac{1}{p} - \frac{1}{r+1}\Big)\int\limits_\Omega |u_n|^{r+1} + \Big(\frac{1}{q} - \frac{1}{s+1}\Big)\int\limits_\Omega |v_n|^{s+1} - C
\end{aligned}
\tag{3.0}
$$

Furthermore

$$
\Big|\int\limits_\Omega |\nabla u_n|^p - \int\limits_\Omega |u_n|^{r+1} - \int\limits_\Omega u_n G_u(u_n,v_n)\Big| \leq \varepsilon||u_n||_{W_0^{1,p}(\Omega)}
\tag{3.1}
$$

$$
\Big|\int\limits_\Omega |\nabla v_n|^q - \int\limits_\Omega |v_n|^{s+1} - \int\limits_\Omega v_n G_v(u_n,v_n)\Big| \leq \varepsilon||v_n||_{W_0^{1,q}(\Omega)}
\tag{3.2}
$$

We continue with the inequality (3.1)

$$
\begin{aligned}
&\int\limits_\Omega |\nabla u_n|^p - \varepsilon||u_n||_{W_0^{1,p}(\Omega)} \leq \\
&\Big|\int\limits_\Omega |u_n|^{r+1} + \int\limits_\Omega u_n G_u(u_n,v_n)\Big| \leq \\
&\int\limits_\Omega |u_n|^{r+1} + \int\limits_\Omega |u_n||G_u(u_n,v_n)| \leq \\
&\int\limits_\Omega |u_n|^{r+1} + c\int\limits_\Omega |u_n| + C\int\limits_\Omega |u_n|(|u_n|^{d+1} + |v_n|^{d+1})
\end{aligned}
\tag{3.3}
$$

and similarly for (3.2)

$$
\begin{aligned}
&\int\limits_\Omega |\nabla v_n|^q - \varepsilon||v_n||_{W_0^{1,q}(\Omega)} \leq \\
&\int\limits_\Omega |v_n|^{s+1} + c\int\limits_\Omega |v_n| + C\int\limits_\Omega |v_n|(|u_n|^{d+1} + |v_n|^{d+1})
\end{aligned}
\tag{3.4}
$$

Combining (3.3) and (3.4) and using (g2) yields

$$\int_\Omega |\nabla u_n|^p + \int_\Omega |\nabla v_n|^q - \varepsilon ||z_n||_{\mathbf{E}} \leq$$

$$\int_\Omega |u_n|^{r+1} + \int_\Omega |v_n|^{s+1} + c \int_\Omega (|u_n| + |v_n|) + C \int_\Omega (|u_n|^{d+1} + |v_n|^{d+1}) \leq$$

$$C + C \int_\Omega |u_n|^{d+1} + \int_\Omega |v_n|^{d+1}$$

Eventually this gives

$$||z_n||_{\mathbf{E}}^{\alpha} - \varepsilon ||z_n||_{\mathbf{E}} \leq C + C \int_\Omega |u_n|^{d+1} + \int_\Omega |v_n|^{d+1} \tag{3.5}$$

for some $\alpha > 1$. Next we substitute (3.0) in (3.5) which gives

$$||z_n||_{\mathbf{E}}^{\alpha} - \varepsilon ||z_n||_{\mathbf{E}} \leq C + \varepsilon ||z_n||_{\mathbf{E}}$$

proving that $||z_n||_{\mathbf{E}} \leq C$. Because $\mathbf{E}$ is compactly embedded into $L^{r+1}(\Omega) \times L^{s+1}(\Omega)$ this provides the following statements

i) $z_n \rightharpoonup z$ weakly in $\mathbf{E}$

ii) $z_n \to z$ strongly in $L^{r+1}(\Omega) \times L^{s+1}(\Omega)$

iii) $z_n \to z$ a.e.

Because $\mathbf{E}$ is uniformly convex and $||z_n||_{\mathbf{E}} \to ||z||_{\mathbf{E}}$, we obtain strong convergence of $\{z_n\}$ in $\mathbf{E}$. Thus

$$J(z_n) \to J(z) = c$$

Finally we need to verify whether or not z is a critical point of J. We fix a number M. Then

$$| < J'(z_n), \phi_m > | \leq \varepsilon_n ||\phi_m||$$

for $m \leq M \leq n$, provided M is large, and $\varepsilon_n \to 0$ as $n \to \infty$. Along a subsequence this yields

$$< J'(z), \phi_m >= 0 \quad \forall \phi_m \in P_m(\mathbf{E}).$$

The latter holds for all $M \in \mathbf{N}$. Since $\cup E_n$ is dense in $\mathbf{E}$ we get that

$$< J'(z), \phi >= 0 \quad \forall \phi \in \mathbf{E}$$

which proves the lemma.

Remark 3.3. If we take a (PS)-sequence in a fixed E_n it follows from the proof of the lemma 3.2 that the functionals J_n satisfy the (PS)-condition in any finite dimensional subspace E_n. This observation will be useful later on.

Next we investigate some geometric properties of J in order to find critical points by means of a mini-max characterization. Define

$$\mathcal{S} = \{(u,0);\ ||u||_{W_0^{1,p}(\Omega)} = \rho\}$$

Let $e^+ \in E_1^+$ with $\|e^+\| = 1$ and fixed $\rho, r_1, r_2 > 0$, $r_1 > \rho$. Consider

$$\mathcal{Q} = \{(te^+, v);\ e^+ \in W_0^{1,p}(\Omega),\ 0 \le t \le r_1,\ v \in W_0^{1,q}(\Omega),\ ||v||_{W_0^{1,q}(\Omega)} \le r_2\}$$

Then $\partial\mathcal{Q}$ is given by

$$\{(0,w)|;\ |v||_{W_0^{1,q}(\Omega)} \le r_2\} \cup \{(r_1e^+, w);\ ||v||_{W_0^{1,q}(\Omega)} < r_2\} \cup \{(te^+, w);\ 0 \le t \le r_1,\ ||v||_{W_0^{1,q}(\Omega)} = r_2\}.$$

We can prove the following lemma.

Lemma 3.4. *Let Ω, p, q, r and s as before and suppose that G satisfies $(g1)-(g3)$. Then there are numbers $0 < \rho < r_1$ and $r_1 \le r_2$, such that*

$$J|_{\mathcal{S}} \ge \alpha > 0, \quad J|_{\partial\mathcal{Q}} \le 0, \quad J|_{\mathcal{Q}} < \infty$$

Proof of Lemma 3.4. We start with $\mathcal{S}$:

$$J(u,0) = \frac{1}{p}\int_\Omega |\nabla u|^p dx - \int_\Omega G(u,0)dx - \frac{1}{r+1}\int_\Omega |u|^{r+1}dx$$

then

$$J(u,0) \ge \frac{1}{p}\int_\Omega |\nabla u|^p dx - \varepsilon\int_\Omega |u|^p dx - C(\varepsilon)\int_\Omega |u|^{d+1}dx - C(\int_\Omega |\nabla u|^p dx)^{(r+1)/p} \ge \alpha > 0,$$

provided $\rho > 0$ is small enough.

As for $\partial\mathcal{Q}$ we proceed as follows.

a)

$$J(0,v) = -\frac{1}{q}\int_\Omega |\nabla u|^q dx - \int_\Omega G(0,v)dx - \frac{1}{s+1}\int_\Omega |v|^{s+1} \le 0.$$

b)

$$\begin{aligned}
&J(r_1e^+, v) = \\
&\frac{1}{p}r_1^p - \frac{1}{q}\int_\Omega |\nabla v|^q dx - \int_\Omega G(r_1e^+, v)dx - \frac{1}{r+1}\int_\Omega |r_1e^+|^{r+1} - \frac{1}{s+1}\int_\Omega |v|^{s+1} \le \\
&\frac{1}{p}r_1^p - c\frac{1}{r+1}r_1^{r+1} - \frac{1}{q}\int_\Omega |\nabla v|^q dx - \int_\Omega G(r_1e^+, v)dx - \frac{1}{s+1}\int_\Omega |v|^{s+1} \le \\
&\frac{1}{p}r_1^p - c\frac{1}{r+1}r_1^{r+1} \le 0.
\end{aligned}$$

if $r_1 > 0$ is sufficiently large.

c)

$$J(te^+, v) =$$
$$\frac{1}{p}t^p - \frac{1}{q}r_2^q - \int_\Omega G(te^+, v)dx - \frac{1}{r+1}\int_\Omega |te^+|^{r+1} - \frac{1}{s+1}\int_\Omega |v|^{s+1} \leq$$
$$\frac{1}{p}t^p - \frac{1}{q}r_2^q - \int_\Omega G(te^+, v)dx \leq 0.$$

if $r_2 \geq r_1$.

Finally

$$J(te^+, v) \leq \frac{1}{p}r_1^p < \infty$$

and

$$J(z) > -\infty, \quad z \in \mathcal{Q}.$$

This completes the proof.

We observe now that by the choice of the subspaces E_n the geometric properties of J are preserved after the restriction to E_n. To be more precise:

i) $S_n = \mathcal{S} \cap E_n$

ii) $Q_n = \mathcal{Q} \cap E_n$ and

iii) $J_n|_{S_n} \geq \alpha > 0$, $J_n|_{\partial Q_n} \leq 0$, $J_n|_{Q_n} < \infty$.

Proof of Theorem 2.1. From Remark 3.3 it follows that J_n satisfies (PS) for every $n \geq 2$ and for Lemma 3.4 it follows that J_n satisfies the conditions of the *Linking Theorem* (see [1], [5] and [14]) and this gives that for every n, J_n has a *nontrivial critical point*, $z_n \in E_n$ with critical value

$$0 < \alpha \leq J_n(z_n) = c_n \leq C$$

We thus have a sequence $\{z_n\} \subset E_n$ for which $||J_n'(z_n)|| = 0$ and $|J_n(z_n)| \leq C$. Then, since J satisfies the $(PS)^*$-condition, it follows that there exists a subsequence $\{z_{n_k}\} \subset \mathbf{E}$ converging to $z \in \mathbf{E}$, which is a nontrivial critical point of J ($u \not\equiv 0$ or $v \not\equiv 0$).

As for the proof that both components are nontrivial we argue as follows. Suppose that $v \equiv 0$, then u satisfies the equation

$$-\Delta_p u = |u|^r u + G_u(u, 0).$$

¿From regularity theory for this equation (see [10],[15]) if follows that $u \in C^0(\overline{\Omega}) \cap C^{1,\alpha}(\Omega)$. On the other hand we know from (g5) that the equation

$$G_v(u, 0) = 0,$$

has only finitely many solutions and thus due to the continuity of u this gives that $u = C$, a constant, in $\overline{\Omega}$. The boundary condition on u then determines the constant to be zero, which would imply that $(u, v) \equiv (0, 0)$, a contradiction. The same argument holds for v. This proves that the solution has non-trivial components.

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