Slow Motion in Higher-Order Systems and Γ-Convergence in One Space Dimension

William D. Kalies[1] *Robert C.A.M. VanderVorst*[1]
Thomas Wanner[2]
Center for Dynamical Systems and Nonlinear Studies
Georgia Institute of Technology
Atlanta, Georgia 30332, USA

February 2, 1997

Abstract

We present a new variational approach for proving exponentially slow motion in singularly-perturbed partial differential equations in one space dimension, which builds on the energy approach due to Bronsard and Kohn [11] and Grant [34]. As well as covering the known applications, this approach is also capable of proving slow motion in equations whose corresponding Lyapunov functionals contain higher-order derivatives. Moreover, we obtain results on Γ-convergence of these functionals.

Keywords: Slow motion, gradient system, metastability, transition layers, phase separation, heteroclinic orbits, bistable evolution equations, Γ-convergence

1 Introduction

Perhaps one of the simplest mathematical models for the temporal evolution of a mixture of two phases of the same substance at the transition temperature is the Allen-Cahn equation

$$\begin{aligned} u_t = \varepsilon^2 u_{xx} - F'(u) \quad &\text{in} \quad (a,b)\ , \\ u_x = 0 \quad &\text{on} \quad \{a,b\}\ , \end{aligned} \tag{1}$$

[1]Partially supported by grants ARO DAAH-0493G0199 and NIST G-06-605.
[2]On leave from the Universität Augsburg. Work supported by the Deutsche Forschungsgemeinschaft, “Forschungsstipendium” Wa 960/3-1, Wa 960/3-2.

where for the purposes of this introduction $F(u) = (u^2 - 1)^2/4$ is the usual double-well potential with equal depth wells. This equation is the L^2-gradient flow equation associated with the van der Waals free energy functional [52]

$$\mathcal{E}_\varepsilon[u] = \int_a^b \left(\frac{\varepsilon^2}{2} u_x^2 + F(u) \right) dx \tag{2}$$

and generates a nonlinear semiflow on $H^1(a, b)$, cf. Hale [36] or Henry [39].

Following Fusco and Hale [31] the value $u(x)$ can be interpreted as an order parameter which is related to the microscopic structure of the substance in such a way that $u(x) \approx +1$ corresponds to one of the two phases and $u(x) \approx -1$ corresponds to the other. The two equal global minima of F at ± 1 reflect the fact that at the transition temperature neither of these phases is physically preferred. The first term in $\mathcal{E}_\varepsilon[u]$ penalizes high derivatives of the function u and therefore models the tendency of the substance to minimize the number of interfaces separating the two phases. Since the relative importance of this effect is assumed to be very small, the same applies to the parameter $\varepsilon > 0$.

Obviously, the energy functional $\mathcal{E}_\varepsilon[u]$ has the two constant global minimizers $u \equiv \pm 1$ on $H^1(a, b)$, which are in fact the only local minimizers. Since (1) is the associated L^2-gradient flow equation, the orbits corresponding to "generic" initial conditions converge to one of these two global minimizers.

The dynamical behavior of generic orbits of (1) can be described in more detail as follows. Starting with a smooth initial condition which has only finitely many simple zeros, the orbit quickly evolves to a state which is close to a step function with values in $\{\pm 1\}$ and has sharp transition layers. Diffusive effects then drive the transition layers towards one another or the endpoints of the interval (a, b) at an extremely low speed. Once some of the layers are close enough to a, b, or one another, they disappear quickly, and again the orbit enters the regime of extremely slow motion of transition layers. If ε is very small, the last two steps can be repeated many times, until finally all the transition layers have disappeared and the orbit converges to either $+1$ or -1. For a more precise dynamical description the reader is referred to Fusco and Hale [31].

Numerical experiments by Pego and McKinney (cf. Fusco and Hale [31]) confirmed the above description some ten years ago. They also indicated that the speed of the slow motion should be of the order $e^{-C/\varepsilon}$, for sufficiently small $\varepsilon > 0$. Of course, from a numerical point of view, this extremely low speed raises questions concerning the reliability of such numerical calculations. For the case of slow motion in the Cahn-Hilliard equation this was pointed out by Elliott and French [22]. Only recently has it been demonstrated that meaningful computation of slow motion is actually possible, compare for example Estep [23], Estep, Verduyn Lunel, and Williams [24], Estep and Williams [25], as well as Reyna and Ward [49].

As the number of references attests, much analysis has been devoted to the study of slow motion of transition layers in the solutions to one-dimensional bistable partial differential equations. There are basically two approaches to a mathematical understanding of this phenomenon. However a third, *stochastic approach* due to Freidlin [28, 29] yields similar, yet weaker results.

The *dynamical approach* is due to Carr and Pego [15, 16, 17], Fusco and Hale [31], and Fusco [30]. They construct invariant manifolds consisting of functions which are close to step functions with values in $\{\pm 1\}$. Furthermore, they derive equations describing the flow of (1) on these "slow motion manifolds" and obtain that the speed indeed is of order $O(e^{-C/\varepsilon})$. Based on these ideas slow motion results have been proved for several other equations and situations: Alikakos, Bates, and Fusco [1], as well as Bates and Xun [8, 9], consider the one-dimensional Cahn-Hilliard equation, Alikakos and Fusco [2, 3, 4] treat the case of bubbles in the higher-dimensional Cahn-Hilliard equation, and Pinto [46] proves the existence of slow motion manifolds for the Allen-Cahn equation with a tristable potential. Recently Sandstede [50] has presented a general result which is applicable to many different situations.

In contrast, there is the *energy approach* due to Bronsard and Kohn [11]. Using rather simple energy estimates they prove that for initial conditions which are sufficiently close to a step function with values in $\{\pm 1\}$, the transition layers move with a speed slower than $O(\varepsilon^k)$ for every $k \in \mathbb{N}$. Grant [34] improves their method to obtain the exponential upper bound $O(e^{-C/\varepsilon})$ for the speed of the motion. Bronsard and Hilhorst [10] have also applied this method successfully to the one-dimensional Cahn-Hilliard equation, and Grant [34] has proven exponentially slow motion for Cahn-Morral systems. Other applications can be found in Hattori and Mischaikow [38] and in Grant and Van Vleck [35].

Each of these methods has its advantages and drawbacks. Whereas the dynamical approach yields very exact results, including the exact order of the speed of slow motion, the proofs are complicated and lengthy. The energy approach is fairly simple and provides a rather clear and intuitive explanation for slow motion, but it gives only an upper bound for the speed. Common to both approaches is that, apart from Sandstede's work [50], all of the above results are restricted to energy functionals of the form (2) (or slightly more generally to the functionals considered by Owen [45]) with different types of associated dynamics.

To illustrate why the energy approach can handle only these energy functionals we briefly present the main ideas contained in Bronsard and Kohn [11]. Rather than considering the free energy functional (2), they define the scaled energy functional

$$E_\varepsilon[u] = \frac{1}{\varepsilon} \int_a^b \left(\frac{\varepsilon^2}{2} u_x^2 + F(u) \right) dx \; . \tag{3}$$

Results of Modica [43] and Sternberg [51] show that asymptotically there is a minimum energy

$$c_0 = \int_{-1}^{1} \sqrt{2F(s)}ds = \frac{1}{\sqrt{2}} \int_{-1}^{1} \left(1 - s^2\right) ds = \frac{2\sqrt{2}}{3}$$

needed to make a transition between -1 and $+1$ in the following sense. For any family of $H^1(a,b)$-functions v^ε with $L^1 - \lim_{\varepsilon\to 0} v^\varepsilon = v^0$, where v^0 denotes a step function with values in $\{\pm 1\}$ and exactly N jump discontinuities, we have

$$\liminf_{\varepsilon\to 0} E_\varepsilon[v^\varepsilon] \geq N c_0 \ .$$

The main step in the argument of Bronsard and Kohn is the following improvement of this estimate:

- For any $k \in \mathbb{N}$ there exist constants $\delta_k, c_k > 0$ such that, if $w \in H^1(a,b)$ satisfies both $||w - v^0||_{L^1(a,b)} \leq \delta_k$ and $E_\varepsilon[w] \leq N c_0 + \varepsilon^k$ with $\varepsilon \in (0,1]$, then $E_\varepsilon[w] \geq N c_0 - c_k \varepsilon^k$.

This estimate, together with the fact that (1) is the L^2-gradient flow associated with $\varepsilon E_\varepsilon$, readily implies that on a time scale of order ε^{-k} there is no appreciable change in the position of the transition layers, cf. Bronsard and Kohn [11] or Section 2 below. The proof of the estimate itself is not too difficult, since it relies mainly on the fact that for $y, z \in (a,b)$ with $y < z$ the arithmetic-geometric mean inequality yields

$$\int_y^z \left(\frac{\varepsilon}{2} w_x^2 + \frac{1}{\varepsilon} F(w)\right) dx \geq \int_y^z \sqrt{2}\,|w_x|\,\sqrt{F(w)}dx \geq |g(w(z)) - g(w(y))| \ ,$$

where $g'(s) = \sqrt{2F(s)} = |1 - s^2|/\sqrt{2}$. Under the above hypotheses there are points $y < z$ near a given transition layer such that $w(y)$ and $w(z)$ are in $O(\varepsilon^{k/2})$-neighborhoods of ± 1. Together with $c_0 = g(1) - g(-1)$ this proves the lower bound.

Similarly, Grant's refinement of the above approach (cf. [34]), which yields the exponential upper bound on the speed, crucially relies on the above application of the arithmetic-geometric mean inequality, as well as the maximum principle. These tools however cannot be used if the energy functional contains additional higher-order terms of the form $\varepsilon^{2k} u^{(k)}$. As shown for example in Bates and Ren [7], Coullet, Elphick, and Repaux [18], and Dee and van Saarloos [19], such functionals arise in the theory of phase transitions and are therefore of interest both from a physical and mathematical point of view.

To handle these more general classes of energy functionals somewhat different methods have to be employed. The first step in this direction is marked by the following two observations concerning the above arguments:

- If $u \in H^1(a,b)$, and we consider the rescaled mapping $u_\varepsilon \in H^1(a/\varepsilon, b/\varepsilon)$ defined by $u_\varepsilon(x) := u(\varepsilon x)$, then obviously $E_\varepsilon[u] = E[u_\varepsilon; (a/\varepsilon, b/\varepsilon)]$, where for every interval $I \subset \mathbb{R}$ and any $w \in H^1(I)$ we define

$$E[w; I] = \int\limits_I \left(\frac{1}{2} w_x^2 + F(w) \right) dx \ .$$

- The functional $E[\cdot; \mathbb{R}]$ has a unique minimizer q in the affine space $\chi + H^1(\mathbb{R})$, where $\chi \in C^\infty(\mathbb{R})$ is a fixed arbitrary function with $\chi(x) = 1$ for $x \geq 1$ and $\chi(x) = -1$ for $x \leq -1$. Note that proving this assertion mainly consists of applying the arithmetic-geometric mean inequality as above. The function q solves the Euler-Lagrange equation

$$-q_{xx} + F'(q) = 0 \ ,$$

and $c_0 = E[q; \mathbb{R}]$. For the particular choice of $F(u) = (u^2 - 1)^2/4$ the minimizer q is given by $q(x) = \tanh(x/\sqrt{2})$.

These observations show that there is a connection between the minimum energy c_0 for making a transition from -1 to $+1$ and the energy $E[q; \mathbb{R}]$ of the heteroclinic orbit q. This provides us with a different way for obtaining the energy estimate of Bronsard, Kohn, and Grant.

Consider a function $u \in H^1(a,b)$ "close" to a given step function v which is -1 on (a, γ) and $+1$ on (γ, b), for some $\gamma \in (a,b)$. Then the first of the above two observations implies that the rescaled mapping u_ε satisfies the identity $E[u_\varepsilon; (a/\varepsilon, b/\varepsilon)] = E_\varepsilon[u]$. If we now extend u_ε to a function $w \in \chi + H^1(\mathbb{R})$ by "efficiently" adding tails on both $(-\infty, a/\varepsilon)$ and $(b/\varepsilon, \infty)$, the energy is increased by some small amount $\delta > 0$ so that $E[w; \mathbb{R}] = E[u_\varepsilon; (a/\varepsilon, b/\varepsilon)] + \delta$. Furthermore, since q is a global minimizer of $E[\cdot; \mathbb{R}]$ on $\chi + H^1(\mathbb{R})$, the inequality $E[w; \mathbb{R}] \geq E[q; \mathbb{R}]$ holds. Finally we deduce

$$E_\varepsilon[u] = E[u_\varepsilon; (a/\varepsilon, b/\varepsilon)] = E[w; \mathbb{R}] - \delta \geq E[q; \mathbb{R}] - \delta = c_0 - \delta \ ,$$

which is the desired estimate if δ can be chosen sufficiently small.

Our *variational approach* for proving slow motion is based on the above heuristic description. Although there are several technicalities which have to be overcome, the approach will prove exponentially slow motion for all of the known examples, as well as equations which could not be handled by the previous energy methods. The basic ingredients needed to apply the variational approach to a given semiflow can roughly be summarized as follows:

- The ε-dependent semiflow under consideration has a Lyapunov function E_ε defined on some space $H^n(a,b)$, and the constants ± 1 are the unique global minimizers of E_ε on $H^n(a,b)$ (this can be generalized, cf. Section 3).

- There exists an ε-independent energy functional E defined on some subset of $W^n(\mathbb{R})$, the set of all functions on $\mathbb{R}$ with weak derivatives up to order n, such that for any $u \in H^n(a,b)$ and the corresponding rescaled function u_ε the identity $E_\varepsilon[u] = E[u_\varepsilon; (a/\varepsilon, b/\varepsilon)]$ holds. As before, for any interval $I \subset \mathbb{R}$ the functional $E[\cdot; I]$ is obtained from E by integrating over I rather than over $\mathbb{R}$.

- The functional E has (not necessarily unique) global minimizers $q_\pm$ on the affine spaces $\chi_\pm + H^n(\mathbb{R})$, where $\chi_\pm \in C^\infty(\mathbb{R})$ are arbitrary but fixed functions satisfying $\chi_\pm(x) = \mp 1$ for $x \le -1$ and $\chi_\pm(x) = \pm 1$ for $x \ge 1$.

- Finally, the equilibria ± 1 are hyperbolic stationary solutions of the Euler-Lagrange equation for E, and their local stable and unstable manifolds satisfy certain geometric conditions which will be specified in more detail below.

The paper is organized as follows. In Section 2 we will present our variational approach for a specific, yet new situation. More precisely, we consider ε-dependent energy functionals of the form

$$E_\varepsilon[u] = \frac{1}{\varepsilon} \int_a^b \left(\sum_{k=1}^n \frac{\gamma_k \varepsilon^{2k}}{2} \left(D^k u\right)^2 + F(u) \right) dx \tag{4}$$

for $n \ge 2$, where $D^k u := \partial^k u / \partial x^k$ denotes the k-th derivative of the function u with respect to x. The precise assumptions on the constants γ_k and the potential F are given in Subsection 2.1 below. We note however that they are satisfied in particular if all the γ_k are positive and F is the standard double-well potential $F(u) = (u^2-1)^2/4$.

There are several possibilities for assigning gradient dynamics to the energy functional E_ε. For simplicity in the presentation of our method we consider the simplest one given by the L^2-gradient, which leads to the singularly-perturbed parabolic equation

$$u_t = \sum_{k=1}^n (-1)^{k+1} \varepsilon^{2k} \gamma_k D^{2k} u - F'(u) \quad \text{in } (a,b) \tag{5}$$

subject to suitable boundary conditions. Then (5) generates a nice semiflow on a subspace of $H^n(a,b)$, cf. for example Henry [39].

Section 2 is divided into several subsections. The first is devoted to giving the precise assumptions on the parameters in (4) and deriving certain properties of the ε-independent energy functional E associated with E_ε, given by

$$E[u; I] = \int_I \left(\sum_{k=1}^n \frac{\gamma_k}{2} \left(D^k u\right)^2 + F(u) \right) dx \ , \quad I \subset \mathbb{R} \ . \tag{6}$$

We establish the existence of globally minimizing heteroclinic solutions $q_\pm$ of the Euler-Lagrange equation of $E[\cdot;\mathbb{R}]$. This equation is given by

$$\sum_{k=1}^{n}(-1)^k\gamma_k D^{2k}u + F'(u) = 0\ . \tag{7}$$

Subsection 2.1 concludes with some results about fixed points of (7), i.e. we verify hyperbolicity and establish certain geometric properties of the local stable and unstable manifolds at these points.

The main result of our variational approach is the subject of Subsection 2.2. Here an energy estimate in the spirit of Bronsard, Kohn [11] and Grant [34] will be derived, following the heuristic description outlined above.

In the following subsection, this energy estimate is interpreted within the theory of Γ-convergence of functionals. Thereby we generalize certain results of Modica [43], Owen [45], Sternberg [51], and Fonseca and Tartar [27] to functionals of the form (4).

Both Subsection 2.2 and 2.3 contain only results concerning the energy functionals E_ε. In the final Subsection 2.4 these are used to prove exponentially slow motion for the partial differential equation (5). The main result given in Proposition 2.12 states that for initial data sufficiently close to a step function the transition layers do not significantly move on a time scale of order $\exp(C/\varepsilon)$. This relies crucially on the identity

$$E_\varepsilon[u(0,\cdot)] - E_\varepsilon[u(T,\cdot)] = \frac{1}{\varepsilon}\int_0^T ||u_t(\tau,\cdot)||^2_{L^2(a,b)}\,d\tau \tag{8}$$

which holds for every solution u of (5), provided the boundary conditions are chosen appropriately.

As we already mentioned above, our variational approach for proving slow motion also applies to several other situations. Therefore we briefly summarize the main parts of the approach in Subsection 3.1, thus pointing out where changes can or have to be made. The remaining subsections of Section 3 address several specific examples such as equations of Allen-Cahn and Cahn-Hilliard type, the phase field model, as well as others.

Throughout the paper, the letter C denotes a generic positive constant which is always independent of ε and whose value may change from line to line, sometimes even within one line. The constants $C_1, C_2, \ldots$ and $C_a, C_b, \ldots$ however, refer to specific ε-independent constants which will be used several times.

2 The Variational Approach

2.1 Preliminaries

It has already been mentioned in the introduction that our approach for proving slow motion will be presented for functionals of the form (4). The specific assumptions on the constants γ_k and the potential F are as follows:

(H1) There exist positive constants d_0 and η such that for every interval $I \subset \mathbb{R}$ with length $|I| \geq d_0$ and all $u \in H^n(I)$ we have

$$\int\limits_I \left(\sum_{k=1}^n \gamma_k \left(D^k u\right)^2 \right) dx \geq \eta \int\limits_I \left(\left(D^n u\right)^2 + \left(D^1 u\right)^2 \right) dx \; .$$

(H2) The function $F : \mathbb{R} \to \mathbb{R}$ is a smooth double-well potential with equal depth wells at ± 1. More precisely, we assume $F(\pm 1) = 0$, $F''(\pm 1) > 0$, and $F(u) > 0$ for all $u \in \mathbb{R} \setminus \{\pm 1\}$. Furthermore, we require $\inf\{F(u) : |u| \geq 2\} > 0$.

The above two hypotheses are satisfied in particular if all the constants γ_k are positive and if $F(u) = (u^2 - 1)^2/4$.

Notice also that (H1) implies the following fact which will be used frequently in the sequel without further mention. For any interval $(a, b) \subset \mathbb{R}$, all $u \in H^n(a, b)$, and every $0 < \varepsilon \leq (b - a)/d_0$ the estimate

$$\int\limits_a^b \left(\sum_{k=1}^n \gamma_k \varepsilon^{2k} \left(D^k u\right)^2 \right) dx \geq \eta \int\limits_a^b \left(\varepsilon^{2n} \left(D^n u\right)^2 + \varepsilon^2 \left(D^1 u\right)^2 \right) dx$$

holds.

Besides E_ε we further consider the ε-independent energy functional $E[\cdot; I]$, which was defined in (6), and the remainder of this subsection is devoted to establishing certain properties of this functional and its Euler-Lagrange equation given in (7). The first and most important one is the existence of global minimizers of $E[\cdot; \mathbb{R}]$.

Lemma 2.1 *Let $\chi_\pm \in C^\infty(\mathbb{R})$ denote arbitrary but fixed functions satisfying both $\chi_\pm(x) = \mp 1$ for all $x \leq -1$ and $\chi_\pm(x) = \pm 1$ for $x \geq 1$. Furthermore, assume that both* (H1) *and* (H2) *hold.*

Then the functional E has (not necessarily unique) global minimizers $q_\pm$ on the affine spaces $\chi_\pm + H^n(\mathbb{R})$ with corresponding energies $E_\pm := E[q_\pm; \mathbb{R}]$. Moreover, the functions $q_\pm$ solve the Euler-Lagrange equation (7).

Proof: For $n = 2$ the proof follows immediately from arguments in Kalies and VanderVorst [42, Section 2]. For the case of general n we employ methods of Rabinowitz [47, Section 3], which will be outlined briefly for the sake of completeness. We only prove the existence of q_+. Consider the space

$$X := \{ w \in H^n_{loc}(\mathbb{R}) : ||D^1 w||_{H^{n-1}(\mathbb{R})} < \infty \} .$$

Then the norm

$$||w||_X := \sqrt{w(0)^2 + ||D^1 w||^2_{H^{n-1}(\mathbb{R})}}$$

makes X a Hilbert space. Furthermore, if $w \in X$ satisfies $E[w; \mathbb{R}] < \infty$ then (H1), together with Lemma 3.6 and Proposition 3.11 of [47], implies that $w(x)$ converges for both $x \to +\infty$ and $x \to -\infty$, and the limits are contained in $\{\pm 1\}$.

Define $E_+ := \inf\{E[w; \mathbb{R}] : w \in \chi_+ + H^n(\mathbb{R})\}$ and let $(q_k)_{k \in \mathbb{N}}$ denote a corresponding minimizing sequence. This minimizing sequence can be chosen to be bounded in X (note that $\chi_+ + H^n(\mathbb{R}) \subset X$), hence we may assume, possibly after passing to a subsequence, that there is a $q_+ \in X$ such that (q_k) converges weakly to q_+. Standard arguments, together with the weak lower semicontinuity of $E[\cdot; \mathbb{R}]$, then yield $E[q_+; \mathbb{R}] \le E_+$, which in turn implies that the limits $\lim_{x \to \pm\infty} q_+(x)$ exist and are contained in $\{\pm 1\}$, by the remark above.

In order to conclude the proof we only have to verify $\lim_{x \to \pm\infty} q_+(x) = \pm 1$. For this we may assume (after passing to subsequences if necessary) that the minimizing sequence converges to q_+ in $C^{n-1}[a, b]$ for all $a < b$, and that for every $k \in \mathbb{N}$ we have $q_k(0) = 0$, as well as $q_k(x) < 0$ for every $x < 0$. This immediately yields the identity $\lim_{x \to -\infty} q_+(x) = -1$.

Now suppose that $\lim_{x \to +\infty} q_+(x) = -1$, and let $0 < \delta < 1/2$. Then there exists a point $x_\delta > 0$ such that

$$|D^\ell (q_+ + 1)(x_\delta)| \le \frac{\delta}{2} \quad \text{for} \quad \ell = 0, \ldots, n-1 .$$

Since (q_k) converges to q_+ in $C^{n-1}[x_\delta - 1, x_\delta + 1]$ this leads to

$$|D^\ell (q_k + 1)(x_\delta)| \le \delta \quad \text{for} \quad \ell = 0, \ldots, n-1 \quad \text{and} \quad k \ge k_0 .$$

Energy estimates as in Rabinowitz [47, Proposition 3.12] (cf. also Lemma 2.3 below) further imply that there exists a constant $c_* > 0$, which is independent of δ, such that for all $k \ge k_0$ we have

$$E[q_k; \mathbb{R}] \ge c_* + E[q_k; [x_\delta, \infty)] .$$

Now let $p_k(x) := q_k(x)$ for $x > x_\delta$, and on $(-\infty, x_\delta]$ let p_k denote the unique polynomial of degree $n-2$ satisfying $D^\ell p_k(x_\delta) = D^\ell q_k(x_\delta)$, for $\ell = 0, \ldots, n-1$. Finally, let ψ denote a $C^\infty(\mathbb{R})$-function which vanishes on $(-\infty, -1]$ and is equal to 1 on $[0, \infty)$. Then the definition

$$Q_k(x) := (p_k(x) + 1) \cdot \psi(x - x_\delta) - 1$$

furnishes a sequence (Q_k) which is contained in $\chi_+ + H^n(\mathbb{R})$. Furthermore, if $\delta > 0$ is sufficiently small we have

$$\begin{aligned} E[Q_k;\mathbb{R}] &= E[Q_k;[x_\delta - 1, x_\delta]] + E[q_k;[x_\delta;\infty)] \leq C\delta + E[q_k;[x_\delta,\infty)] \\ &\leq C\delta - c_* + E[q_k;\mathbb{R}]\ , \end{aligned}$$

where $C > 0$ is independent of δ— contradicting the fact that (q_k) is a minimizing sequence. Finally, standard arguments show that q_+ solves the Euler-Lagrange equation (7). $\diamond$

Apart from the existence of these globally minimizing heteroclinics, we only need some simple properties of the Euler-Lagrange equation (7) associated with the functional $E[\cdot;\mathbb{R}]$. As usual, we think of (7) as a $2n$-dimensional system of first order by introducing the variables $v_k := D^{k-1}u$ for $k = 1,\ldots,2n$. Then due to (H2) the points $p_\pm := (\pm 1, 0, \ldots, 0)$ are fixed points of (7). Additionally, the following lemma holds.

Lemma 2.2 *Under hypotheses* (H1) *and* (H2) *the fixed points* $p_\pm := (\pm 1, 0, \ldots, 0)$ *are hyperbolic. Their local stable and unstable manifolds are of dimension* n *and can be represented as graphs of functions in the variables* $v_1,\ldots,v_n$*. Furthermore, there exists a strictly positive constant* λ *such that* $\lambda < |Re\ \mu|$*, for all eigenvalues* μ *of the linearization of (7) at the points* $p_\pm$*.*

Proof: Since (H1) implies the estimate $\sum_{k=1}^n \gamma_k \xi^{2k} \geq \eta(\xi^{2n} + \xi^2)$ for all $\xi \in \mathbb{R}$, the hyperbolicity of $p_\pm$ is an easy calculation. The simple proof of the remaining assertions will be omitted. $\diamond$

It will become clear in the next subsections that the two lemmas above are the main ingredients for our variational approach to slow motion.

2.2 The Energy Estimate

In this subsection we derive the main result (Proposition 2.6) of our variational approach— an energy estimate in the spirit of Bronsard, Kohn [11] and Grant [34]. Similar to their method we consider initial data u_ε depending on ε. The following lemma shows that the boundedness of the scaled energy $E_\varepsilon[u_\varepsilon]$ alone suffices to ensure the closeness of u_ε to the potential wells everywhere in the interval (a,b) except for a set of measure $O(\varepsilon)$, provided $\varepsilon > 0$ is sufficiently small.

Lemma 2.3 *Assume that both* (H1) *and* (H2) *hold. Let* $M > 0$ *and* $\varrho > 0$*. Then there exists a constant* $C_1 > 0$ *such that for any* $0 < \varepsilon \leq (b-a)/d_0$ *and every* $u \in H^n(a,b)$ *with* $E_\varepsilon[u] \leq M$ *the following property holds. There is a set* $J \subset (a,b)$ *with Lebesgue measure* meas $J \leq C_1\varepsilon$ *such that*

$$\operatorname{dist}(u(x),\{\pm 1\}) < \varrho \qquad \textit{and} \qquad \left|\varepsilon^k D^k u(x)\right| < \varrho$$

hold for all $x \in (a,b) \setminus J$ *and* $k = 1, \ldots, n-1$. *Here* d_0 *is the constant introduced in* (H1) *and* $\mathrm{dist}(y, P) := \inf\{|y-p| : p \in P\}$.

Proof: Using (H1) together with the interpolation inequalities of Lemma 4.1 in the appendix, it can easily be seen that for all $0 < \varepsilon \le (b-a)/d_0$ and $w \in H^n(a,b)$ the estimate

$$E_\varepsilon[w] \ge \frac{1}{\varepsilon} \int_a^b \left(\sum_{k=1}^n \frac{\eta_k \varepsilon^{2k}}{2} \left(D^k w\right)^2 + F(w) \right) dx \tag{9}$$

holds, with positive constants η_k. Now let $J_0 := \{x \in (a,b) : \mathrm{dist}(u(x), \{\pm 1\}) \ge \varrho\}$. From hypothesis (H2) we have $c := \inf\{F(w) : \mathrm{dist}(w, \{\pm 1\}) \ge \varrho\} > 0$. Then the assumptions of the lemma, together with the positivity of the η_k, imply

$$M \ge E_\varepsilon[u] \ge \frac{1}{\varepsilon} \int_a^b F(u) dx \ge \frac{1}{\varepsilon} \int_{J_0} F(u) dx \ge \frac{c}{\varepsilon} \cdot \ \mathrm{meas}\ J_0 \ ,$$

and therefore meas $J_0 \le \varepsilon M / c$. Similarly, for $J_k := \{x \in (a,b) : |\varepsilon^k D^k u(x)| \ge \varrho\}$ the estimate

$$M \ge E_\varepsilon[u] \ge \frac{1}{\varepsilon} \int_a^b \left(\frac{\eta_k \varepsilon^{2k}}{2} \left(D^k u\right)^2 \right) dx \ge \frac{\eta_k \varrho^2}{2\varepsilon} \cdot \mathrm{meas}\ J_k$$

furnishes meas $J_k \le 2M\varepsilon / (\eta_k \varrho^2)$. Defining $J := J_0 \cup \ldots \cup J_{n-1}$ completes the proof of the lemma. $\diamond$

Remark 2.4 In fact, it is possible to choose the set J in such a way that it consists of finitely many intervals whose number is bounded by a constant C_2. As C_1, this constant depends only on M, ϱ, and the energy functional E. This can be achieved by a slightly less straightforward definition of the sets J_k, in combination with (9) and the arithmetic-geometric mean inequality, but will not be needed later on. $\square$

Whereas the preceding lemma proves the closeness of the function u to the potential wells on a large set, it obviously can not specify the position of the transition layers of u. It is somewhat surprising that a very weak notion of closeness to a given step function u_0 with values in $\{\pm 1\}$ already suffices to fix these positions, as well as to yield the desired exponential energy estimate from below. This will be demonstrated in the following main result of this section. First, however, we have to introduce a notion of *energy* for certain step functions.

Definition 2.5 *Let* $u_0 : (a,b) \to \{\pm 1\}$ *denote a right-continuous step function with jumps at* $x_1 < \ldots < x_N$, *let* N_+ *be the number of increasing, and* $N_- = N - N_+$ *the number of decreasing jumps of* u_0. *Then its* energy $E_0[u_0]$ *is defined by*

$$E_0[u_0] := N_+ E_+ + N_- E_- \ ,$$

where $E_\pm$ *are the energies of the heteroclinics of Lemma 2.1.*

Proposition 2.6 *Let u_0 denote a step function as above and choose $d > 0$ such that the intervals $(x_k - d, x_k + d)$, $k = 1, \ldots, N$, are contained in (a, b) and pairwise disjoint. Moreover, assume that the hypotheses* (H1) *and* (H2) *hold, and let $M > 0$ be arbitrary.*

Then there exist constants $\delta > 0$, $\varepsilon_* > 0$, and $C_3 > 0$ such that the following property holds. For any $\varepsilon \in (0, \varepsilon_*)$ and $u \in H^n(a, b)$ satisfying both

$$E_\varepsilon[u] \le M \quad \textit{and} \quad ||u - u_0||_{L^1(a,b)} \le \delta \, ,$$

the estimate

$$E_\varepsilon[u] \ge E_0[u_0] - C_3 \exp\left(-\frac{d\lambda}{3\varepsilon}\right)$$

holds, where $\lambda > 0$ is the constant introduced in Lemma 2.2.

Proof: Without loss of generality we consider the case of a step function u_0 with exactly one jump at $x_1 = (a + b)/2$ which is increasing, and let $d := (b - a)/2$. The general case can be proved by separately considering the N pairwise disjoint subintervals $(x_k - d, x_k + d)$ of (a, b) as in the proposition.

The proof follows the outline given at the end of the introduction. Starting with the given function u, we will construct a function $w \in \chi_+ + H^n(\mathbb{R})$ such that $E[w; \mathbb{R}] \le E_\varepsilon[u] + C_3 \exp(-d\lambda/(3\varepsilon))$. This construction is divided into five steps.

First however, the constants δ and ε_* have to be chosen appropriately. To that end, define $\sigma := d/12$ and let $\varepsilon_* := \min\{\sigma/(2C_1), \sigma/d_0\}$, where C_1 denotes the constant of Lemma 2.3 and d_0 is given in (H1). The correct choice of δ is more involved and depends on properties of the Euler-Lagrange equation (7), as well as two results which will be given in the appendix (Proposition 4.2, Lemma 4.3). Choose neighborhoods $U_\pm$ of the hyperbolic fixed points $p_\pm$ such that the following holds:

- $U_\pm$ is a subset of the neighborhood of $p_\pm$ guaranteed by Lemma 4.3.
- For every $r \in U_\pm$ there exist points $r^{s/u} \in U_\pm$ which are contained in the local stable/unstable manifold of $p_\pm$, respectively, such that the first n components of r, r^s, and r^u coincide. Moreover, there exists a constant C depending only on $U_\pm$ and (7) such that $||r^{s/u} - p_\pm|| \le C||r - p_\pm||$. This is possible due to Lemma 2.2.

According to Proposition 4.2 there now exists a $\varrho > 0$ such that for any $p \in \mathbb{R}^{2n}$ satisfying $\max_{i=1,\ldots,2n} |p^i - p^i_\pm| \le C_b\varrho$ we have $p \in U_\pm$, where C_b denotes the constant of Proposition 4.2 and $p = (p^1, \ldots, p^{2n})$. Finally, we define $\delta := \varrho\sigma/2$.

(I) *Construct four points in suitable parts of the interval (a, b) at which u, together with its derivatives up to order $n-1$, is sufficiently close to the potential wells $p_\pm$.*

We construct four points $y_{1,1}$, $y_{1,2}$, $y_{2,1}$, and $y_{2,2}$ in $(a+\sigma, a+2\sigma)$, $(x_1-2\sigma, x_1-\sigma)$, $(x_1+\sigma, x_1+2\sigma)$, and $(b-2\sigma, b-\sigma)$, respectively, such that

$$\begin{array}{lcl} |u(y_{1,i})+1| < \varrho & \text{and} & \left|\varepsilon^k D^k u(y_{1,i})\right| < \varrho \ , \\ |u(y_{2,i})-1| < \varrho & \text{and} & \left|\varepsilon^k D^k u(y_{2,i})\right| < \varrho \ , \end{array}$$

for $i=1,2$ and $k=1,\ldots,n-1$. This construction will be presented only for $y_{1,1}$, since the other cases can be treated analogously. Due to $||u-v||_{L^1(a,b)} \leq \delta$ and the above choice of δ we get

$$\delta \geq \int\limits_{a+\sigma}^{a+2\sigma} |u(x)+1| dx \geq \varrho \cdot \text{meas}\{x \in (a+\sigma, a+2\sigma) : |u(x)+1| \geq \varrho\} \ ,$$

as well as

$$\text{meas}\{x \in (a+\sigma, a+2\sigma) : |u(x)+1| < \varrho\} \geq \sigma - \frac{\delta}{\varrho} = \frac{\sigma}{2} \ .$$

However, if J denotes the set introduced in Lemma 2.3, then the definition of ε_*, together with $\varepsilon \in (0, \varepsilon_*)$, implies the estimate meas $J < \sigma/2$, and hence

$$\text{meas}\left((a+\sigma, a+2\sigma) \setminus J\right) > \frac{\sigma}{2} \ .$$

Thus there is a point $y_{1,1} \in (a+\sigma, a+2\sigma) \setminus J$ satisfying $|u(y_{1,1})+1| < \varrho$, and this point has all the properties required above.

(II) *Define a function $\tilde{u}$ on $(a/\varepsilon, b/\varepsilon)$ such that $E[\tilde{u}; (a/\varepsilon, b/\varepsilon)] = E_\varepsilon[u]$.*

As we already pointed out, we want to relate the ε-dependent energy $E_\varepsilon[u]$ to the ε-independent energy $E[q_+; \mathbb{R}]$ of the heteroclinic q_+. Therefore we begin by replacing u by a function $\tilde{u}$ in the following way. For $x \in (a/\varepsilon, b/\varepsilon)$ define

$$\tilde{u}(x) := u(\varepsilon x) \ .$$

Then it is clear that $E[\tilde{u}; (a/\varepsilon, b/\varepsilon)] = E_\varepsilon[u]$. Furthermore, due to the choice of the points $y_{1,i}$ and $y_{2,i}$ in (I) we have for $\tilde{y}_{1,i} := y_{1,i}/\varepsilon$ and $\tilde{y}_{2,i} := y_{2,i}/\varepsilon$

$$\begin{array}{lcl} |\tilde{u}(\tilde{y}_{1,i})+1| < \varrho & \text{and} & \left|D^k \tilde{u}(\tilde{y}_{1,i})\right| < \varrho \ , \\ |\tilde{u}(\tilde{y}_{2,i})-1| < \varrho & \text{and} & \left|D^k \tilde{u}(\tilde{y}_{2,i})\right| < \varrho \ , \end{array}$$

for $i=1,2$ and $k=1,\ldots,n-1$.

(III) *Modify the function $\tilde{u}$ on the intervals $(\tilde{y}_{1,1}, \tilde{y}_{1,2})$ and $(\tilde{y}_{2,1}, \tilde{y}_{2,2})$ such that the newly-defined function v has an energy no greater than that of $\tilde{u}$, as well as points $z_1 \in (\tilde{y}_{1,1}, \tilde{y}_{1,2})$ and $z_2 \in (\tilde{y}_{2,1}, \tilde{y}_{2,2})$ at which it is exponentially close to the potential wells $p_\mp$, respectively.*

We consider only the interval $[\tilde{y}_{1,1}, \tilde{y}_{1,2}]$; completely analogous arguments apply to the interval $[\tilde{y}_{2,1}, \tilde{y}_{2,2}]$. Let

$$\mathcal{M} := \left\{ v \in H^n(\tilde{y}_{1,1}, \tilde{y}_{1,2}) : D^k v(\tilde{y}_{1,i}) = D^k \tilde{u}(\tilde{y}_{1,i}) \, , \, i = 1, 2 \, , \, k = 0, \ldots, n-1 \right\} \, .$$

According to Proposition 4.2, which will be proved in the appendix, the functional $E[\cdot; (\tilde{y}_{1,1}, \tilde{y}_{1,2})]$ has a unique global minimizer $\hat{v}$ on $\mathcal{M}$, and this minimizer solves the Euler-Lagrange equation (7). Then Proposition 4.2, together with our choice of ϱ, implies that $(\hat{v}(t), \ldots, D^{2n-1}\hat{v}(t))$ is contained in U_- for all $t \in (\tilde{y}_{1,1}, \tilde{y}_{1,2})$. Since the length of the interval $(\tilde{y}_{1,1}, \tilde{y}_{1,2})$ is bounded from below by $(d-4\sigma)/\varepsilon = 2d/(3\varepsilon)$, Lemma 4.3 in the appendix establishes the existence of a $z_1 \in [\tilde{y}_{1,1}, \tilde{y}_{1,2}]$ such that

$$||(\hat{v}(z_1), \ldots, D^{2n-1}\hat{v}(z_1)) - p_-|| \le C_c \exp\left(-\frac{d\lambda}{3\varepsilon} \right) \, ,$$

where C_c denotes the constant of Lemma 4.3, which depends only on U_- and (7). Finally define $v(x) := \hat{v}(x)$ for $x \in [\tilde{y}_{1,1}, \tilde{y}_{1,2}]$, apply analogous arguments to the interval $[\tilde{y}_{2,1}, \tilde{y}_{2,2}]$, and let $v(x) := \tilde{u}(x)$ for all remaining values of $x \in (a/\varepsilon, b/\varepsilon)$. Since $\hat{v}$ is a minimizer in $\mathcal{M}$ we have $E[v; (a/\varepsilon, b/\varepsilon)] \le E[\tilde{u}; (a/\varepsilon, b/\varepsilon)]$. This completes step (III).

(IV) *There exists a function $w \in \chi_+ + H^n(\mathbb{R})$ such that the estimate $0 \le E[w; \mathbb{R}] - E[v; [z_1, z_2]] \le C \exp(-d\lambda/(3\varepsilon))$ holds.*

According to (III), the point $(v(z_1), \ldots, D^{2n-1}v(z_1))$ is contained in the neighborhood U_-. Furthermore, the choice of U_- and Lemma 2.2 imply the existence of a point $r^u \in \mathbb{R}^{2n}$ on the local unstable manifold of p_- such that the first n components of r^u coincide with $(v(z_1), \ldots, D^{n-1}v(z_1))$ and

$$||r^u - p_-|| \le C||(v(z_1), \ldots, D^{2n-1}v(z_1)) - p_-|| \le C \exp\left(-\frac{d\lambda}{3\varepsilon} \right) \, .$$

Now let r^- denote the solution of (7) to the initial condition $r^-(0) = r^u$. Then due to the hyperbolicity of the fixed point p_- we have (cf. Lemma 2.2)

$$||r^-(t) - p_-|| \le Ce^{\lambda t}||r^u - p_-|| \quad \text{for } t \le 0 \, ,$$

and from this it can easily be deduced that

$$E[r^-; (-\infty, 0]] \le C||r^u - p_-|| \le C \exp\left(-\frac{d\lambda}{3\varepsilon} \right) \, .$$

Moreover, all of the above constants C depend only on U_- and (7).

Finally, analogous arguments furnish a solution $r^+ : \mathbb{R}_0^+ \to \mathbb{R}^{2n}$ of (7) in the local stable manifold of p_+ such that the first n components of $r^+(0)$ coincide with the vector $(v(z_2), \ldots, D^{n-1}v(z_2))$ and that $E[r^+; [0, \infty)] \le C \exp(-d\lambda/(3\varepsilon))$. If we now define w by

$$w(x) := \begin{cases} r^-(x - z_1) & \text{for} \quad x < z_1 \\ v(x) & \text{for} \quad z_1 \le x \le z_2 \\ r^+(x - z_2) & \text{for} \quad x > z_2 \end{cases}$$

then w has all the properties claimed above.

(V) *Finally,* $E_\varepsilon[u] \ge E_+ - C \exp(-d\lambda/(3\varepsilon))$.

The verification of this last estimate follows easily from the previous parts of the proof. More precisely we deduce

$$\begin{aligned} E_+ - E_\varepsilon[u] &= E[q_+; \mathbb{R}] - E[\tilde{u}; (a/\varepsilon, b/\varepsilon)] \\ &\le E[q_+; \mathbb{R}] - E[v; [z_1, z_2]] \\ &\le E[w; \mathbb{R}] - E[v; [z_1, z_2]] \\ &\le C \exp\left(-\frac{d\lambda}{3\varepsilon}\right) , \end{aligned}$$

recalling that the first estimate is a consequence of our choice of ε_*, $y_{1,1}$, and $y_{2,2}$, together with (H1). This completes the proof of the proposition. ◇

Before we use the above energy estimate to prove exponentially slow motion for (5), let us point out that the exponent $-d\lambda/(3\varepsilon)$ is similar to the ones derived by Alikakos, Bates, Fusco [1], Bates, Xun [8], or Grant [34], and in particular to the one obtained by Carr and Pego [15].

2.3 Γ-Convergence

The energy estimate of the last subsection has a very natural interpretation within the theory of Γ-convergence of functionals. For functionals of the form (3) this has been demonstrated by Modica [43], Owen [45], Sternberg [51], and Fonseca and Tartar [27]. A comprehensive treatment of this subject can be found in Attouch [5], where Γ-convergence is referred to as epi-convergence. For completeness we provide the following definition.

Definition 2.7 *Let $(E_\varepsilon)_{\varepsilon \ge 0}$ denote a family of functionals on $L^1(a, b)$ with values in $\mathbb{R} \cup \{+\infty\}$, and let $u \in L^1(a, b)$ be arbitrary. Then $(E_\varepsilon)_{\varepsilon > 0}$* Γ-converges to E_0 at u as $\varepsilon \to 0$ *if the following is satisfied:*

(a) For any family $(u_\varepsilon)_{\varepsilon > 0} \subset L^1(a, b)$ with $||u_\varepsilon - u||_{L^1(a,b)} \to 0$ as $\varepsilon \to 0$ we have

$$E_0[u] \le \liminf_{\varepsilon \to 0} E_\varepsilon[u_\varepsilon] .$$

(b) *There exists a family* $(u_\varepsilon)_{\varepsilon>0} \subset L^1(a,b)$ *with* $||u_\varepsilon - u||_{L^1(a,b)} \to 0$ *as* $\varepsilon \to 0$ *and*

$$E_0[u] = \lim_{\varepsilon \to 0} E_\varepsilon[u_\varepsilon] \ .$$

If we extend the definition of the functionals E_ε given in (4) to the whole of $L^1(a,b)$ by defining $E_\varepsilon[u] := +\infty$ for all $u \in L^1(a,b) \setminus H^n(a,b)$, it is natural to ask whether they Γ-converge to some limiting functional E_0. The central result of this subsection will show that this actually is true, if we consider the functional E_0 from Definition 2.5 and extend it to the whole of $L^1(a,b)$ by defining $E_0[u] := +\infty$ for all $u \in L^1(a,b)$ which are not of the form required in Definition 2.5.

But first of all, we need the following auxiliary result. Apart from its application here, this lemma will be important for proving the existence of exponentially slow motion for (5) in the next subsection.

Lemma 2.8 *Let* $u \in L^1(a,b)$ *be a function with* $E_0[u] < \infty$*, and let* $d, \lambda > 0$ *be defined as in Proposition 2.6. Then there exists a family* $(u_\varepsilon)_{0<\varepsilon\le\varepsilon_0} \subset C^{2n}[a,b]$ *satisfying both*

$$\lim_{\varepsilon \to 0} ||u_\varepsilon - u||_{L^1(a,b)} = 0 \quad \textit{and} \quad E_\varepsilon[u_\varepsilon] \le E_0[u] + C_4 \exp\left(-\frac{2d\lambda}{\varepsilon}\right) \ ,$$

where $C_4 > 0$ *is independent of* $\varepsilon \le \varepsilon_0$*. Furthermore, for* $x \in \{a,b\}$ *we have both*

$$u_\varepsilon(x) \in \{\pm 1\} \quad \textit{and} \quad D^k u_\varepsilon(x) = 0 \ , \quad k = 1, \ldots, 2n \ ,$$

i.e. all functions u_ε *satisfy reasonable boundary conditions.*

Proof: We consider only the special case of the step function $u : (a,b) \to \{\pm 1\}$ which is -1 on (a, x_1) and $+1$ on (x_1, b), where $x_1 := (a+b)/2$ and $d = (b-a)/2$. The general case can easily be reduced to this situation by considering suitable subintervals as in the proof of Proposition 2.6 and concatenating the functions obtained below.

Let q_+ denote the heteroclinic of Lemma 2.1 and assume without loss of generality that $q_+(0) = 0$. Furthermore, let $\psi : \mathbb{R} \to [0,1]$ be a C^∞-function satisfying $\psi(x) = 1$ for $x \le -1$ and $\psi(x) = 0$ for $x \ge -1/2$. Finally, for any $r > 1$ define a C^∞-function ψ_r with values in $[0,1]$ by

$$\psi_r(x) := \psi(|x| - r) \quad \text{for all} \quad x \in \mathbb{R} \ .$$

In order to construct the family (u_ε) we first use the cut-off functions ψ_r to define a family $(w_r)_{r>1}$ of C^{2n}-functions such that the following holds. The function w_r coincides with q_+ on the interval $[-r+1, r-1]$, is equal to -1 for $x \le -r + 1/2$, and equal to $+1$ for $x \ge r - 1/2$. More precisely, we define w_r by

$$w_r(x) := (q_+(x) - \mathrm{sgn} x)\psi_r(x) + \mathrm{sgn} x \ .$$

Then the definition of the functions ψ_r easily implies that for every $x \in \mathbb{R} \setminus \{0\}$ and $k = 0, \ldots, n$ the estimate

$$|D^k(w_r(x) - \mathrm{sgn} x)| \leq C|D^k(q_+(x) - \mathrm{sgn} x)|$$

holds, where the constant $C > 0$ is independent of r. Recalling that due to the hyperbolicity of the fixed point p_+ of (7) (cf. Lemma 2.2), we further have

$$|D^k(q_+(x) - \mathrm{sgn} x)| \leq Ce^{-\lambda|x|} \quad \text{for all} \quad x \in \mathbb{R} \setminus \{0\} \ , \ k = 0, \ldots, n \ ,$$

and we deduce

$$|D^k(w_r(x) - \mathrm{sgn} x)| \leq Ce^{-\lambda|x|} \quad \text{for all} \quad x \in \mathbb{R} \setminus \{0\} \ , \ k = 0, \ldots, n \ .$$

A straightforward calculation now yields for all $r > 1$ both

$$E[w_r; (-\infty, -r+1)] + E[w_r; (r-1, \infty)] \leq Ce^{-2\lambda r}$$

and

$$E[w_r; [-r, r]] \leq E[w_r; [-r+1, r-1]] + Ce^{-2\lambda r} \leq E[q_+; \mathbb{R}] + Ce^{-2\lambda r} \ .$$

Finally, define the function u_ε by

$$u_\varepsilon(x) := w_{d/\varepsilon}\left(\frac{x - x_1}{\varepsilon}\right) \quad \text{for} \quad x \in [a, b] \ .$$

Then $E_\varepsilon[u_\varepsilon] = E[w_{d/\varepsilon}; [-d/\varepsilon, d/\varepsilon]] \leq E_+ + C\exp(-2d\lambda/\varepsilon)$, and it is easy to verify that the family (u_ε) has all the remaining properties claimed in the formulation of the lemma. $\diamond$

The idea of constructing u_ε by modifying the heteroclinics $q_\pm$ of Lemma 2.1 is not new. It was used already in the above-cited papers by Modica [43], Owen [45], Sternberg [51], and Fonseca and Tartar [27] for functionals of the form (3).

After this preliminary lemma we can prove the following central result of this subsection.

Proposition 2.9 *Let the functionals* $(E_\varepsilon)_{\varepsilon>0}$ *and* E_0 *be defined as above and assume that* (H1) *and* (H2) *are satisfied. Then the family* $(E_\varepsilon)_{\varepsilon>0}$ Γ*-converges to* E_0 *at every* $u \in L^1(a, b)$.

Proof: We begin by verifying Definition 2.7*(a)*. Let a family $(u_\varepsilon)_{\varepsilon>0}$ and a function u be given such that $||u_\varepsilon - u||_{L^1(a,b)} \to 0$ as $\varepsilon \to 0$. Obviously, we only have to consider the case $\liminf_{\varepsilon\to 0} E_\varepsilon[u_\varepsilon] < \infty$. Then (H1) and Lemma 2.3 imply the existence of a sequence $(u_{\varepsilon_k})_{k\in\mathbb{N}} \subset H^n(a, b)$ with

$$\sup_{k\in\mathbb{N}} \int_a^b \left|D^1 u_{\varepsilon_k}\right| \, dx \ < \ \infty \tag{10}$$

which converges pointwise to a function with values in $\{\pm 1\}$. Due to (10) and Giusti [33, Theorem 1.9] this limit function is of bounded variation. Thus, u is equal to a step function with values in $\{\pm 1\}$ and finitely many jumps, perhaps after modifying u on a set of measure 0. Now Proposition 2.6 immediately yields the estimate $E_0[u] \le \liminf_{\varepsilon \to 0} E_\varepsilon[u_\varepsilon]$. Since Definition 2.7*(b)* is an easy consequence of Lemma 2.8, this completes the proof of the proposition. $\diamond$

Remark 2.10 Let $\mu \in \mathbb{R}$ be arbitrary. If we define functionals $I_\mu : L^1(a,b) \to \mathbb{R}$ by

$$I_\mu(u) := \begin{cases} 0 & \text{for} \quad \int_a^b u\,dx = \mu \\ +\infty & \text{for} \quad \int_a^b u\,dx \neq \mu \end{cases}$$

then it can be deduced from the above proposition that the functionals $E_\varepsilon + I_\mu$ Γ-converge to $E_0 + I_\mu$ as $\varepsilon \to 0$. In particular, this implies that if u_ε is a minimizer of $E_\varepsilon + I_\mu$ for every $\varepsilon > 0$ and if $(u_\varepsilon)_{\varepsilon>0}$ converges in $L^1(a,b)$ to some function u_0 as $\varepsilon \to 0$, then u_0 is a minimizer of $E_0 + I_\mu$ and $\lim_{\varepsilon\to 0} E_\varepsilon[u_\varepsilon] = E_0[u_0]$.

In this sense, our above result generalizes certain results of Modica [43], Owen [45], Sternberg [51], and Fonseca and Tartar [27] to functionals involving higher-order derivatives. $\square$

2.4 Slow Motion of the Transition Layers

That an energy estimate of the form given in Proposition 2.6, together with the identity (8), implies exponentially slow motion for suitable classes of initial conditions was already demonstrated by Bronsard and Kohn [11]. Their results can be applied immediately to our situation as well. Yet in order to make this paper as self-contained as possible we briefly present the necessary derivations. We begin with the following lemma which can be found in Grant [34].

Lemma 2.11 *Assume that all the assumptions of Proposition 2.6 are satisfied and let u_0, d, λ, and δ be as introduced there. Furthermore, let $(u_0^\varepsilon)_{\varepsilon>0} \subset H^n(a,b)$ denote a family of initial conditions such that*

$$||u_0^\varepsilon - u_0||_{L^1(a,b)} \le \frac{\delta}{2} \quad \textit{and} \quad E_\varepsilon[u_0^\varepsilon] \le E_0[u_0] + \frac{1}{g(\varepsilon)} \quad \textit{for all} \quad \varepsilon > 0\ ,$$

where $g : \mathbb{R}^+ \to \mathbb{R}^+$ is an arbitrary function with positive infimum. Then

$$\sup_{0 \le t \le \min\{g(\varepsilon), \exp(d\lambda/(3\varepsilon))\}} ||u^\varepsilon(\cdot,t) - u_0^\varepsilon||_{L^1(a,b)} \to 0 \quad \textit{as} \quad \varepsilon \to 0\ .$$

Here $u^\varepsilon(x,t)$ denotes the solution of (5) to the initial condition u_0^ε.

Proof: If we can choose $T_\varepsilon > 0$ such that the estimate

$$\int_0^{T_\varepsilon} ||u_t^\varepsilon(\cdot,t)||_{L^1(a,b)}\,dt \le \frac{\delta}{2}$$

holds, then obviously $||u_0^\varepsilon - u^\varepsilon(\cdot,T_\varepsilon)||_{L^1(a,b)} \le \delta/2$, and the triangle inequality implies $||u^\varepsilon(\cdot,T_\varepsilon) - u_0||_{L^1(a,b)} \le \delta$. Recalling that due to (8) the energy $E_\varepsilon[u^\varepsilon(\cdot,t)]$ decreases, we may apply Proposition 2.6 to $u^\varepsilon(\cdot,T_\varepsilon)$. This immediately furnishes the estimate $E_\varepsilon[u^\varepsilon(\cdot,T_\varepsilon)] \ge E_0[u_0] - C_3\exp(-d\lambda/(3\varepsilon))$, and together with (8) we further deduce (assuming without loss of generality $C_3 \ge 1$)

$$\begin{aligned}
\int_0^{T_\varepsilon} ||u_t^\varepsilon(\cdot,t)||^2_{L^2(a,b)}\,dt &\le \varepsilon\left(\frac{1}{g(\varepsilon)} + C_3\exp\left(-\frac{d\lambda}{3\varepsilon}\right)\right) \\
&\le \frac{2C_3\varepsilon}{\min\{g(\varepsilon),\exp(d\lambda/(3\varepsilon))\}}\,.
\end{aligned}$$

Now Hölder's inequality yields

$$\begin{aligned}
\left(\int_0^{T_\varepsilon} ||u_t^\varepsilon(\cdot,t)||_{L^1(a,b)}\,dt\right)^2 &\le T_\varepsilon(b-a)\int_0^{T_\varepsilon} ||u_t^\varepsilon(\cdot,t)||^2_{L^2(a,b)}\,dt \\
&\le \frac{2C_3(b-a)\varepsilon T_\varepsilon}{\min\{g(\varepsilon),\exp(d\lambda/(3\varepsilon))\}}\,. \qquad (11)
\end{aligned}$$

After these preliminary estimates we have to show that it is actually possible to choose $T_\varepsilon = \min\{g(\varepsilon),\exp(d\lambda/(3\varepsilon))\}$, at least for sufficiently small $\varepsilon > 0$. Assuming that $\int_0^\infty ||u_t^\varepsilon(\cdot,t)||_{L^1(a,b)}\,dt < \delta/2$, this is obviously possible. Alternatively if

$$\int_0^\infty ||u_t^\varepsilon(\cdot,t)||_{L^1(a,b)}\,dt \ge \frac{\delta}{2}$$

we can choose T_ε such that $\int_0^{T_\varepsilon} ||u_t^\varepsilon(\cdot,t)||_{L^1(a,b)}\,dt = \delta/2$. However (11) then implies

$$\begin{aligned}
T_\varepsilon &\ge \frac{\delta^2}{8C_3(b-a)\varepsilon}\cdot\min\{g(\varepsilon),\exp(d\lambda/(3\varepsilon))\} \\
&\ge \min\{g(\varepsilon),\exp(d\lambda/(3\varepsilon))\} \quad \text{for} \quad 0 < \varepsilon \le \frac{\delta^2}{8C_3(b-a)}\,.
\end{aligned}$$

In either case, setting $T_\varepsilon := \min\{g(\varepsilon),\exp(d\lambda/(3\varepsilon))\}$ in (11) we derive for all sufficiently small $\varepsilon > 0$ the estimate

$$\begin{aligned}
\sup_{0\le t\le T_\varepsilon} ||u^\varepsilon(\cdot,t) - u_0^\varepsilon||_{L^1(a,b)} &\le \int_0^{T_\varepsilon} ||u_t^\varepsilon(\cdot,t)||_{L^1(a,b)}\,dt \\
&\le \sqrt{2C_3(b-a)\varepsilon} \;\to\; 0 \quad \text{as} \quad \varepsilon \to 0\,,
\end{aligned}$$

which proves the lemma. $\diamond$

The above result pertains only to changes in the solution with respect to the $L^1(a,b)$-norm. It is however also possible to obtain results on the speed of the transition layers itself. This is the subject of the following proposition, which again is based on an analogous result of Grant [34].

Before stating the proposition let us recall some definitions. If u_0 denotes a step function as in Proposition 2.6 with jumps at $x_1, \ldots, x_N$, then its *interface* $I[u_0]$ is defined by

$$I[u_0] := \{x_1, \ldots, x_N\} \; .$$

Now fix some closed subset $K \subset \mathbb{R} \setminus \{\pm 1\}$ and let $u : (a,b) \to \mathbb{R}$ be an arbitrary function. Then the *interface* $I_K[u]$ of the function u is defined as

$$I_K[u] := u^{-1}(K) \; .$$

Finally, if $A, B \subset \mathbb{R}$ then the *Hausdorff distance* $d(A,B)$ between A and B is defined by $d(A,B) := \max\{\sup_{\alpha \in A} d(\alpha, B), \sup_{\beta \in B} d(\beta, A)\}$.

Proposition 2.12 *Assume again that all the assumptions of Proposition 2.6 are satisfied and let u_0, x_k, d, λ, and δ be as introduced there. Then there exists a positive $\delta_* \le \delta$ such that for any function $g : \mathbb{R}^+ \to \mathbb{R}^+$ with $\lim_{\varepsilon \to 0} g(\varepsilon) = +\infty$ and any family of initial conditions $(u_0^\varepsilon)_{\varepsilon > 0} \subset H^n(a,b)$ satisfying both*

$$||u_0^\varepsilon - u_0||_{L^1(a,b)} \le \frac{\delta_*}{2} \quad \textit{and} \quad E_\varepsilon[u_0^\varepsilon] \le E_0[u_0] + \frac{1}{g(\varepsilon)} \quad \textit{for all} \quad \varepsilon > 0$$

we have

$$\sup_{0 \le t \le \min\{g(\varepsilon), \exp(d\lambda/(3\varepsilon))\}} d\left(I_K[u^\varepsilon(\cdot,t)], I[u_0]\right) \to 0 \quad \textit{as} \quad \varepsilon \to 0 \; ,$$

for every closed subset K of $\mathbb{R} \setminus \{\pm 1\}$. Here $u^\varepsilon(x,t)$ denotes the solution of (5) to the initial condition u_0^ε.

Proof: Let $r \in (0, d/2)$ be arbitrary, where d denotes the constant introduced in the formulation of Proposition 2.6. Define $J_k := B_r(x_k)$ for every $k = 1, \ldots, N$, and let $J := J_1 \cup \ldots \cup J_N$. Furthermore, choose $\varrho > 0$ such that the sets $B_\varrho(\pm 1)$ are contained in $\mathbb{R} \setminus K$, and set $T_\varepsilon := \min\{g(\varepsilon), \exp(d\lambda/(3\varepsilon))\}$. It will be shown below that for sufficiently small $\varepsilon > 0$ we have

$$\operatorname{dist}\left(u^\varepsilon(x,t), \{\pm 1\}\right) < \varrho \quad \text{for every} \quad x \in (a,b) \setminus J \quad \text{and} \quad t \in [0, T_\varepsilon] \; . \qquad (12)$$

This obviously implies the inclusion $I_K[u^\varepsilon(\cdot,t)] \subset J$ for every $t \in [0, T_\varepsilon]$, and therefore $d(I_K[u^\varepsilon(\cdot,t)], I[u_0]) \le r$. Since $r > 0$ can be chosen arbitrarily small this already proves the proposition. The proof of (12) is divided into several steps.

(I) *For* $\varepsilon \to 0$ *we have* $\sup_{t\in[0,T_\varepsilon]} E_\varepsilon[u^\varepsilon(\cdot,t);(a,b)\setminus J] \to 0$.

According to Lemma 2.11 and the properties of g there exist $\varepsilon_0 > 0$ and $M > 0$ such that for all $\varepsilon \in (0,\varepsilon_0)$ and all $t \in [0,T_\varepsilon]$ we have both

$$||u^\varepsilon(\cdot,t) - u_0||_{L^1(a,b)} \le \delta \quad \text{and} \quad E_\varepsilon[u^\varepsilon(\cdot,t);(a,b)] \le M \,.$$

These estimates remain valid if we replace (a,b) by J_k for $k = 1,\ldots,N$ and let ε be sufficiently small. An application of Proposition 2.6 yields an $\varepsilon_1 \in (0,\varepsilon_0)$ such that for all $\varepsilon \in (0,\varepsilon_1)$, $t \in [0,T_\varepsilon]$, and $k = 1,\ldots,N$ the estimate

$$E_\varepsilon[u^\varepsilon(\cdot,t);J_k] \ge E_k - C\exp\left(-\frac{r\lambda}{3\varepsilon}\right)$$

holds, where $E_k = E_+$ if u_0 has an increasing jump at x_k, and $E_k = E_-$ otherwise. This finally implies

$$E_\varepsilon[u^\varepsilon(\cdot,t);J] \ge E_0[u_0] - C\exp\left(-\frac{r\lambda}{3\varepsilon}\right)$$

for all $\varepsilon \in (0,\varepsilon_1)$ and $t \in [0,T_\varepsilon]$, as well as

$$E_\varepsilon[u^\varepsilon(\cdot,t);(a,b)\setminus J] = E_\varepsilon[u^\varepsilon(\cdot,t)] - E_\varepsilon[u^\varepsilon(\cdot,t);J] \le \frac{1}{g(\varepsilon)} + C\exp\left(-\frac{r\lambda}{3\varepsilon}\right) \to 0$$

as $\varepsilon \to 0$, uniformly with respect to $t \in [0,T_\varepsilon]$.

(II) *There exists an* $\varepsilon_2 > 0$ *such that for all* $\varepsilon \in (0,\varepsilon_2)$ *and all* $t \in [0,T_\varepsilon]$ *the following holds. In every component of* $(a,b)\setminus J$ *we can find a point* x *satisfying the estimate* $|u^\varepsilon(x,t) - u_0(x)| \le \varrho/2$.

This follows immediately from Lemma 2.3 together with the L^1-closeness of $u^\varepsilon(\cdot,t)$ to u_0 as in the proof of Proposition 2.6(I).

(III) *There exists an* $\varepsilon_3 > 0$ *such that for all* $\varepsilon \in (0,\varepsilon_3)$, $t \in [0,T_\varepsilon]$, *and* $x \in (a,b)\setminus J$ *we have* $|u^\varepsilon(x,t) - u_0(x)| < \varrho$.

Let $c := \min\{\int_{\pm1-\varrho}^{\pm1-\varrho/2}\sqrt{F(s)}\,ds, \int_{\pm1+\varrho/2}^{\pm1+\varrho}\sqrt{F(s)}\,ds\} > 0$, and let η denote the constant introduced in (H1). Furthermore, choose $0 < \varepsilon_3 < \varepsilon_2$ in such a way that for every $\varepsilon \in (0,\varepsilon_3)$, all $t \in [0,T_\varepsilon]$, and every component $\tilde{J}$ of $(a,b)\setminus J$ we have

$$E_\varepsilon[u^\varepsilon(\cdot,t);\tilde{J}] < c\sqrt{2\eta}\,. \tag{13}$$

This is possible according to (I). We claim that (III) holds for this choice of ε_3. Assuming the contrary, we can find $\varepsilon \in (0,\varepsilon_3)$, $t \in [0,T_\varepsilon]$, and $x_1 \in (a,b)\setminus J$ such that $|u^\varepsilon(x_1,t) - u_0(x_1)| \ge \varrho$. Let $\tilde{J}$ denote the component of $(a,b)\setminus J$ containing

x_1 and set $p := u_0(x_1)$. Because of (II) there exists a point $x_2 \in \tilde{J}$ satisfying the inequality $|u^\varepsilon(x_2,t) - p| \le \varrho/2$. Then (H1) and the arithmetic-geometric mean inequality furnish

$$\begin{aligned}
E_\varepsilon[u^\varepsilon(\cdot,t);\tilde{J}] &\ge \int\limits_{\min\{x_1,x_2\}}^{\max\{x_1,x_2\}} \left(\frac{\varepsilon\eta}{2} \left(D^1 u^\varepsilon(x,t)\right)^2 + \frac{1}{\varepsilon} F(u^\varepsilon(x,t)) \right) dx \\
&\ge \sqrt{2\eta} \int\limits_{\min\{x_1,x_2\}}^{\max\{x_1,x_2\}} \sqrt{F(u^\varepsilon(x,t))}\, \left|D^1 u^\varepsilon(x,t)\right| dx \\
&= \sqrt{2\eta} \int\limits_{\min\{u^\varepsilon(x_1,t),u^\varepsilon(x_2,t)\}}^{\max\{u^\varepsilon(x_1,t),u^\varepsilon(x_2,t)\}} \sqrt{F(s)} ds \ \ge\ c\sqrt{2\eta}\,.
\end{aligned}$$

This however contradicts (13), and concludes the proof of the proposition. $\diamond$

Together with Lemma 2.8 the following result is immediate. It shows that for suitable initial data u_0^ε the solution u^ε does not seem to change on a time interval of length $\exp(d\lambda/(3\varepsilon))$, i.e. this solution exhibits exponentially slow motion.

Corollary 2.13 *There exists a family of initial conditions $(u_0^\varepsilon)_{\varepsilon>0}$ such that the results of Lemma 2.11 and Proposition 2.12 hold with $g(\varepsilon) = \exp(d\lambda/(3\varepsilon))$.*

3 Further Applications of the Method

3.1 Discussion of the Main Parts

We already mentioned several times that our variational approach for proving slow motion is applicable to more situations than just the one discussed in the previous section. It will be the aim of this subsection to briefly summarize the crucial parts of the approach, thereby also pointing out the main assumptions needed in each step. Further applications will be presented in subsequent subsections. We hope that this not only demonstrates the range of possible applications, but that it also gives the reader a good understanding of what is a possible driving force for slow motion. Our approach can be divided into five parts.

1. Existence of global minimizing heteroclinics

Perhaps the most important ingredient for the approach is the existence of globally minimizing heteroclinics of the ε-independent energy functional $E[\cdot;\mathbb{R}]$. Based on results of Rabinowitz [47] we could treat the energy functionals given by (6) in Lemma 2.1, and some generalizations are immediate. Yet for other situations the existence proof may be difficult.

2. Geometric properties of equilibria of the Euler-Lagrange equation

This is the easiest part of the whole approach. It has to be checked that the fixed points of the Euler-Lagrange equation of $E[\cdot;\mathbb{R}]$ which correspond to the potential wells are hyperbolic, with stable and unstable manifolds of equal dimension. Furthermore, both of these local manifolds have to be graphs over the variables $v_1, \ldots, v_n$, cf. Lemma 2.2. Since this is only a linear algebra problem, it can easily be verified for a specific situation.

3. Closeness to the potential wells on a large set

As a first step towards proving the energy estimate contained in Proposition 2.6, Lemma 2.3 gives conditions implying the closeness of a function u to a step function with values in $\{\pm 1\}$ on a large subset of (a,b), provided $\varepsilon > 0$ is sufficiently small. All that is needed for the proof are estimates of the form

$$E_\varepsilon[u] \geq \frac{C}{\varepsilon} \int_a^b G_k(\varepsilon^k D^k u)dx$$

for $k = 0, \ldots, n-1$. Here the functions $G_k : \mathbb{R} \to \mathbb{R}$ have to be nonnegative, the only zeros being the potential wells ± 1 for G_0, and the origin for the remaining G_k. Furthermore, all the G_k must have a strictly positive infimum outside a small neighborhood of their zero set. Once these requirements are met, the assertions of Lemma 2.3 are immediate. We like to mention however that the stronger result of Remark 2.4 holds only if an estimate of the form (9) is satisfied.

4. The main energy estimate and Γ-convergence

The central result of our variational approach is Proposition 2.6, and in a way its proof combines variational results with certain facts about hyperbolic fixed points of ordinary differential equations to yield the desired energy estimate. Part (I) of the proof uses Lemma 2.3 to construct points at which the function u is close to a given step function u_0. Since this lemma yields only the closeness of u on a large set to either $+1$ or -1, without further specifying to which one, an additional, yet weaker, notion of "closeness" is needed. In Proposition 2.6 this notion is the $L^1(a,b)$-closeness of u to the step function u_0, but this is only one possible choice. Choosing the right notion depends also on the semiflow associated with the energy functional, cf. the discussion under "*The associated gradient flow and slow motion results*" below.

The remaining parts of the proof of Proposition 2.6 require only local arguments, which basically rely on the fact that at its potential wells $p = \pm 1$ the potential F satisfies both $F(p) = F'(p) = 0$ and $F''(p) > 0$. The only exception is part (V) where the nonnegativity of the energies $E[v;(a/\varepsilon, z_1)]$ and $E[v;(z_2, b/\varepsilon)]$ is needed. This however will be satisfied under reasonable assumptions on the energy functional E.

The results on Γ-convergence in Subsection 2.3 follow immediately from the assertions of Proposition 2.6.

5. The associated gradient flow and slow motion results

Establishing the energy estimate of Proposition 2.6 is only one step toward proving slow motion. One still has to relate it to the dynamics of the underlying differential equation.

In Subsection 2.4 this was accomplished with the help of (8), which is due to the fact that (5) is the L^2-gradient flow associated with the energy functional. For the proof of our first slow motion result in Lemma 2.11 the special form of the integral term in (8) is very important. Since the integrand contains the L^2-norm of u_t, and the L^1-norm is the notion of closeness to the step function u_0 which was needed in Proposition 2.6, an application of Hölder's inequality furnishes the slow motion result.

Further generalizations are however possible. For example, assume there exists a bounded linear operator $L : L^2(a,b) \to L^2(a,b)$ such that for any solution u of the semiflow under consideration the estimate

$$E_\varepsilon[u(0,\cdot)] - E_\varepsilon[u(T,\cdot)] \le \frac{C}{\varepsilon} \int_0^T ||Lu_t(\tau,\cdot)||^2_{L^2(a,b)}\, d\tau$$

holds. Furthermore, suppose that Proposition 2.6 remains valid if we replace the condition $||u - u_0||_{L^1(a,b)} \le \delta$ by $||Lu - Lu_0||_{L^1(a,b)} \le \delta$. Then it can easily be checked that all the assertions of Lemma 2.11 hold, if we replace $||u_0^\varepsilon - u_0||_{L^1(a,b)}$ and $||u^\varepsilon(\cdot,t) - u_0^\varepsilon||_{L^1(a,b)}$ by $||Lu_0^\varepsilon - Lu_0||_{L^1(a,b)}$ and $||Lu^\varepsilon(\cdot,t) - Lu_0^\varepsilon||_{L^1(a,b)}$, respectively.

Although at first glance this generalization might seem to be artificial, it may be needed if we associate other gradient dynamics with E_ε. As an example, consider the Cahn-Hilliard equation, which will be discussed in Subsection 3.3. This equation is the H^{-1}-gradient of the functional (2), and the estimate given above holds with $(Lu)(x) := \int_a^x u(s)ds$.

Once Lemma 2.11 has been established, the second slow motion result of Subsection 2.4 can be derived easily. It has to be checked however that the energy functional E_ε allows for the application of the arithmetic-geometric mean inequality in part (III) of the proof.

3.2 Allen-Cahn Type Equations

In Section 2 we used our variational approach to derive slow motion results for equations of the form (5), which generalize the Allen-Cahn equation (1). We already pointed out that these equations are obtained by considering the L^2-gradient flow associated with the energy functional (4).

Equations of this type have been proposed and studied in several contexts. For example, Coullet, Elphick, and Repaux [18] and Dee and van Saarloos [19] consider the particular case $n = 2$ and $F(u) = (u^2 - 1)^2/4$ as a model for phase transitions and other bistable phenomena. In this case, (5) is commonly referred to as *extended Fisher-Kolmogorov equation*. Another important example is given by the *phase field model*, which will be described in more detail in Subsection 3.4.

Although our exposition in Section 2 concentrated on the specific equation (5) with potential F satisfying (H2), several generalizations are immediate. For the sake of completeness, we would like to briefly indicate three possible generalizations in this subsection.

1. Multiple-well potentials:

Our method extends to smooth nonnegative potentials F of the following form. Assume that the zero set $P := F^{-1}(0)$ consists of isolated points, that for every sufficiently small $\delta > 0$ the infimum of F outside $B_\delta(P)$ is strictly positive, and that F is uniformly superquadratic at $p \in P$, i.e. there exist constants $d_1, d_2 > 0$ such that for every $p \in P$ the estimate $F(p + u) \geq d_2 u^2$ holds for all $|u| \leq d_1$.

The reader may verify that all parts of our method can be applied to these potentials, with one exception. In general, it is not possible to choose an arbitrary step function u_0 with values in P in Proposition 2.6. This is due to the fact that there might be potential wells $p_1, p_2 \in P$ which are not connected by a heteroclinic solution of the Euler-Lagrange equation (7). If we assume however that u_0 is chosen in such a way that for every jump discontinuity of u_0, say from p_- to p_+, there exists a heteroclinic solution with limits $p_\pm$ at $\pm\infty$, then all results of Section 2 remain valid.

For example, the 2-periodic potential $F(u) = 1 + \cos \pi u$ has been considered in Kalies and VanderVorst [42] for the extended Fisher-Kolmogorov equation. They prove that for any $k \in \mathbb{Z}$ there exist heteroclinic solutions $q_k^\pm$ of the Euler-Lagrange equation (7) with limit $2k \mp 1$ at $-\infty$ and $2k \pm 1$ at $+\infty$. Thus, if u_0 denotes a step function taking values in the odd integers such that each jump is of absolute magnitude 2, then any initial condition close to u_0 leads to exponentially slow motion in the extended Fisher-Kolmogorov equation with the above potential.

2. More general energy functionals:

The particular form of the energy functional (4) is not decisive. In fact, it is possible to consider more general energy functionals of the form

$$E_\varepsilon[u] = \frac{1}{\varepsilon} \int_a^b g\left(\varepsilon^n D^n u, \varepsilon^{n-1} D^{n-1} u, \ldots, \varepsilon D^1 u, u\right)\, dx$$

and its associated L^2-gradient flows. In order for our approach to work one has to impose certain conditions on the density function g, in the spirit of our discussion in Subsection 3.1. For the case $n = 1$ the conditions used by Owen [45] are sufficient, generalizations to other values of n are straightforward.

3. Systems of Allen-Cahn type:

Similar to the case of Cahn-Morral systems, which will be discussed briefly in the next subsection, our method also extends to systems of Allen-Cahn type. These can be obtained by considering the energy functional

$$E_\varepsilon[u] = \frac{1}{\varepsilon} \int_a^b \left(\sum_{k=1}^{n} \frac{\gamma_k \varepsilon^{2k}}{2} \left| D^k u \right|^2 + F(u) \right) dx \ , \tag{14}$$

where $u = (u^1, \ldots, u^d)$ is $\mathbb{R}^d$-valued and $F : \mathbb{R}^d \to \mathbb{R}$. We assume that the coefficients γ_k satisfy (H1), and that F is a smooth nonnegative potential whose zero set consists of isolated points. The associated L^2-gradient flow is given by the system

$$\begin{aligned} u_t^1 &= \sum_{k=1}^{n} (-1)^{k+1} \varepsilon^{2k} \gamma_k D^{2k} u^1 - \frac{\partial F}{\partial u^1} \left(u^1, \ldots, u^d \right) \ , \\ &\vdots \\ u_t^d &= \sum_{k=1}^{n} (-1)^{k+1} \varepsilon^{2k} \gamma_k D^{2k} u^d - \frac{\partial F}{\partial u^d} \left(u^1, \ldots, u^d \right) \ . \end{aligned}$$

All the results of Subsections 2.2 through 2.4 remain valid, the existence of heteroclinic solutions of the Euler-Lagrange equation of (14) between certain wells of F could be obtained by adapting the methods of Lemma 2.1, cf. also Rabinowitz [47].

For each of the above generalizations it is not difficult to give precise conditions which imply the validity of our results, and we leave this to the interested reader.

3.3 Cahn-Hilliard Type Equations

We already mentioned in Subsection 3.1 that there are many ways of assigning gradient dynamics to an energy functional. Until now we have only considered the L^2-gradient flow associated with the energy functional (4). In this subsection we will discuss Cahn-Hilliard type equations, which are obtained by associating H^{-1}-gradient dynamics with (4) for $n = 1$. Consider the energy functional

$$E_\varepsilon[u] = \frac{1}{\varepsilon} \int_a^b \left(\frac{\varepsilon^2}{2} \left| D^1 u \right|^2 + F(u) \right) dx \ , \tag{15}$$

where for the sake of simplicity we assume $F(u) = (u^2 - 1)^2/4$. It was shown for example in Fife [26] that the H^{-1}-gradient flow associated with (15) is given by the fourth-order parabolic *Cahn-Hilliard equation*

$$\begin{aligned} u_t &= -(\varepsilon^2 u_{xx} - F'(u))_{xx} \quad \text{in} \quad (a,b) \, , \\ & \qquad u_x = u_{xxx} = 0 \quad \text{on} \quad \{a,b\} \, . \end{aligned}$$

This equation is one of the standard mathematical models for phase separation and coarsening in binary alloys, compare [13, 14]. If in (15) we consider $u = (u^1, \ldots, u^d)$ as an $\mathbb{R}^d$-valued function and let $F : \mathbb{R}^d \to \mathbb{R}$ denote a suitable potential, then the H^{-1}-gradient flow associated with (15) is the *Cahn-Morral system*

$$\begin{aligned} u_t^1 &= -\left(\varepsilon^2 u_{xx}^1 - \frac{\partial F}{\partial u^1}\left(u^1, \ldots, u^d\right)\right)_{xx} \, , \\ &\vdots \\ u_t^d &= -\left(\varepsilon^2 u_{xx}^d - \frac{\partial F}{\partial u^d}\left(u^1, \ldots, u^d\right)\right)_{xx} \, . \end{aligned}$$

This system is a model for phase separation of alloys with $d+1$ components, and was proposed by Cahn and Morral [44] and DeFontaine [20, 21].

Obviously, the results of Subsections 2.1 through 2.3 immediately apply to the functional (15), in either of the above two forms. However, these results cannot be used in Subsection 2.4. We have to take into account that the identity (8) in the Allen-Cahn case is replaced by

$$E_\varepsilon[u(0,\cdot)] - E_\varepsilon[u(T,\cdot)] = \frac{1}{\varepsilon} \int_0^T ||Lu_t(\tau,\cdot)||_{L^2(a,b)}^2 \, d\tau \tag{16}$$

in the Cahn-Hilliard case, where the bounded linear operator L on $L^2(a,b)$ is defined by $(Lu)(x) := \int_a^x u(s)ds$.

We already indicated in Subsection 2.1 that our approach can easily be modified to cover this situation as well. Thus, we recover the results of Grant [34], and generalize those of Bronsard and Hilhorst [10]. We close this subsection with the following remark.

Remark 3.1 In the L^2-gradient case the Euler-Lagrange equation of the energy functional (4) happened to be exactly the equation determining the stationary solutions of the associated partial differential equation. This is not the case for the Cahn-Hilliard equation. □

3.4 Equations of Phase Field Model Type

Higher-order variants of the phase field model were originally proposed by Caginalp and Fife [12]. In the isothermal case, their systems lead to even-order parabolic equations of the type

$$u_t = a_{2n} D^{2n} u + a_{2n-2} D^{2n-2} u + \ldots + a_2 D^2 u - F'(u) \, , \tag{17}$$

with for instance $F(u) = (u^2-1)^2/4$. In this model, the coefficients $a_{2k} > 0$ are given by $a_{2k} = \int_{-\infty}^{\infty} \tilde{J}(x)x^{2k}dx/(2k)!$, where the function $\tilde{J}$ is nonnegative and rapidly decaying (cf. Bates et. al. [6], or Caginalp and Fife [12]). For the case $n = 3$ Gardner and Jones [32] study traveling waves for (17). Notice that one has to restrict attention to odd values of n, since otherwise the partial differential equation (17) is not well-posed.

In order to apply the results of the previous section to the phase field model, we consider the following rescaled version of (17). Replacing $\tilde{J}(x)$ in the formula for a_{2k} by $J(x/\varepsilon)/\varepsilon$ it can easily be verified that $a_{2k} = \varepsilon^{2k}A_{2k}$, with

$$A_{2k} = \frac{1}{(2k)!}\int\limits_{-\infty}^{\infty} J(x)x^{2k}dx > 0 \; . \tag{18}$$

Thus, (17) takes the form

$$u_t = \varepsilon^{2n}A_{2n}D^{2n}u + \varepsilon^{2n-2}A_{2n-2}D^{2n-2}u + \ldots + \varepsilon^2 A_2 D^2 u - F'(u) \; . \tag{19}$$

This is exactly equation (5) if we set $\gamma_k := (-1)^{k+1}A_{2k}$ for $k = 1,\ldots,n$, and therefore our results apply whenever (H1) and (H2) are satisfied. Since (H2) can easily be obtained by choosing an appropriate potential F, we only have to verify hypothesis (H1). Using the interpolation inequalities of Lemma 4.1 it can actually be shown that there are choices of the constants $A_{2k} > 0$ in (19) for which (H1) is satisfied. For these choices all slow motion results of Subsection 2.4 apply to (19). However, in order to verify (H1) for the specific phase field model constants A_{2k} defined in (18) one needs more information about the constants C_a in Lemma 4.1.

3.5 Other Applications and Remarks

The more or less straightforward applications given in the last three subsections already demonstrated the scope of the above method. There are however other applications which we will indicate below. These will be treated elsewhere.

1. Fine-structure equation:

In order to investigate dynamical phase transformations in nonlinear elasticity Kalies and Holmes [40] considered the equation

$$\begin{array}{rcll} u_{tt} & = & (u_x^3 - u_x + \beta u_{xt} - \varepsilon^2 u_{xxx})_x - \alpha u & \text{in} \quad (0,1) \; , \\ & & u = u_{xx} = 0 & \text{on} \quad \{0,1\} \; . \end{array} \tag{20}$$

This equation generates gradient dynamics, and the associated energy functional is given by

$$\mathcal{E}_\varepsilon[u] = \int\limits_0^1 \left(\frac{\varepsilon^2}{2}u_{xx}^2 + \frac{1}{4}\left(u_x^2 - 1\right)^2 \right) dx + \int\limits_0^1 \left(\frac{1}{2}u_t^2 + \frac{\alpha}{2}u^2 \right) dx \; . \tag{21}$$

For $\alpha = 0$ Hattori and Mischaikow [37] characterized the global attractor of (20) explicitly, in [38] the same authors proved exponentially slow motion of phase boundaries using energy methods. This latter result can also be obtained with our variational approach.

Much less is known about the dynamics of (20) for $\alpha > 0$, cf. Kalies and Holmes [40]. One particularly interesting open question is whether there is still exponentially slow motion of the phase boundaries. Notice that in the above model it is u_x which develops transition layers between ± 1. When $\alpha = 0$, this equation in essence reduces to the Allen-Cahn equation by a change of variables, cf. Hattori and Mischaikow [37, 38]. For $\alpha > 0$, the minimization of the rescaled functional $E[\cdot;\mathbb{R}]$ associated to (21) no longer represents the energy required to make a transition for ε small, and so our variational method does not directly apply. Due to the presence of the term $\alpha u^2/2$, which promotes the formation of finely layered structures, the motion of layers in solutions to (20) will be governed not only by the energy associated to each transition layer, which is a local quantity, but also by the global arrangement of the layers.

2. Bubble solutions in higher-dimensional equations:

We already mentioned in the introduction that Alikakos and Fusco [2, 3, 4] proved exponentially slow motion of bubbles for the higher-dimensional Cahn-Hilliard equation. Whereas they use the dynamical approach to derive this result, it is also possible to obtain the exponentially slow motion with our variational approach. Similarly, one can prove exponentially slow motion of bubbles for the constrained Allen-Cahn equation, cf. Ward [53]. These results will be presented elsewhere.

3. Stable equilibria of the extended Fisher-Kolmogorov equation:

In the introduction we described that for the Allen-Cahn equation (1) generic initial conditions will quickly develop transition layers, which subsequently disappear through several stages of exponentially slow motion. For the higher-order generalizations (5) of the Allen-Cahn equation we proved in Section 2 that solutions with a sharp transition layer structure still move at a speed which is at most exponentially slow. Let us point out however that there might even be no motion at all. For example, the extended Fisher-Kolmogorov equation ($n = 2$ in equation (5)) subject to suitable boundary conditions has stable equilibria with such a transition layer structure, cf. Kalies, Kwapisz, and VanderVorst [41, Section 8.1]. For this result one has to assume that the hyperbolic equilibria ± 1 of (7) are saddle-foci.

4 Appendix

In this appendix we present technical results which were needed in Section 2. The first lemma addresses interpolation inequalities for H^n-functions. It differs from known results in that it provides control over the domain-dependence of the constants appearing in these estimates.

Lemma 4.1 *Let $\delta > 0$ and $T_0 > 0$ be arbitrary and assume that the integers n and k satisfy $1 \leq k \leq n-1$. Then there exists a constant $C_a = C_a(\delta, T_0, n, k)$ such that for all $T \geq T_0$ and every $u \in H^n(0,T)$ the estimate*

$$||D^k u||_{L^2(0,T)} \leq C_a ||u||_{L^2(0,T)} + \delta ||D^n u||_{L^2(0,T)}$$

holds.

Proof: The proof is divided into several steps. In (I) and (II) we verify the above estimate for the special case $n = 2$ and $k = 1$, and in (III) the general case will be deduced.

(I) *Let $T_0 > 0$. Then there is a constant $C^* > 0$ such that the following holds. For every $T \geq T_0$ there exists an extension map $P : H^2(0,T) \to H^2(\mathbb{R})$ with*

$$||Pu||_{H^j(\mathbb{R})} \leq C^* ||u||_{H^j(0,T)} \quad \textit{for} \quad j = 0, 1, 2$$

and $Pu|_{(0,T)} = u$ for any $u \in H^2(0,T)$.

It is easy to show that there exists an extension map $P_+ : H^2(\mathbb{R}^+) \to H^2(\mathbb{R})$ and a constant $c_* > 0$ such that for any $u \in H^2(\mathbb{R}^+)$ and every $j = 0, 1, 2$ the estimate

$$||P_+ u||_{H^j(\mathbb{R})} \leq c_* ||u||_{H^j(\mathbb{R}^+)}$$

holds, cf. for example Renardy and Rogers [48, Theorem 6.86]. Let $\psi : \mathbb{R} \to [0,1]$ be a C^∞-function which is equal to 1 on $(-\infty, 1/4]$ and vanishes on $[3/4, \infty)$. Furthermore, define ψ_T by $\psi_T(x) := \psi(x/T)$ for $x \in \mathbb{R}$.

Now let $T \geq T_0$ and $u \in H^2(0,T)$ be given. If we extend u to the whole of $\mathbb{R}$ by defining $u(x) := 0$ for $x \notin (0,T)$ and let $u_+ := \psi_T \cdot u$, $u_- := (1 - \psi_T) \cdot u$, then obviously $u = u_+ + u_-$. A straightforward calculation yields the estimate

$$||u_+||_{H^j(\mathbb{R}^+)} \leq 6 \left(\sum_{i=0}^{j} ||D^i \psi_T||_{L^\infty(\mathbb{R})} \right) \cdot ||u||_{H^j(0,T)} \quad \text{for} \quad j = 0, 1, 2 \,,$$

and with $||D^i \psi_T||_{L^\infty(\mathbb{R})} = T^{-i} ||D^i \psi||_{L^\infty(\mathbb{R})}$ and $C^* := 12 c_* \sum_{i=0}^{2} (T_0^{-i} ||D^i \psi||_{L^\infty(\mathbb{R})})$ this further implies

$$||P_+ u_+||_{H^j(\mathbb{R})} \leq \frac{C^*}{2} \cdot ||u||_{H^j(0,T)} \quad \text{for} \quad j = 0, 1, 2 \,.$$

Using completely analogous arguments, we can extend the function $u_-|_{(-\infty,T)}$ to an $H^2(\mathbb{R})$-function P_-u_- such that $||P_-u_-||_{H^j(\mathbb{R})} \le (C^*/2) \cdot ||u||_{H^j(0,T)}$ for $j = 0, 1, 2$. Since $P_\pm u_+|_{(0,T)} = u_\pm|_{(0,T)}$ and $u = u_+ + u_-$ immediately imply $Pu|_{(0,T)} = u$, the definition $Pu := P_+u_+ + P_-u_-$ completes the proof of (I).

(II) *The assertions of the lemma hold for $n = 2$ and $k = 1$.*

Let $\delta > 0$ and $T_0 > 0$ be arbitrary, and let $C^* > 0$ be the constant of (I) corresponding to T_0. Then for any $\varrho > 0$ we may apply (I) and the Fourier transform to get

$$\begin{aligned} ||D^1u||_{L^2(0,T)} &\le ||D^1(Pu)||_{L^2(\mathbb{R})} \le \varrho||D^2(Pu)||_{L^2(\mathbb{R})} + \frac{1}{4\varrho}||Pu||_{L^2(\mathbb{R})} \\ &\le \varrho C^*||u||_{H^2(0,T)} + \frac{C^*}{4\varrho}||u||_{L^2(0,T)} \\ &\le \varrho C^*||D^2u||_{L^2(0,T)} + \varrho C^*||D^1u||_{L^2(0,T)} + C^*\left(\varrho + \frac{1}{4\varrho}\right)||u||_{L^2(0,T)} . \end{aligned}$$

For $\varrho := \delta/(C^*(\delta+1))$ this yields the desired estimate.

(III) *The assertions of the lemma hold for arbitrary $n \ge 2$ and $1 \le k < n$.*

The proof uses induction over n. Assume that $m \ge 2$ and that Lemma 4.1 holds for all $n \le m$. Let $u \in H^{m+1}(0,T)$ and $2 \le k < m+1$. Then two applications of the inductive hypothesis furnish

$$\begin{aligned} ||D^ku||_{L^2(0,T)} &= ||D^{k-1}D^1u||_{L^2(0,T)} \le \frac{\delta}{2}||D^mD^1u||_{L^2(0,T)} + c_1||D^1u||_{L^2(0,T)} \\ &\le \frac{\delta}{2}||D^{m+1}u||_{L^2(0,T)} + c_1\left(\frac{1}{2c_1}||D^ku||_{L^2(0,T)} + c_2||u||_{L^2(0,T)}\right) , \end{aligned}$$

thus we get $||D^ku||_{L^2(0,T)} \le \delta||D^{m+1}u||_{L^2(0,T)} + 2c_1c_2||u||_{L^2(0,T)}$. The case $k = 1$ can be proved by similar arguments. $\diamond$

The next result is a generalization of a theorem due to Kalies, Kwapisz, and VanderVorst [41, Theorem 4.1] which characterizes minimizers of the functional (6) on finite intervals subject to small boundary conditions. Whereas they consider only the case $n = 2$, the following proposition treats general n.

Proposition 4.2 *Consider the energy functional (6) and the corresponding Euler-Lagrange equation (7). Assume that* (H1) *holds and that $F : \mathbb{R} \to \mathbb{R}$ is a C^2-function with $F(0) = F'(0) = 0$ and $F''(0) > 0$. Finally, for $a = (a_1, \ldots, a_n) \in \mathbb{R}^n$, $b = (b_1, \ldots, b_n) \in \mathbb{R}^n$ and $T \in \mathbb{R}^+$ define*

$$\mathcal{M}_{a,b,T} := \{w \in H^n(0,T) : D^kw(0) = a_{k+1}, D^kw(T) = b_{k+1}, k = 0, \ldots, n-1\} .$$

Then there exist constants $\delta^* > 0$, $T^* > 0$, and $C_b > 0$ such that the following holds. If $T \geq T^*$ and $||a||, ||b|| \leq \delta \leq \delta^*$ then the functional $E[\cdot; [0,T]]$ has a global minimizer $\hat{w}$ on $\mathcal{M}_{a,b,T}$. This minimizer $\hat{w}$ solves the Euler-Lagrange equation (7), and the estimate $||\hat{w}||_{W^{2n-1,\infty}(0,T)} \leq C_b\delta$ holds.

Proof: Completely analogous to the proof of Theorem 4.1 in Kalies, Kwapisz, and VanderVorst [41] the following two assertions can be verified.

(I) *There exist constants $\delta_0 > 0$ and $c_0 > 0$, which are independent of T, such that if $||a||, ||b|| \leq \delta \leq \delta_0$, then $\inf_{\mathcal{M}_{a,b,T}} E[\cdot; [0,T]] \leq c_0\delta^2$.*

(II) *There exist constants $T^* > 0$ and $c_1 > 0$ such that for $T \geq T^*$ and $||a||, ||b|| \leq \delta < \delta_0/2$, and any $w \in \mathcal{M}_{a,b,T}$ we have $E[w; [0,T]] \geq c_1 \min\{||w||^2_{L^\infty(0,T)}, \delta_0^2\}$.*

Since $E[\cdot; [0,T]]$ is weakly lower semicontinuous on $\mathcal{M}_{a,b,T}$, standard arguments imply the existence of a minimizer $\hat{w} \in \mathcal{M}_{a,b,T}$. Furthermore, $\hat{w}$ solves the Euler-Lagrange equation (7). Then the following claim can be obtained from (I) and (II) as in Kalies, Kwapisz, and VanderVorst [41].

(III) *There exist constants $\delta^* > 0$ and $c_2 > 0$ such that for any $T \geq T^*$ and $||a||, ||b|| \leq \delta < \delta^*$ the following holds. If $\hat{w}$ is a minimizer of $E[\cdot; [0,T]]$ on $\mathcal{M}_{a,b,T}$, then we have both $||\hat{w}||_{L^2(0,T)} \leq c_2\delta$ and $||\hat{w}||_{L^\infty(0,T)} \leq c_2\delta$.*

In order to complete the proof of Proposition 4.2 we only have to verify the following assertion.

(IV) *There exists a constant $C_b > 0$ such that for $T \geq T^*$ and $||a||, ||b|| \leq \delta \leq \delta^*$ the following holds. The minimizer $\hat{w}$ of $E[\cdot; [0,T]]$ on $\mathcal{M}_{a,b,T}$ satisfies the estimate $||\hat{w}||_{W^{2n-1,\infty}(0,T)} \leq C_b\delta$.*

Since $\hat{w}$ satisfies the Euler-Lagrange equation (7) and $\gamma_n > 0$ due to (H1), there exists a constant c_3 which depends only on the γ_k and F such that for $\hat{w}$ as above we have

$$||D^{2n}\hat{w}||_{L^2(0,T)} \leq c_3 \cdot \sum_{i=0}^{n-1} ||D^{2i}\hat{w}||_{L^2(0,T)} \, .$$

Applying the interpolation inequalities of Lemma 4.1 furnishes

$$||D^{2n}\hat{w}||_{L^2(0,T)} \leq \frac{1}{2}||D^{2n}\hat{w}||_{L^2(0,T)} + c_4||\hat{w}||_{L^2(0,T)} \, ,$$

where $c_4 > 0$ depends only on the γ_k, F, δ^*, and T^*. Recalling that $||\hat{w}||_{L^2(0,T)} \leq c_2\delta$ due to (III) we further deduce $||D^{2n}\hat{w}||_{L^2(0,T)} \leq 2c_2c_4\delta$. Another application of Lemma 4.1 now implies $||\hat{w}||_{H^{2n}(0,T)} \leq c_5\delta$, and together with Sobolev's embedding theorem we get the desired estimate for $||\hat{w}||_{W^{2n-1,\infty}(0,T)}$. $\diamond$

The proof of the following lemma follows from standard arguments, hence it will be omitted.

Lemma 4.3 *Consider the ordinary differential equation*

$$\dot{x} = g(x) \tag{22}$$

with a C^1*-mapping* $g : \mathbb{R}^d \to \mathbb{R}^d$*. Furthermore, assume that* $x_0 \in \mathbb{R}^d$ *is a hyperbolic fixed point of (22) and let the constant* $\lambda > 0$ *be strictly less than* $|Re\ \mu|$ *for all eigenvalues* μ *of the Jacobian* $Dg(x_0)$*.*

Then there exists a neighborhood $U \subset \mathbb{R}^d$ *of* x_0*, as well as a constant* $C_c > 0$*, such that the following holds. If* $x : [0, T] \to U$ *is a solution of (22), then the inequality*

$$||x(t^*) - x_0|| \leq C_c e^{-\lambda T/2}$$

holds for some $t^* \in [0, T]$*.*

References

[1] N. D. Alikakos, P. W. Bates, and G. Fusco. Slow motion for the Cahn-Hilliard equation in one space dimension. *Journal of Differential Equations*, 90:81–135, 1991.

[2] N. D. Alikakos and G. Fusco. Equilibrium and dynamics of bubbles for the Cahn-Hilliard equation. In *International Conference on Differential Equations, Vol. 1, 2 (Barcelona, 1991)*, pages 59–67, 1993.

[3] N. D. Alikakos and G. Fusco. Slow dynamics for the Cahn-Hilliard equation in higher space dimensions. Part I: Spectral estimates. *Communications in Partial Differential Equations*, 19:1397–1447, 1994.

[4] N. D. Alikakos and G. Fusco. Slow dynamics for the Cahn-Hilliard equation in higher space dimensions: The motion of bubbles. *Preprint*, 1996.

[5] H. Attouch. *Variational Convergence for Functions and Operators*. Pitman, Boston – London – Melbourne, 1984.

[6] P. W. Bates, P. C. Fife, R. A. Gardner, and C. K. Jones. Phase field models for hypercooled solidification. *Physica D*, 1996. To appear.

[7] P. W. Bates and X. Ren. Transition layer solutions of a higher order equation in an infinite tube. *Communications in Partial Differential Equations*, 21:195–220, 1996.

[8] P. W. Bates and J. Xun. Metastable patterns for the Cahn-Hilliard equation. I. *Journal of Differential Equations*, 111:421–457, 1994.

[9] P. W. Bates and J. Xun. Metastable patterns for the Cahn-Hilliard equation. II. Layer dynamics and slow invariant manifold. *Journal of Differential Equations*, 117:165–216, 1995.

[10] L. Bronsard and D. Hilhorst. On the slow dynamics for the Cahn-Hilliard equation in one space dimension. *Proceedings of the Royal Society, London, Series A*, 439:669–682, 1992.

[11] L. Bronsard and R. V. Kohn. On the slowness of phase boundary motion in one space dimension. *Communications on Pure and Applied Mathematics*, 43:983–997, 1990.

[12] G. Caginalp and P. C. Fife. Higher-order phase field models and detailed anisotropy. *Physical Review B*, 34:4940–4943, 1986.

[13] J. W. Cahn. Free energy of a nonuniform system. II. Thermodynamic basis. *Journal of Chemical Physics*, 30:1121–1124, 1959.

[14] J. W. Cahn and J. E. Hilliard. Free energy of a nonuniform system I. Interfacial free energy. *Journal of Chemical Physics*, 28:258–267, 1958.

[15] J. Carr and R. L. Pego. Metastable patterns in solutions of $u_t = \varepsilon^2 u_{xx} - f(u)$. *Communications on Pure and Applied Mathematics*, 42:523–576, 1989.

[16] J. Carr and R. L. Pego. Very slow phase separation in one dimension. In *PDEs and continuum models of phase transitions (Nice, 1988), Lecture Notes in Physics 344*, pages 216–226, 1989.

[17] J. Carr and R. L. Pego. Invariant manifolds for metastable patterns in $u_t = \varepsilon^2 u_{xx} - f(u)$. *Proceedings of the Royal Society of Edinburgh*, 116A:133–160, 1990.

[18] P. Coullet, C. Elphick, and D. Repaux. The nature of spatial chaos. *Physical Review Letters*, 58:431–434, 1987.

[19] G. T. Dee and W. van Saarloos. Bistable systems with propagating front leading to pattern formation. *Physical Review Letters*, 60:2641–2644, 1988.

[20] D. DeFontaine. An analysis of clustering and ordering in multicomponent solid solutions. I. Stability criteria. *Journal of Physics and Chemistry of Solids*, 33:297–310, 1972.

[21] D. DeFontaine. An analysis of clustering and ordering in multicomponent solid solutions. II. Fluctuations and and kinetics. *Journal of Physics and Chemistry of Solids*, 34:1285–1304, 1973.

[22] C. M. Elliott and D. A. French. Numerical studies of the Cahn-Hilliard equation for phase separation. *IMA Journal of Applied Mathematics*, 38:97–128, 1987.

[23] D. J. Estep. An analysis of numerical approximations of metastable solutions of the bistable equation. *Nonlinearity*, 7:1445–1462, 1994.

[24] D. J. Estep, S. M. Verduyn Lunel, and R. D. Williams. Error estimation for numerical differential equations. *Preprint*, 1995.

[25] D. J. Estep and R. D. Williams. Accurate parallel integration of large sparse systems of differential equations. *Preprint*, 1995.

[26] P. C. Fife. Models for phase separation and their mathematics. In *Proc. Taniguchi Int. Symp. on Nonlinear Partial Differential Equations and Applications*, 1991.

[27] I. Fonseca and L. Tartar. The gradient theory of phase transitions for systems with two potential wells. *Proceedings of the Royal Society of Edinburgh*, 111A:89–102, 1989.

[28] M. I. Freidlin. Limit theorems for large deviations and reaction-diffusion equations. *Annals of Probability*, 13:639–675, 1985.

[29] M. I. Freidlin. Geometric optics approach to reaction-diffusion equations. *SIAM Journal on Applied Mathematics*, 46:222–232, 1986.

[30] G. Fusco. A geometric approach to the dynamics of $u_t = \varepsilon^2 u_{xx} + f(u)$ for small ε. In *Problems involving change of type (Stuttgart, 1988), Lecture Notes in Physics 359*, pages 53–73, 1990.

[31] G. Fusco and J. K. Hale. Slow-motion manifolds, dormant instability, and singular perturbations. *Journal of Dynamics and Differential Equations*, 1:75–94, 1989.

[32] R. A. Gardner and C. K. Jones. Traveling waves of a perturbed diffusion equation arising in a phase field model. *Indiana University Mathematics Journal*, 39:1197–1222, 1990.

[33] E. Giusti. *Minimal Surfaces and Functions of Bounded Variation.* Birkhäuser, Basel – Boston – Stuttgart, 1984.

[34] C. P. Grant. Slow motion in one-dimensional Cahn-Morral systems. *SIAM Journal on Mathematical Analysis*, 26:21–34, 1995.

[35] C. P. Grant and E. S. Van Vleck. Slowly-migrating transition layers for the discrete Allen-Cahn and Cahn-Hilliard equations. *Nonlinearity*, 8:861–876, 1995.

[36] J. K. Hale. *Asymptotic Behavior of Dissipative Systems.* American Mathematical Society, Providence, Rhode Island, 1988.

[37] H. Hattori and K. Mischaikow. A dynamical system approach to a phase transition problem. *Journal of Differential Equations*, 94:340–378, 1991.

[38] H. Hattori and K. Mischaikow. On the slow motions of phase boundaries in the Korteweg theory of capillarity. *Dynamic Systems and Applications*, 1:441–452, 1992.

[39] D. Henry. *Geometric Theory of Semilinear Parabolic Equations*, volume 840 of *Lecture Notes in Mathematics.* Springer-Verlag, Berlin –Heidelberg – New York, 1981.

[40] W. D. Kalies and P. J. Holmes. On a dynamical model for phase transformation in nonlinear elasticity. *Fields Institute Communications*, 5:255–269, 1996.

[41] W. D. Kalies, J. Kwapisz, and R. C. VanderVorst. Homotopy classes for stable connections between Hamiltonian saddle-focus equilibria. *Preprint*, 1996.

[42] W. D. Kalies and R. C. VanderVorst. Multitransition homoclinic and heteroclinic solutions of the extended Fisher-Kolmogorov equation. *Journal of Differential Equations*, 131:209–228, 1996.

[43] L. Modica. The gradient theory of phase transitions and the minimal interface criterion. *Archive for Rational Mechanics and Analysis*, 98:123–142, 1987.

[44] J. E. Morral and J. W. Cahn. Spinodal decomposition in ternary systems. *Acta metallurgica*, 19:1037–1045, 1971.

[45] N. C. Owen. Nonconvex variational problems with general singular perturbations. *Transactions of the American Mathematical Society*, 310:393–404, 1988.

[46] J. T. Pinto. *Slow motion manifolds for a class of evolutionary equations.* PhD thesis, Georgia Institute of Technology, 1995.

[47] P. H. Rabinowitz. Periodic and heteroclinic orbits for a periodic Hamiltonian system. *Annales de l'Institut Henri Poincaré. Analyse Non Linéaire*, 6(5):331–346, 1989.

[48] M. Renardy and R. C. Rogers. *An Introduction to Partial Differential Equations.* Springer-Verlag, New York, 1993.

[49] L. G. Reyna and M. J. Ward. Metastable internal layer dynamics for the viscous Cahn-Hilliard equation. *Methods and Applications of Analysis*, 2:285–306, 1995.

[50] B. Sandstede. Interaction of pulses in dissipative systems. *In preparation*, 1996.

[51] P. Sternberg. The effect of a singular perturbation on nonconvex variational problems. *Archive for Rational Mechanics and Analysis*, 101:209–260, 1988.

[52] J. D. van der Waals. The thermodynamic theory of capillarity flow under the hypothesis of a continuous variation in density. *Verh. Konink. Akad. Wetensch. Amsterdam*, 1:1–56, 1893.

[53] M. J. Ward. Metastable bubble solutions for the Allen-Cahn equation with mass conservation. *Preprint*, 1996.