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TRAVELLING WAVES FOR FOURTH ORDER PARABOLIC EQUATIONS*

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Abstract. We study travelling wave solutions for a class of fourth order parabolic equations. Travelling wave fronts of the form $u(x,t) = U(x+ct)$, connecting homogeneous states, are proven to exist in various cases: connections between two stable states, as well as connections between an unstable and a stable state, are considered.

Key words. travelling wave, fourth order equation, nonlinear parabolic equation, bistable equation, Poincaré transformation, global analysis

AMS subject classifications. 34A34, 34C20, 34C37, 34C23, 35A18, 35K25, 35K55, 37B25, 37C29

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1. Introduction.

Fourth order parabolic equations of the form

$$u_t = -\gamma u_{xxxx} + u_{xx} + f(u), \qquad \gamma > 0, \tag{1.1}$$

where $x \in \mathbb{R}$, $t > 0$, occur in many physical models such as the theory of phase-transitions [9], nonlinear optics [1], shallow water waves [7], etc. Usually the potential $F(u) = \int f(s)ds$ has at least two local maxima (stable state) and one local minimum (unstable state).[1] A prototypical example is $f_a(u) = (u+a)(1-u^2)$ with $-1 < a < 1$.

For a thorough understanding of (1.1), the stationary problem is of great importance. An extensive literature on this subject exists (see, e.g., [3, 29, 7, 16, 17, 18, 25, 22, 23, 24]). Typically, depending on the parameter γ, the stationary problem displays a multitude of periodic, homoclinic, and heteroclinic solutions. The stationary equation is Hamiltonian, which restricts the possible connections between the equilibrium points. As an example we mention that when the maximum of F is attained in two points, e.g., $F(u) = -\frac{1}{4}(u^2-1)^2$, a solution connecting these maxima exists for all $\gamma > 0$. One could regard this solution as a standing wave. The heteroclinic solution is unique (modulo the obvious symmetries) for small values of γ, say, $\gamma \le \gamma_1(f)$ [28, 29, 19]. On the other hand, for large γ, say, $\gamma > \gamma_2(f)$, there is a multitude of (multibump/transition) solutions connecting the two maxima [17, 18, 24]. This is due to the fact that as γ crosses the critical value $\gamma = \gamma_2(f)$, the eigenvalues of the linearized stationary equation around the two maxima of F become complex.

In the special case $f(u) = u - u^3$, corresponding to $F(u) = -\frac{1}{4}(u^2-1)^2$, it holds that $\gamma_1(f) = \gamma_2(f) = \frac{1}{8}$. Although in many simple cases equality holds, generally

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[1]Sometimes the potential is denoted by $-F$ so that the stable states correspond to local minima.

there will be a gap between $\gamma_1(f)$ and $\gamma_2(f)$. The critical value γ_1 is not necessarily small, and a lower bound on γ_1 can in general be explicitly determined (see [29] for more details).

For the time-dependent problem travelling fronts of the form $u(x,t) = U(x + ct)$, connecting extrema of the potential F, play a prominent role in most models. Results on travelling waves for (1.1) have previously been obtained in [6], where nonlinearities of the form $f(u) = f_a(u) = (u + a)(1 - u^2)$, $a \approx 0$, are considered using transversality arguments and perturbing near a standing wave. Moreover, in [2] singular perturbations techniques were applied near $\gamma = 0$. In both cases travelling waves between local maxima (stable states) are studied. A recent work [27] deals with singular perturbations techniques for travelling waves connecting an unstable and a stable state; the stability of these waves for very small γ is also established. Furthermore, in the context of singular perturbation theory, travelling waves for higher order parabolic equations have been studied in [15].

The objective of this paper is to obtain existence results for a large range of parameter values. We therefore study travelling waves of (1.1) via topological arguments rather than perturbation methods. To illustrate the underlying ideas of the method, let us consider the related second order parabolic equation, i.e., $\gamma = 0$. Such equations arise as models in, for example, population genetics and combustion theory [4]. In the special case where $f(u) = f_a(u)$, (1.1) with $\gamma = 0$ admits a travelling wave solution $u(x,t) = \tanh\left(\frac{x+a\sqrt{2}t}{\sqrt{2}}\right)$. This travelling wave connects the two stable homogeneous states $u = -1$ and $u = +1$. The literature on this problem is extensive and we will not attempt to give a complete list. However, a few key references are of importance for explaining the similarities of the second and fourth order problems. In the case $\gamma = 0$ the equation for travelling waves $u(x,t) = U(x + ct)$ is given by $cU' = U'' + f(U)$. A phase plane analysis for both $0 < c \ll 1$ and $c \gg 1$ shows two topologically different phase portraits, from which the conclusion may be drawn that a global bifurcation has to take place for some intermediate c-value(s). In this way a wave speed c_0 can be found for which a travelling wave exists which connects the two local maxima of F. In this context we mention the work by Fife and McLeod [13] based on an analytic approach and Conley's more topological approach [8].

From the second order problem we learn that for the present problem it is sensible to look for topologically different phase portraits (in $\mathbb{R}^4$) for small and large values of c. A big part of our analysis will be to do just that.

In order to simplify the exposition of the main results we reformulate (1.1) as

$$u_t = -u_{xxxx} + \alpha u_{xx} + f(u), \tag{1.2}$$

via the rescaling $x \mapsto \gamma^{\frac{1}{4}} x$, with $\alpha = \frac{1}{\sqrt{\gamma}}$. Notice that (1.2) also has meaning for $\alpha \leq 0$.

Let us start now with the hypotheses on the nonlinearity:

$$(\mathrm{H}_0)\quad \begin{cases} \bullet\ F'(u) = f(u) \in C^1(\mathbb{R}); \\ \bullet\ f(u) = 0 \Leftrightarrow u \in \{\pm 1, -a\} \text{ for some } a \in (-1,1), \text{ and } f'(\pm 1) \neq 0,\ f'(-a) \neq 0; \\ \bullet\ F(-1) < F(+1); \\ \bullet\ F(u) \to -\infty \text{ as } u \to \pm\infty; \\ \bullet\ \text{for some } M > 0 \text{ it holds that } f'(u) \leq M \text{ for all } u \in \mathbb{R}.^2 \end{cases}$$

Of course, the prototypical example $f_a(u) = (u+a)(1-u^2)$ satisfies (H_0). We remark that the third condition excludes the existence of a standing wave which connects

[2]Note that $f'(u)$ may be unbounded from below.

two different equilibria. The last condition is a technical one, which we use to obtain certain a priori bounds. Without loss of generality we set

$$F(u) = \int_1^u f(s)ds,$$

so that $F(1) = 0$.

Denote the wave speed by c, and, searching for a travelling wave, we set $u(x,t) = U(x+ct)$, which, switching to lower case again, reduces (1.2) to the ordinary differential equation

$$cu' = -u'''' + \alpha u'' + f(u). \tag{1.3}$$

An important ingredient of our analysis is a conserved quantity for (1.3) when $c = 0$, which is a Lyapunov function when $c \neq 0$. Define

$$\mathcal{E}(u, u', u'', u''') \stackrel{\text{def}}{=} -u'u''' + \frac{1}{2}u''^2 + \frac{\alpha}{2}u'^2 + F(u). \tag{1.4}$$

Multiplying (1.3) by u' we find that

$$\mathcal{E}'(u, u', u'', u''') = cu'^2, \tag{1.5}$$

so that $\mathcal{E}$, which will be referred to as the *energy* of the solution, is increasing along orbits if $c > 0$, constant if $c = 0$, and decreasing if $c < 0$. When we are looking for a solution of (1.3) connecting $u = -1$ to $u = 1$, we see that we can restrict our attention to $c > 0$.

The first theorem deals with the connection between the two stable states $u = -1$ and $u = +1$. This connection is nongeneric with respect to the wave speed c. Noting that $F(u) \leq 0$ for all $u \in \mathbb{R}$ if f satisfies hypothesis (H_0), we define

$$\sigma(f) \stackrel{\text{def}}{=} \min_{-1<u<-a} \frac{-F(u)}{2f(u)^2}. \tag{1.6}$$

THEOREM 1.1. *Let f satisfy hypothesis* (H_0) *and let* $\alpha > \frac{1}{\sqrt{\sigma(f)}}$. *Then, for some wave speed* $c = c_0(f) > 0$, *there exists a travelling wave solution of* (1.2) *connecting* $u = -1$ *to* $u = +1$.

The analogous condition on γ for (1.1) reads $0 < \gamma < \sigma(f)$.

At the minimum in (1.6) the equality $\frac{-F(u)}{2f(u)^2} = \frac{-1}{4f'(u)}$ holds. We easily derive that for our model nonlinearity f_a we have $\sigma(f_a) > \frac{1}{8(1-a)}$ for all $0 < a < 1$. Although this estimate is sharp for $a \to 0$, it is not sharp at all for larger values of a.

For general nonlinearities $f(u)$ satisfying (H_0), a lower bound on σ is

$$\sigma \geq \min\left\{ \frac{-1}{4f'(u)} \;\middle|\; u \in (-1, -a) \text{ and } f'(u) < 0 \right\}. \tag{1.7}$$

This estimate is often easier to compute than σ itself, but it is in general a rather blunt estimate. Finally, we remark that the critical value σ is also encountered in the study of homoclinic orbits for $c = 0$ (see [22, Theorem B]). This originates from the similarity of that problem with the proof of Lemma 5.1, which is in fact the only instance in our analysis where γ is required to be smaller than σ.

We do not obtain much insight in the shape of the travelling wave from Theorem 1.1. Because Theorem 1.1 does not give information about the wave speed, it is not known whether the connected equilibrium points are approached monotonically or in an oscillatory manner. The linearized equation around the equilibrium points leads to the following characteristic equation for the eigenvalues: $c\lambda = -\lambda^4 + \alpha\lambda + f'(\pm 1)$. A few conclusions can be drawn from analyzing this equation. It follows that for $\alpha \geq \sqrt{-4f'(1)}$ the travelling wave tends to $+1$ monotonically as $x \to \infty$. Besides, for $\alpha \leq \sqrt{-4f'(-1)}$ the travelling wave tends to -1 in an oscillatory way as $x \to -\infty$. For other cases the behavior in the limits depends on the value of c.

The travelling wave solution found in Theorem 1.1 connects the two maxima of F. Theorem 1.1 can be extended to potentials F having many local extrema, i.e., $f(u)$ having many zeros. In that case we find a travelling wave connecting the global maximum and the second largest local maximum of F. The other conditions on F remain the same, but we also need that $f(u)u < 0$ for large values of $|u|$. The definition of σ in this case is, setting $\max_{u\in\mathbb{R}} F(u) = 0$,

$$\sigma(f) \stackrel{\text{def}}{=} \inf\left\{ \frac{-F(u)}{2f(u)^2} \;\middle|\; u \in \mathbb{R} \text{ and } f(u)f'(u) > 0 \right\}.$$

The travelling wave solution found in Theorem 1.1 connects the two stable states. The following theorems deal with travelling waves connecting the unstable state $u = -a$ to one of the stable states $u = \pm 1$. These theorems also apply to the parameter regime where $\alpha \geq 0$, but for these parameter values we need an additional condition on f:

$$(\mathrm{H}_1) \quad f \text{ satisfies } (\mathrm{H}_0) \quad \text{and} \quad \lim_{|u|\to\infty} \frac{f(u)}{u} = -\infty.$$

THEOREM 1.2. *Let $\alpha \in \mathbb{R}$ and let f satisfy hypothesis* (H_0) *if $\alpha < 0$ and* (H_1) *if $\alpha \geq 0$.*[3] *Then for every $c > 0$ there exists a travelling wave solution of* (1.2) *connecting $u = -a$ to $u = -1$.*

The limiting behavior of the travelling waves can be determined from the characteristic equations. For $\alpha \geq \sqrt{-4f'(-1)}$ the solution tends to -1 monotonically for $x \to \infty$ regardless of the speed c. On the other hand, for $\alpha < \sqrt{-4f'(-1)}$ the limit behavior is oscillatory for small c and monotonic for large c. The limit behavior near $u = -a$ as $x \to -\infty$ is more complicated. For small c the behavior is generically oscillatory, while for large c the solutions generically tends to $-a$ monotonically. We do not know whether the behavior is indeed generic. However, for $\alpha > \sqrt{12f'(-a)}$ there is an intermediate range of c-values for which the travelling wave certainly tends to $-a$ monotonically.

For general potentials F this result applies to any pair of consecutive nondegenerate extrema u_- (a minimum) and u_+ (a maximum), for which the interval $\big(F(u_-), F(u_+)\big)$ contains no critical values and either u_- or u_+ is the only critical point at level $F(u_\pm)$. The other conditions on F remain the same. The method of proof of Theorem 1.2 requires only one of the two extrema -1 or $-a$ to be nondegenerate.

The next theorem deals with the case of travelling waves from $-a$ to $+1$.

THEOREM 1.3. *Let $\alpha \in \mathbb{R}$ and let f satisfy hypothesis* (H_0) *if $\alpha < 0$ and* (H_1) *if $\alpha \geq 0$. Then there exists a constant $c^*(f) > 0$, such that for every $c > c^*$ there exists a travelling wave solution of* (1.2) *connecting $u = -a$ to $u = +1$.*

[3] The result also holds when $F(-1) = F(+1)$.

Theorem 1.3 extends to general potentials, giving travelling waves between any pair of consecutive nondegenerate extrema u_- (a minimum) and u_+ (a maximum), provided the local minimum $\tilde{u}_-$ on the other side of u_+, if it exists, satisfies $F(\tilde{u}_-) > F(u_-)$. Of course, if the opposite inequality holds then one can exchange u_- and $\tilde{u}_-$. If equality holds, i.e., $F(\tilde{u}_-) = F(u_-)$, then one obtains for every $c > c^*$ a travelling wave connecting either u_- or $\tilde{u}_-$ to u_+. Again, the other conditions on F remain the same.

In certain cases one obtains information about the constant c^* in Theorem 1.3. In that case the situation is very much analogous to the second order equation.

COROLLARY 1.4. *Let f satisfy hypothesis* (H$_0$) *and let $\alpha > \frac{1}{\sqrt{\sigma(f)}}$. Then there exists a $c^*(f) > 0$, such that c^* is the largest speed for which there exists a travelling wave solution of* (1.2) *connecting $u = -1$ to $u = +1$. Moreover, for all $c > c^*$ there exists a travelling wave solution of* (1.2) *connecting $u = -a$ to $u = +1$.*

Finally, we discuss nonlinearities with different behavior for $u \to \pm\infty$. Assume that f has *two* zeros and satisfies

$$(\mathrm{H}_2) \begin{cases} \bullet\ F'(u) = f(u) \in C^1(\mathbb{R}); \\ \bullet\ f(u) = 0 \Leftrightarrow u \in \{0,1\}, \text{ and } f'(0) \neq 0,\ f'(1) \neq 0; \\ \bullet\ \text{for some } D < 0 \text{ it holds that } F(u) > F(1) \text{ for all } u < D; \\ \bullet\ F(u) \to -\infty \text{ as } u \to \infty; \\ \bullet\ \text{if } \alpha \geq 0, \text{ then } \lim_{|u|\to\infty} \frac{f(u)}{u} = -\infty. \end{cases}$$

A typical example is $f(u) = u(1-u)$. The following theorem is analogous to Theorem 1.2.

THEOREM 1.5. *Let $\alpha \in \mathbb{R}$ and let f satisfy hypothesis* (H$_2$)*. Then for every $c > 0$ there exists a travelling wave solution of* (1.2) *connecting $u = 0$ to $u = 1$.*

This last theorem is just an example of how the methods in this paper can also be applied when $F(u)$ does not tend to $-\infty$ as $u \to \pm\infty$. The theorem holds under weaker conditions, but we leave this to the interested reader.

Of the results in this paper, the proof of Theorem 1.1 is by far the most involved. This is caused by the fact that connections between local maxima are nongeneric with respect to the wave speed c. Hence, part of the problem is to determine the wave speed c. The idea behind the proof is that one can detect a change in the phase portrait (in $\mathbb{R}^4$) of (1.3) as c goes from small values to large values. In particular, looking for a travelling wave which connects -1 to $+1$, we investigate the global behavior of the orbits in the stable manifold $W^s(1)$ of the equilibrium point $u = +1$.

The analysis for $c > 0$ large is based on a continuation argument deforming the nonlinearity $f(u)$ into a function which is linear on some interval containing $u = 1$.

For $c > 0$ small the analysis is much more involved. A crucial step is that for $c = 0$ all orbits in $W^s(1)$ are unbounded. A first result in this direction was already proved in [29]. There it was shown that, for γ not too large, the bounded stationary solutions of (1.1) correspond exactly to the bounded stationary solutions of the second order equation ($\gamma = 0$). This excludes the existence of bounded orbits in $W^s(1)$. However, since the analysis comprises *all* bounded solutions, this result is limited to a restricted parameter regime. In particular, the equilibrium points $u = \pm 1$ need to be real saddles. In the present situation we want to exclude bounded solutions in the stable manifold of $u = 1$, i.e., we can restrict the analysis to the energy level $\mathcal{E} = 0$. This allows us to cover a larger range of α-values; to be precise, $\alpha > \frac{1}{\sqrt{\sigma(f)}}$. This parameter regime includes cases where both equilibrium points $u = \pm 1$ are saddle-foci. To give an example, for our model nonlinearity $f_a = (u+a)(1-u^2)$ with $0 < a < 1$

the result from [29] holds for $\alpha \geq \sqrt{8(1+a)}$. The equilibrium points $u = 1$ and $u = -1$ become saddle-foci for $\alpha < \sqrt{8(1+a)}$ and $\alpha < \sqrt{8(1-a)}$, respectively. One may compare this to the estimate $\sigma(f_a) > \frac{1}{8(1-a)}$. Notice that this estimate, although sharp for $a \to 0$, is very blunt for a close to 1.

For the description of unbounded orbits we use a modified Poincaré transformation which we believe is of independent interest. We investigate the unbounded orbits, and we will show that, in an appropriate compactification of the phase space, these orbits must converge to a unique periodic orbit lying at infinity in the phase space. The analysis at infinity largely relies on a global analysis of bounded and unbounded solutions of the family of equations

$$u'''' + u^s = 0 \quad \text{with the convention that} \quad u^s = |u|^{s-1}u, \ s \geq 1.$$

This equation is invariant under the scaling $u(t) \mapsto \kappa u(\kappa^{\frac{s-1}{4}} t)$ for all $\kappa > 0$. The analysis of this equation is in particular used in the proof of finite time blow-up of unbounded solutions, and, more importantly, to determine the behavior of unbounded orbits for $0 \leq c \ll 1$.

From this analysis we conclude that the phase portrait for c positive but small is different from the phase portrait for c large, which in turn is used to prove the existence of a connection between -1 and $+1$ for some intermediate wave speed c_0.

The organization of the paper is as follows. We start with some a priori bounds in section 2. In section 3 we give the proof of Theorem 1.1, and in sections 4 to 6 the details of this proof are filled in. In particular, in section 4 we perform an analysis of the flow "at infinity." Sections 5 and 6 deal with the analysis of the orbits in $W^s(1)$ for small c and large c, respectively. Section 7 discusses the existence of travelling waves connecting $u = -a$ to $u = \pm 1$; Theorems 1.2 to 1.5 are proved here. We conclude with some remarks on open problems in section 8.

2. A priori estimates.

We establish a priori bounds on the wave speed c and the profile u for any travelling wave connecting -1 and $+1$. The bound on the wave speed c holds for all $\alpha \in \mathbb{R}$.

LEMMA 2.1. *Let f satisfy hypothesis* (H_0) *and let $\alpha \in \mathbb{R}$. There exists a constant c_0, depending only on α, $F(-1)$, $F(-a)$, and the upper bound M for $f'(u)$, such that when $c > 0$ is a speed for which there exists a travelling wave solution of* (1.3) *connecting -1 to $+1$, then $c \leq c_0$.*

Proof. Suppose u is a solution of (1.3) connecting -1 to $+1$. Integrating (1.5), we have

$$-F(-1) = F(1) - F(-1) = c \int_{-\infty}^{\infty} u'^2. \tag{2.1}$$

Multiplying (1.3) by u'' and integrating (by parts) we obtain

$$\int_{-\infty}^{\infty} u'''^2 + \alpha \int_{-\infty}^{\infty} u''^2 = \int_{-\infty}^{\infty} (f(u))'u' = \int_{-\infty}^{\infty} f'(u)u'^2 \leq M \int_{-\infty}^{\infty} u'^2 = M \frac{-F(-1)}{c}. \tag{2.2}$$

Let $u_1 \in (-a, 1)$ be defined by

$$F(u_1) = \frac{F(-a) + F(-1)}{2}.$$

There must be points $t_0, t_1 \in \mathbb{R}$, $t_0 < t_1$, such that $u(t_0) = -a$, $u(t_1) = u_1$, and $u(t) \in [-a, u_1]$ for $t \in [t_0, t_1]$. The length of this interval is estimated from below by

$$(u_1 + a)^2 = \left(\int_{t_0}^{t_1} u'(t)dt\right)^2 \leq (t_1 - t_0)^2 \int_{t_0}^{t_1} u'(t)^2 dt \leq (t_1 - t_0)^2 \frac{-F(-1)}{c}.$$

On the one hand, because the energy $\mathcal{E}$ increases along orbits, we have

$$\begin{aligned}
&\int_{t_0}^{t_1} \left(-u'''(t)u'(t) + \frac{1}{2}u''(t)^2 + \frac{\alpha}{2}u'(t)^2\right)dt \\
&\qquad\qquad \geq \int_{t_0}^{t_1} \big(F(-1) - F(u(t))\big)dt \\
&\qquad\qquad \geq (F(-1) - F(u_1))(t_1 - t_0) = \frac{F(-1) - F(-a)}{2}(t_1 - t_0) \\
&\qquad\qquad \geq \frac{F(-1) - F(-a)}{2}(u_1 + a)\sqrt{\frac{c}{-F(-1)}}. \qquad (2.3)
\end{aligned}$$

We now first restrict to the case that $\alpha > 0$, and come back to the other case later on. Using (2.1) and (2.2), we obtain the estimate

$$\begin{aligned}
&\int_{t_0}^{t_1} \left(-u'''(t)u'(t) + \frac{1}{2}u''(t)^2 + \frac{\alpha}{2}u'(t)^2\right)dt \\
&\qquad\qquad \leq \int_{t_0}^{t_1} \left(\frac{1}{2}\big(u'''(t)^2 + u''(t)^2\big) + \frac{1+\alpha}{2}u'(t)^2\right)dt \\
&\qquad\qquad \leq \left(M \max\left\{\frac{1}{\alpha}, 1\right\} + 1 + \alpha\right)\frac{-F(-1)}{2c}. \qquad (2.4)
\end{aligned}$$

By combining (2.3) and (2.4) we obtain

$$\frac{F(-1) - F(-a)}{2}(u_1 + a)\sqrt{\frac{c}{-F(-1)}} \leq \left(M \max\left\{\frac{1}{\alpha}, 1\right\} + 1 + \alpha\right)\frac{-F(-1)}{2c}.$$

Since also

$$\frac{F(-1) - F(-a)}{2} = F(u_1) - F(-a) \leq \frac{M}{2}(u_1 + a)^2,$$

it follows that

$$c \leq M^{\frac{1}{3}}\left(M \max\left\{\frac{1}{\alpha}, 1\right\} + 1 + \alpha\right)^{\frac{2}{3}} \frac{-F(-1)}{F(-1) - F(-a)}.$$

This completes the proof of the lemma for the case that $\alpha > 0$.

We now deal with the case $\alpha \leq 0$. The first part of estimate 2.4 is replaced by

$$\begin{aligned}
&\int_{t_0}^{t_1} \left(-u'''(t)u'(t) + \frac{1}{2}u''(t)^2 + \frac{\alpha}{2}u'(t)^2\right)dt \\
&\qquad \leq \int_{-\infty}^{\infty} \left(\frac{1}{2}u'''(t)^2 + \frac{1}{2}u''(t)^2 + \frac{1}{2}u'(t)^2\right)dt
\end{aligned}$$

$$= \int_{-\infty}^{\infty} \left(u'''(t)^2 + \alpha u''(t)^2 + \left(\frac{1}{2} - \alpha\right) u''(t)^2 - \frac{1}{2} u'''(t)^2 + \frac{1}{2} u'(t)^2 \right) dt$$

$$\le \int_{-\infty}^{\infty} \left(u'''(t)^2 + \alpha u''(t)^2 + \frac{4\alpha^2 - 4\alpha + 5}{8} u'(t)^2 \right) dt,$$

where we have used that $\int_{-\infty}^{\infty} u''^2 \le \lambda \int_{-\infty}^{\infty} u'''^2 + \frac{1}{4\lambda} \int_{-\infty}^{\infty} u'^2$ for all $\lambda > 0$. The remainder of the proof is the same as above. □

The L^∞-bound on the profile u holds for $\alpha > 0$, or equivalently, for all $\gamma > 0$.

LEMMA 2.2. *Let f satisfy hypothesis* (H_0) *and let $\alpha > 0$. There exists a constant C_1, depending only on α, $F(-1)$, $F(-a)$, and the upper bound M for $f'(u)$, such that when u is, for some $c > 0$, a travelling wave solution of* (1.3) *connecting -1 to $+1$, then $F(u) \ge C_1$.*

Proof. We may suppose that there is a connection u with range not contained in the bounded interval $\{u \in \mathbb{R} \mid F(u) \ge F(-a)\}$, otherwise we already have our desired uniform bound. Therefore, without loss of generality we may assume that

$$F(u(0)) = \min_{t \in \mathbb{R}} F(u(t)) < F(-a). \tag{2.5}$$

We consider the case where $u(0) < -1$ (the case $u(0) > 1$ is completely analogous). Since

$$\mathcal{E}(u, u', u'', u''')(t) \in \big(F(-1), F(1)\big) = \big(F(-1), 0\big) \qquad \text{for all } t \in \mathbb{R}, \tag{2.6}$$

we clearly have that

$$u(0) < -1, \quad u'(0) = 0, \quad 0 < \sqrt{2\big(F(-1) - F(u(0))\big)} < u''(0) < \sqrt{-2F(u(0))}.$$

We now consider two cases: $u'''(0) \ge 0$ and $u'''(0) < 0$. We start with the latter case. Since $u(t)$ tends to an equilibrium point as $t \to -\infty$, there exists a $t_1 < 0$ such that $u'''(t) < 0$ for $t_1 < t < 0$ and $u'''(t_1) = 0$, (1.5) implies that

$$-u'(t)u'''(t) + F(u(t)) - F(u(0)) = -\frac{1}{2}\big(u''(t)^2 - u''(0)^2\big) - \frac{\alpha}{2} u'(t)^2 + c\int_0^t u'(s)^2 ds. \tag{2.7}$$

By (2.5) we know that $F(u(t_1)) \ge F(u(0))$, so that

$$\frac{1}{2}\big(u''(t_1)^2 - u''(0)^2\big) + \frac{\alpha}{2} u'(t_1)^2 \le -c \int_{t_1}^0 u'(s)^2 ds.$$

Since $u''(t)$ decreases on $(t_1, 0)$ and α is positive, this implies that $c < 0$, a contradiction.

We now deal with the case that $u'''(0) \ge 0$. Since $u''''(0) > 0$ by the differential equation, and since $u(t)$ tends to an equilibrium point as $t \to \infty$, there exists a $t_2 > 0$ such that $u'''(t) > 0$ for $0 < t < t_2$ and $u'''(t_2) = 0$. By (2.5) we know that $F(u(t_2)) \ge F(u(0))$. Since $\alpha > 0$, it follows from (2.7) that

$$\frac{\alpha}{2} u'(t_2)^2 \le c \int_0^{t_2} u'(s)^2 ds \le c \int_{-\infty}^{\infty} u'(s)^2 ds \le -F(-1). \tag{2.8}$$

Furthermore, from the fact that $u''(t)$ increases on $(0, t_2)$ we infer that

$$u''(0)t \le u'(t) \le u'(t_2) \qquad \text{for } t \in [0, t_2]. \tag{2.9}$$

On the one hand it follows from (2.8) and (2.9) that $\frac{\alpha}{2}u'(t_2)^2 \le c\int_0^{t_2} u'(s)^2 ds \le cu'(t_2)^2 t_2$, hence

$$t_2 \ge \frac{\alpha}{2c}. \tag{2.10}$$

On the other hand it follows from (2.8) and (2.9) that $-F(-1) \ge c\int_0^{t_2} u'(s)^2 ds \ge \frac{1}{3}ct_2^3 u''(0)^2$. Combining with (2.10) we thus obtain that

$$u''(0)^2 \le \frac{-24c^2F(-1)}{\alpha^3}.$$

This gives a bound on $u''(0)^2$, because it follows from Lemma 2.1 that the wave speed c is bounded above by a constant $c_0\big(\alpha, M, F(-a), F(-1)\big)$.

Finally, by (2.5) and (2.6) we have

$$F(u(t)) \ge F(u(0)) \ge F(-1) - \frac{1}{2}u''(0)^2 \qquad \text{for all } t \in \mathbb{R}.$$

This completes the proof of Lemma 2.2. $\square$

3. Proof of Theorem 1.1. In this section we give the proof of Theorem 1.1. Some of the major steps, which require a quite involved analysis, are only stated as a proposition in this section and are proved in subsequent sections.

We first use the a priori bounds of section 2 to reduce our analysis to nonlinearities $f(u)$ of the form $f(u) = -u^3 + g(u)$, where $g(u)$ has compact support. The advantage of such nonlinearities is that they behave nicely as $u \to \pm\infty$, and it will thus be possible to analyze the flow near/at infinity.

Let $f(u)$ satisfy hypothesis (H_0). Lemma 2.2 implies that there exists a constant C_0 such that any travelling wave solution u connecting -1 to $+1$ satisfies $\|u\|_\infty < C_0$. Define the cut-off function $\phi \in C_0^\infty$ with $0 \le \phi \le 1$, $\phi(y) = 1$ for $|y| \le C_0$, and $\phi(y) = 0$ for $|y| > C_0 + 1$. We now consider the modified nonlinearity $\tilde{f}(u) = \phi(u)f(u) - u^3(1-\phi(u))$. Lemma 2.2 ensures that u is a travelling wave solution for nonlinearity $f(u)$ if and only if u is a travelling wave solution for nonlinearity $\tilde{f}(u)$. Besides, $\sigma(f) = \sigma(\tilde{f})$. This shows that we may restrict our analysis to nonlinearities $f(u)$ such that

(3.1) $f(u) = -u^3 + g(u)$ *with g compactly supported, and f satisfies hypothesis* (H_0).

The purpose of the reduction to nonlinearities f which satisfy (3.1) is that it makes it possible to analyze the orbits which are unbounded. An important property of unbounded solutions, which we will need in the following, is formulated in the next lemma.

LEMMA 3.1. *Let f satisfy hypothesis* (3.1) *and let $\alpha, c \in \mathbb{R}$. Then any unbounded solution of* (1.3) *blows up in finite time.*

This lemma is proved in section 4.5, Theorem 4.8(b), and is based on the analysis of the flow near/at infinity.

As already discussed in the introduction, denote the wave speed by c. For finding a travelling wave we set $u(x,t) = U(x+ct)$, which reduces (1.1) to the ordinary differential equation (1.3). Written as a four-dimensional system, (1.3) becomes

$$u' = v; \quad v' = w; \quad w' = z; \quad z' = \alpha w - cv + f(u). \tag{3.2}$$

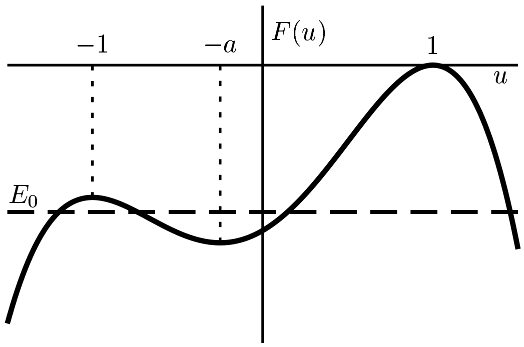

FIG. 3.1. *The potential $F(u)$ and the energy level E_0 separating $u = -a$ from $u = \pm 1$.*

The equilibria of this system are $(u, v, w, z) = (-1, 0, 0, 0)$, $(u, v, w, z) = (-a, 0, 0, 0)$, and $(u, v, w, z) = (1, 0, 0, 0)$ (for short: $u = -1$, $u = -a$, and $u = 1$). To prove Theorem 1.1 we look for a $c \neq 0$ and a corresponding heteroclinic orbit of (3.2) connecting $u = -1$ to $u = 1$. Linearizing around $u = \pm 1$ we find that, irrespective of c, both $u = -1$ and $u = 1$ have two-dimensional stable and unstable manifolds, denoted by $W^s(\pm 1)$ and $W^u(\pm 1)$. Generically $W^s(1)$ and $W^u(-1)$ will not intersect but varying c we expect to pick up a nonempty intersection.

We recall that the *energy* is defined as

$$\mathcal{E}(u, v, w, z) \stackrel{\text{def}}{=} -vz + \frac{1}{2}w^2 + \frac{\alpha}{2}v^2 + F(u),$$

where the potential $F(u) = \int_1^u f(s)ds$ is depicted in Figure 3.1. Since we are looking for a solution of (1.3) which connects $u = -1$ to $u = 1$, we see from (1.5) that we can restrict our attention to $c > 0$. The energy $\mathcal{E}$ thus increases along orbits.

To separate the equilibrium point $u = -a$ from $u = \pm 1$, we choose an energy level E_0 such that (see also Figure 3.1)

$$F(-a) < E_0 < F(-1) < 0,$$

and we define the set

$$K \stackrel{\text{def}}{=} \{(u, v, w, z) \in \mathbb{R}^4 \mid \mathcal{E}(u, v, w, z) \geq E_0\}. \tag{3.3}$$

This allows us to formulate the following lemma.

LEMMA 3.2. *Let f satisfy hypothesis* (3.1) *and let $\alpha \in \mathbb{R}$. If $c > 0$ is such that $W^s(1) \cap W^u(-1) = \emptyset$, then every orbit in $W^s(1)$ enters K through its boundary δK and $\hat{\Gamma} = W^s(1) \cap \delta K$ is a simple closed curve. The set of positive c for which this property holds is open and $\hat{\Gamma}$ varies continuously with c.*

Proof. In view of (1.5) the intersection of $W^s(1)$ and δK must be transversal. Assume that $W^s(1) \cap W^u(-1) = \emptyset$. We need to show that every orbit in $W^s(1)$ can be traced back to δK, for then there is bijection between $W^s(1) \cap \delta K$ and a smooth simple closed curve in $W^s_{\text{loc}}(1)$ winding around $u = 1$ (in $W^s_{\text{loc}}(1)$). Arguing by contradiction we assume that there is an orbit in $W^s(1)$ which is completely contained in K. Let $u(t)$ be a solution representing this orbit. Then $u(t)$ exists on some maximal time interval $(t_{\min}, \infty)$. Since $u(t)$ has energy larger than E_0, it follows from (1.5) and

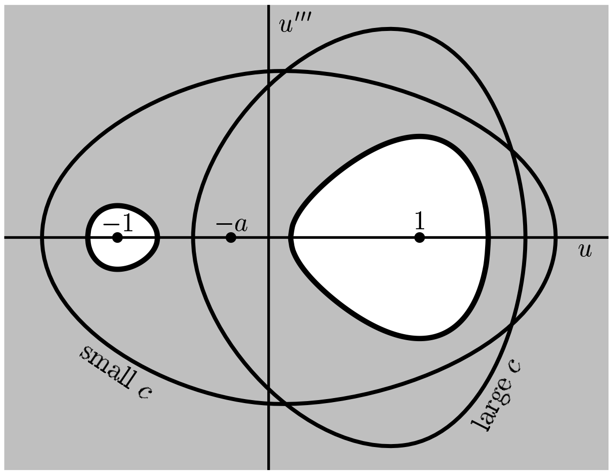

FIG. 3.2. *The projection (in grey) of δK onto the (u, z)-plane. The closed curves which form the boundary of the grey area are given by* (3.6)*. The other two curves depict* Γ *(i.e., the projection of $W^s(1) \cap \delta K$ onto the (u, z)-plane) for small c and large c.*

(3.3) that

$$\int_{t_{\min}}^{\infty} u'^2 \leq \frac{F(1) - E_0}{c} = \frac{-E_0}{c}, \tag{3.4}$$

so that $u(t)$ remains bounded on $(t_{\min}, \infty)$ if $t_{\min}$ is finite. Thus $t_{\min} = -\infty$ and, by Lemma 3.1, $u(t)$ is bounded. It follows from standard arguments that the orbit converges to a limit as $t \to -\infty$. Because $u = -1$ is the only equilibrium in K with energy less than the energy of $u = 1$, we infer that $u(t) \in W^u(-1)$. This contradicts the assumption that $W^s(1) \cap W^u(-1) = \emptyset$. The second statement is an immediate consequence of the (topological) transversality of $W^s(1) \cap \delta K$. $\square$

It now suffices to show that there is a $c > 0$ for which the assumption of Lemma 3.2 fails. Again arguing by contradiction, we assume that Lemma 3.2 applies to all $c > 0$ and search for a topological obstruction. This requires a description of δK that allows us to form a global picture of this set. To this end we write δK as (with $\alpha > 0$)

$$\delta K = \left\{ (u, v, w, z) \in \mathbb{R}^4 \;\middle|\; \frac{\alpha}{2}\left(v - \frac{1}{\alpha} z\right)^2 + \frac{1}{2} w^2 = E_0 - F(u) + \frac{1}{2\alpha} z^2 \right\}. \tag{3.5}$$

In Figure 3.2 we have plotted the projection of δK onto the (u, z)-plane. For (u, z) lying inside one of the two closed curves (see Figure 3.2) defined by

$$E_0 - F(u) + \frac{1}{2\alpha} z^2 = 0, \tag{3.6}$$

every (u, v, w, z) belongs to K, hence there are no points in δK with (u, z) lying inside these two closed curves. For (u, z) lying outside the two closed curves we have that (u, v, w, z) is in K if (v, w) is outside the ellipse defined by $\frac{\alpha}{2}\left(v - \frac{1}{\alpha} z\right)^2 + \frac{1}{2} w^2 = 0$. We conclude that the projection of δK onto the (u, z)-plane is the region outside the two closed curves defined by (3.6); see Figure 3.2.

The projection of δK onto the (u,z)-plane maps $\hat{\Gamma} = W^s(1) \cap \delta K$, which by assumption exists for all $c > 0$, to a closed but not necessarily simple curve Γ in the (u,z)-plane for which the winding numbers[4] $n(\Gamma,-1)$ and $n(\Gamma,1)$ around $(u,z) = (-1,0)$ and $(u,z) = (1,0)$, respectively, are well defined and independent of c (by continuity). However, the following proposition establishes the configuration depicted in Figure 3.2, contradicting the assumption that $W^s(1) \cap W^u(-1) = \emptyset$ for all $c > 0$, and thereby completing the proof of Theorem 1.1.

PROPOSITION 3.3. *Let f satisfy hypothesis* (3.1).

(a) *Let $\alpha > \frac{1}{\sqrt{\sigma(f)}}$. Then there exists a $c_* > 0$ such that $n(\Gamma,-1) = 1$ and $n(\Gamma,1) = 1$ for all $0 < c < c_*$.*

(b) *Let $\alpha \in \mathbb{R}$. Then there exists a $c^* > 0$ such that $n(\Gamma,-1) = 0$ and $n(\Gamma,1) = 1$ for all $c > c^*$.*

Part (a) of Proposition 3.3 will be proved in Theorem 5.3 in section 5, while part (b) is proved in section 6, Theorem 6.1.

4. Classification of unbounded solutions. In this section we investigate the behavior of unbounded solutions, or in other words, we analyze the flow at infinity. This analysis is relevant both for the proof of finite time blow-up of unbounded solutions, and to determine the behavior of unbounded orbits for $0 \le c \ll 1$. We have argued in section 3 that we may restrict our attention to nonlinearities of the form $f(u) = -u^3 + g(u)$, where $g(u)$ has compact support. It turns out that the flow for large u is governed by the *reduced* equation $u'''' + u^3 = 0$, i.e., only the highest order derivative and the highest order term in the nonlinearity play a role at infinity. In the following sections we investigate the reduced equation, and in section 4.5 we come back to the full equation.

4.1. A modified Poincaré transformation. We analyze the reduced equation

$$u'''' + u^s = 0 \quad \text{with the convention that} \quad u^s = |u|^{s-1}u,\ s \ge 1, \tag{4.1}$$

and we use this notational convention throughout. Written as a system, (4.1) reads

$$x_1' = x_2; \quad x_2' = x_3; \quad x_3' = x_4; \quad x_4' = -x_1^s, \tag{4.2}$$

where x_1, x_2, x_3, and x_4 correspond to u, u', u'', and u'''. Note that for this system the energy (or Hamiltonian)

$$H(x_1,x_2,x_3,x_4) \stackrel{\text{def}}{=} -x_2x_4 + \frac{x_3^2}{2} - \frac{|x_1|^{s+1}}{s+1} \tag{4.3}$$

is a conserved quantity.

Introduce five new dependent variables X_1, X_2, X_3, X_4, and $X_5 > 0$ by setting

$$x_i = \frac{X_i}{X_5^{a_i}} \qquad (i = 1,2,3,4), \tag{4.4}$$

where the exponents a_i are to be chosen shortly. Unbounded orbits of (4.2) will correspond to orbits in the new variables with X_5 approaching zero. By substituting

[4] We may choose the orientation of the simple closed curve in $W^s_{\text{loc}}(1)$ winding around $u = 1$ in such a way that its projection onto the (u,z)-plane has winding number equal to $+1$.

(4.4) in (4.2) we obtain the equations

$$X_5 X_1' - a_1 X_1 X_5' = X_2 X_5^{1+a_1-a_2}; \tag{4.5a}$$

$$X_5 X_2' - a_2 X_2 X_5' = X_3 X_5^{1+a_2-a_3}; \tag{4.5b}$$

$$X_5 X_3' - a_3 X_3 X_5' = X_4 X_5^{1+a_3-a_4}; \tag{4.5c}$$

$$X_5 X_4' - a_4 X_4 X_5' = -X_1^s X_5^{1+a_4-sa_1}, \tag{4.5d}$$

with a fifth equation pending. We choose the exponents in such a way that all the exponents in the right-hand sides of (4.5) are the same, i.e.,

$$b \stackrel{\text{def}}{=} 1 + a_1 - a_2 = 1 + a_2 - a_3 = 1 + a_3 - a_4 = 1 + a_4 - sa_1.$$

Solving for a_1, a_2, a_3, a_4, and b we find

$$a_1 = 4\lambda;\ a_2 = (s+3)\lambda;\ a_3 = (2s+2)\lambda;\ a_4 = (3s+1)\lambda;\ b = 1 - (s-1)\lambda, \tag{4.6}$$

where λ is still free and, for the moment, positive. We close system (4.5) by imposing as a fifth equation

$$X_1^s X_1' + X_2 X_2' + X_3 X_3' + X_4 X_4' = 0. \tag{4.7}$$

If we multiply (4.5a)–(4.5d) by X_1^s, X_2, X_3, and X_4 respectively, and add up the resulting equations, we obtain

$$P X_5' = -\frac{1}{\lambda} Q X_5^b. \tag{4.8}$$

Here we have set

$$P \stackrel{\text{def}}{=} 4|X_1|^{s+1} + (3+s)X_2^2 + (2+2s)X_3^2 + (1+3s)X_4^2, \tag{4.9}$$

which is nonnegative, and

$$Q \stackrel{\text{def}}{=} X_1^s(X_2 - X_4) + X_3(X_2 + X_4).$$

Introducing a new independent variable, we write

$$\dot{X}_5 = P X_5^{(s-1)\lambda} X_5' = -\frac{1}{\lambda} Q X_5, \tag{4.10}$$

where the dot denotes derivation with respect to this new independent variable from which the old one may be recovered by integration. Thus, combining (4.10) and (4.5), we arrive at the system

$$\dot{X}_1 = X_2 P - 4X_1 Q\,; \tag{4.11a}$$

$$\dot{X}_2 = X_3 P - (3+s)X_2 Q\,; \tag{4.11b}$$

$$\dot{X}_3 = X_4 P - (2+2s)X_3 Q\,; \tag{4.11c}$$

$$\dot{X}_4 = -X_1^s P - (1+3s)X_4 Q\,. \tag{4.11d}$$

Note that X_5 has been decoupled from the equations. By construction the system (4.11) leaves the surfaces

$$\Sigma \stackrel{\text{def}}{=} \left\{ (X_1, X_2, X_3, X_4) \;\middle|\; \frac{|X_1|^{s+1}}{s+1} + \frac{X_2^2}{2} + \frac{X_3^2}{2} + \frac{X_4^2}{2} = C_0 \right\} \cong S^3 \tag{4.12}$$

invariant for all $C_0 > 0$. The free parameter λ appears only in (4.10) and may be discarded.

The Poincaré transformation (4.4) is used here to blow up the flow near "infinity." As will be explained in section 4.4 this is equivalent to blowing up the flow near the equilibrium point $u = 0$. This blowing-up technique is frequently used in the study of flows in the neighborhood of nonhyperbolic equilibrium points (see, e.g., [10, 11, 21]). The transformation defined by (4.4) and (4.12) is a variant of the standard Poincaré transformation, which has $a_1 = a_2 = a_3 = a_4 = 1$ and imposes as fifth equation that $X_1^2 + X_2^2 + X_3^2 + X_4^2 + X_5^2$ be constant, so that the transformed problem is situated on the Poincaré sphere. The modification presented above, in particular the choice of exponents, is needed to obtain a nontrivial vector field at infinity from which we may derive the qualitative properties of the flow of the system (4.2) near infinity. The values of the exponents are derived from the invariance of (4.1) under the scaling $u(t) \mapsto \kappa u(\kappa^{\frac{s-1}{4}} t)$.

In (4.7) we have chosen not to include a term $X_5 X_5'$ and to modify the exponent of X_1. This simplifies the new vector field and allows the decoupling of the $\dot{X}_5$-equation. Note that instead of a Poincaré sphere we now have a Poincaré cylinder Π, namely, the topological product of the deformed sphere Σ and the positive X_5-axis:

$$\Pi \stackrel{\text{def}}{=} \{(X_1, X_2, X_3, X_4, X_5) \,|\, (X_1, X_2, X_3, X_4) \in \Sigma,\ X_5 \geq 0\} \cong S^3 \times [o, \infty).$$

The flow of (4.2) is completely determined by the flow of (4.11) on Σ. Therefore, we have a reduction from dimension 4 for (4.2) to dimension 3 for (4.11). The role of $X_5 = 0$ and $X_5 = \infty$ can be reversed by changing from positive to negative λ at the expense of a minus sign in (4.10).

Remark 4.1. The choice of $C_0 > 0$ in (4.12) is arbitrary, because the flows on all spheres Σ are C^1-conjugated (modulo the introduction of the new independent variable in (4.10)). This is in fact the very idea of Poincaré transformations, namely, that we divide out the invariance of (4.1) and focus on the resulting flow. From a more abstract point of view one can construct a flow on the quotient manifold $(\mathbb{R}^4 \backslash \{0\})/\mathbb{R}^+ \cong S^3$ via the scaling invariance $u(t) \mapsto \kappa u(\kappa^{\frac{s-1}{4}} t)$ ($\mathbb{R}^+$-action); see [20] for more details. Our construction involves explicit choices of coordinates, for which the flows, by general theory, are all related by conjugation.

To be explicit, let X_i and Y_i be two sets of Poincaré coordinates, i.e.,

$$x_i = \frac{X_i}{X_5^{a_i}} = \frac{Y_i}{Y_5^{a_i}} \qquad \text{for } i = 1, 2, 3, 4,$$

with constraints

$$\frac{|X_1|^{s+1}}{s+1} + \frac{X_2^2}{2} + \frac{X_3^2}{2} + \frac{X_4^2}{2} = C_0, \tag{4.13a}$$

$$\frac{|Y_1|^{s+1}}{s+1} + \frac{Y_2^2}{2} + \frac{Y_3^2}{2} + \frac{Y_4^2}{2} = C_1. \tag{4.13b}$$

When we define $\mu = \frac{X_5}{Y_5}$, then the two sets of coordinates are related by

$$X_5 = \mu Y_5 \qquad \text{and} \qquad X_i = \mu^{a_i} Y_i \quad \text{for } i = 1, 2, 3, 4. \tag{4.14}$$

Substituting this into (4.13a) we obtain

$$G(Y_1, Y_2, Y_3, Y_4, \mu) \equiv \mu^{(s+1)a_1} \frac{|Y_1|^{s+1}}{s+1} + \mu^{2a_2} \frac{Y_2^2}{2} + \mu^{2a_3} \frac{Y_3^2}{2} + \mu^{2a_4} \frac{Y_4^2}{2} = C_0.$$

Since $\frac{\partial G}{\partial \mu} > 0$ for all Y_i that obey (4.13b), it follows from the implicit function theorem that $\mu(Y_1, Y_2, Y_3, Y_4)$ is a differentiable function. It is now easily seen from (4.14) that X_i and Y_i are related by a C^1-conjugacy. Therefore, we may choose the constant C_0 according to our liking to obtain a description of the flow that is most suitable to our needs.

4.2. The flow at infinity. For the analysis of (4.11) we first observe the following.

LEMMA 4.2. *System* (4.11) *has no stationary points on* Σ *for any* $C_0 > 0$.

Proof. Since $X_1 = X_2 = X_3 = X_4 = 0$ is excluded we have that P, defined by (4.9), is positive. Equating the right-hand sides of (4.11) to zero and considering the resulting equations as linear equations in P and Q, it follows that we can only have solutions if every determinant of every pair of two equations vanishes. This would give, for instance, that

$$\begin{aligned} 0 &\le (2+2s)X_3^2 = (3+s)X_2X_4; \\ 0 &\le 4|X_1|^{s+1} = -(1+3s)X_2X_4. \end{aligned}$$

We conclude that $X_2X_4 = 0$ and with any of the $X_i = 0$ the others thus follow immediately. ◻

We next use the conserved quantity to obtain a further reduction from dimension 3 to dimension 2 for the limit sets of orbits of (4.5) which approach infinity ($X_5 \to 0$) or the origin ($X_5 \to \infty$). In the new variables the Hamiltonian is

$$H = \left(-X_2X_4 + \frac{X_3^2}{2} - \frac{|X_1|^{s+1}}{s+1}\right) X_5^{-4\lambda(s+1)}. \tag{4.15}$$

Denote the first factor of H by H_0:

$$H_0 \stackrel{\text{def}}{=} -X_2X_4 + \frac{X_3^2}{2} - \frac{|X_1|^{s+1}}{s+1}. \tag{4.16}$$

Since H is a conserved quantity, we conclude that for $\lambda > 0$

$$X_5 \to 0 \quad \Leftrightarrow \quad H_0 \to 0. \tag{4.17}$$

For the classification of unbounded orbits we have to analyze the flow restricted to the invariant set given by

$$\begin{aligned} T &\stackrel{\text{def}}{=} \{(X_1, X_2, X_3, X_4) \in \Sigma \mid H_0 = 0\} \\ &= \left\{(X_1, X_2, X_3, X_4) \;\middle|\; \frac{|X_1|^{s+1}}{s+1} + \frac{X_2^2}{2} + \frac{X_3^2}{2} + \frac{X_4^2}{2} = C_0, \frac{X_3^2}{2} = X_2X_4 + \frac{|X_1|^{s+1}}{s+1}\right\}. \end{aligned}$$

This set is a topological torus as can be seen by setting

$$X_1 = \xi_1; \quad X_2 = \frac{\xi_2 + \xi_4}{\sqrt{2}}; \quad X_3 = \xi_3; \quad X_4 = \frac{\xi_2 - \xi_4}{\sqrt{2}}, \tag{4.18}$$

so that, in terms of the ξ-variables,

$$T = \left\{(\xi_1, \xi_2, \xi_3, \xi_4) \;\middle|\; \frac{2}{s+1}|\xi_1|^{s+1} + \xi_2^2 = \xi_3^2 + \xi_4^2 = C_0\right\} \cong S^1 \times S^1. \tag{4.19}$$

Clearly we have that T is the product of two topological circles, one in the (ξ_1, ξ_2)-plane, the other in the (ξ_3, ξ_4)-plane.

LEMMA 4.3. *Let $s \geq 1$ and fix the constant $C_0 > 0$. Then there exist precisely two periodic orbits Λ_- and Λ_+ of (4.11) on the torus T.*

Proof. The proof is based on the observation that the coefficient Q in (4.10), which after transforming by (4.18) reads

$$Q = \sqrt{2}(\xi_1^s \xi_4 + \xi_2 \xi_3), \tag{4.20}$$

plays a double role. Obviously it determines which parts of infinity attract solutions toward $X_5 = 0$, in forward and in backward time. We begin by showing that Q can also be seen as minus the divergence of the vector field restricted to the invariant torus T. From (4.11) and (4.18) we derive

$$\dot{\xi}_1 = \frac{\xi_2 + \xi_4}{\sqrt{2}} P - 4\xi_1 Q\,; \tag{4.21a}$$

$$\dot{\xi}_2 = \frac{\xi_3 - \xi_1^s}{\sqrt{2}} P - ((2+2s)\xi_2 + (1-s)\xi_4)Q\,; \tag{4.21b}$$

$$\dot{\xi}_3 = \frac{\xi_2 - \xi_4}{\sqrt{2}} P - (2+2s)\xi_3 Q\,; \tag{4.21c}$$

$$\dot{\xi}_4 = \frac{\xi_3 + \xi_1^s}{\sqrt{2}} P - ((1-s)\xi_2 + (2+2s)\xi_4)Q\,. \tag{4.21d}$$

We parametrize T by "polar coordinates"

$$\xi_1 = f_1(\phi); \quad \xi_2 = g_1(\phi); \quad \xi_3 = f_2(\theta); \quad \xi_4 = g_2(\theta), \tag{4.22}$$

satisfying

$$f_1' = -g_1; \quad g_1' = f_1^s; \quad f_2' = -g_2; \quad g_2' = f_2. \tag{4.23}$$

Note that when $C_0 = 1$ and $s = 1$ we just have

$$\xi_1 = \cos\phi; \quad \xi_2 = \sin\phi; \quad \xi_3 = \cos\theta; \quad \xi_4 = \sin\theta.$$

From (4.21a), (4.21c), (4.22), and (4.23) we derive that on T the flow is given by

$$\dot{\phi} = \frac{P}{\sqrt{2}}\left(-1 - \frac{g_2}{g_1}\right) + 4Q\frac{f_1}{g_1} \equiv w_1(\phi,\theta); \tag{4.24a}$$

$$\dot{\theta} = \frac{P}{\sqrt{2}}\left(1 - \frac{g_1}{g_2}\right) + 2(s+1)Q\frac{f_2}{g_2} \equiv w_2(\phi,\theta), \tag{4.24b}$$

where in terms of f_1, g_1, f_2, g_2,

$$P = 4(s+1)C_0 + 2(1-s)g_1 g_2, \qquad \text{and} \qquad Q = \sqrt{2}(f_1^s g_2 + f_2 g_1).$$

The functions w_1 and w_2, defined in (4.24), appear to have singularities, but using (4.19) they can be written as

$$\begin{aligned} w_1(\phi,\theta) &= \sqrt{2}\big[-2(s+1)C_0 - (s+3)g_1 g_2 + (s-1)g_2^2 + 4f_1 f_2\big], \\ w_2(\phi,\theta) &= \sqrt{2}\big[2(s+1)C_0 - (3s+1)g_1 g_2 + (s-1)g_1^2 + 2(s+1)f_1^s f_2\big]. \end{aligned}$$

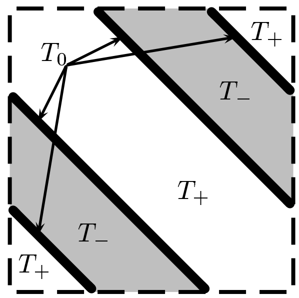

FIG. 4.1. *A fundamental domain of the torus, in which T_-, T_+, and T_0 are indicated (schematically).*

Taking the divergence of the vector field w we obtain (using (4.23),

$$\nabla \cdot w = \frac{\partial w_1}{\partial \phi} + \frac{\partial w_2}{\partial \theta} = \sqrt{2}(-5-3s)(f_1^s g_2 + f_2 g_1) = -(3s+5)Q.$$

Next, we split T into

$$T_+ = \{(X_1, X_2, X_3, X_4) \,|\, Q > 0\} \quad \text{and} \quad T_- = \{(X_1, X_2, X_3, X_4) \,|\, Q < 0\}.$$

These two sets share the boundary

$$T_0 = \{(X_1, X_2, X_3, X_4) \,|\, Q = 0\},$$

which, in view of (4.19) and (4.20), consists of two topological circles, which both wind once around the two homotopically distinct simple loops on the torus (see Figure 4.1). We will show in Lemma 4.4 that, when C_0 is chosen properly, an orbit can only pass through T_0 from T_- to T_+. It then follows from the negativity of $\nabla \cdot w$ in T_+ and the winding properties of T_0 on T that T_+ contains precisely one periodic orbit. The same statement holds for T_- with respect to the backward flow on T.

To be precise, we deduce from (4.22), (4.23), and (4.19) that we may choose $\xi_3 = f_2(\theta) = \sqrt{C_0}\cos\theta$. Define the set $S \stackrel{\text{def}}{=} \{(\theta, \phi) \in T \,|\, \theta = \frac{\pi}{2}\}$, and it follows that

$$\dot{\theta}\Big|_S = \sqrt{2}\left[2(s+1)C_0 - (3s+1)\sqrt{C_0}g_1 + (s-1)g_1^2\right].$$

Since $|g_1| \le \sqrt{C_0}$, it is easy to check that $\dot{\theta}\big|_S \ge 0$, and equality holds only when $g_1 = \sqrt{C_0}$. By continuity arguments the orbit through this point crosses S also in the direction of increasing θ. Thus S is a global section for the flow on T. Moreover, the return map is well defined, since there is no point in T for which the forward orbit is contained in $T \setminus S$. Indeed, such a forward orbit would either be contained in T_- or eventually be in T_+, because T_+ is positively invariant and orbits can only pass through T_0 from T_- to T_+. In the absence of equilibrium points (Lemma 4.2) its ω-limit set would be a periodic orbit. However, there would have to be an equilibrium point inside this periodic orbit, contradicting Lemma 4.2. Hence the return map is well defined. The intersection $S \cap (T_+ \cup T_0)$ consists of the line segment $\{(\theta, \phi) \in T \,|\, \theta = \frac{\pi}{2}, f_1(\phi) \ge 0\}$. The return map maps this line segment into itself, which implies the existence of a periodic orbit in T_+. Similarly, there exists a periodic orbit in T_-. The return map is contracting in T_+ and expanding in T_-, since the divergence of the vector field is negative in T_+ and positive in T_-. This proves the uniqueness of the two period orbits and shows that all other orbits on the torus T have Λ_- as α-limit set and Λ_+ as ω-limit set.

We remark that the same conclusion can be reached by combining the Poincaré–Bendixson theorem for flows on the torus and the Morse theory for Morse–Smale flows.

Finally, note that, although the preceding proof needs C_0 to have a particular value (see Lemma 4.4 and (4.27)), the statement in Lemma 4.3 is true for any choice of $C_0 > 0$ (see Remark 4.1).

Another observation is that the linear case $s = 1$ may be treated by direct computation, i.e., by transforming the general solution of the then linear equation (4.1) to the X-variables. ∎

We still have to show that an orbit can only pass through T_0 from T_- to T_+.

LEMMA 4.4. *Let $s > 1$. There exists a $C_0 > 0$ such that orbits on T can only pass through T_0 in the direction from T_- to T_+.*

Proof. We deduce from (4.20) and (4.21) that

$$\dot{Q}\Big|_{Q=0} = P\Big(|\xi_1|^{2s} + \xi_2^2 + \xi_3^2 + \xi_4^2 + (s|\xi_1|^{s-1} - 1)(\xi_2 + \xi_4)\xi_4\Big). \tag{4.25}$$

Notice that for $s = 1$, P is positive on T (see (4.9)), thus $\dot{Q}\big|_{Q=0} > 0$ on T. For $s > 1$ we define R as the second factor in the right-hand side of (4.25) and simplify it using the expression (4.19) for T:

$$\begin{aligned} R &\overset{\text{def}}{=} |\xi_1|^{2s} + \xi_2^2 + \xi_3^2 + \xi_4^2 + (s|\xi_1|^{s-1} - 1)(\xi_2 + \xi_4)\xi_4 \\ &= 2C_0 + |\xi_1|^{2s} - \frac{2}{s+1}|\xi_1|^{s+1} - (1 - s|\xi_1|^{s-1})(\xi_2 + \xi_4)\xi_4. \end{aligned} \tag{4.26}$$

From (4.19) we infer that

$$(\xi_2+\xi_4)\xi_4 \le \left(\left(C_0 - \frac{2}{s+1}|\xi_1|^{s+1}\right)^{\frac{1}{2}} + C_0^{\frac{1}{2}}\right)C_0^{\frac{1}{2}} = C_0\left(1+\left(1 - \frac{2}{C_0(s+1)}|\xi_1|^{s+1}\right)^{\frac{1}{2}}\right).$$

Fix

$$C_0 = \frac{2}{s+1}\left(\frac{1}{s}\right)^{\frac{s+1}{s-1}}, \tag{4.27}$$

and set

$$|\xi_1| = x\left(\frac{1}{s}\right)^{\frac{1}{s-1}}, \text{ where } 0 \le x \le 1.$$

It follows that

$$\begin{aligned} R &\ge \frac{2}{s+1}\left(\frac{1}{s}\right)^{\frac{s+1}{s-1}}\Big(2 + \frac{s+1}{2s}x^{2s} - x^{s+1} - (1 - x^{s-1})(1 + (1 - x^{s+1})^{\frac{1}{2}})\Big) \\ &= \frac{2}{s+1}\left(\frac{1}{s}\right)^{\frac{s+1}{s-1}}\Big(1 + \frac{s+1}{2s}x^{2s} - x^{s+1} + x^{s-1} - (1 - x^{s-1})(1 - x^{s+1})^{\frac{1}{2}}\Big) \\ &= \frac{2}{s+1}\left(\frac{1}{s}\right)^{\frac{s+1}{s-1}}\Big((1 - x^{s+1})^{\frac{1}{2}}\big((1 - x^{s+1})^{\frac{1}{2}} - (1 - x^{s-1})^{\frac{1}{2}}\big) + x^{s-1} + \frac{s+1}{2}x^{2s}\Big). \end{aligned}$$

Since $0 \le x \le 1$ we see that $R > 0$ unless $x = 0$. Looking at (4.26) we infer that R can only be zero if $\xi_1 = \xi_3 = 0$ and $\xi_2 = \xi_4 = \pm\sqrt{C_0}$, or, in terms of the X_i, if

$X_1 = X_3 = X_4 = 0$. By continuity arguments it follows that also in these two points the orbits go from T_- to T_+. Thus, with the particular choice of C_0 given by (4.27) we have indeed that T_+ is positively invariant and T_- is negatively invariant. □

Having proven the existence of precisely two periodic orbits, Λ_- and Λ_+, on the torus T, we analyze some of their properties.

LEMMA 4.5. *The three nontrivial Floquet multipliers of Λ_+ are contained in the interval $(0,1)$, and the three nontrivial Floquet multipliers of Λ_- are contained in the interval $(1,\infty)$.*

Proof. Restricted to T the nontrivial Floquet multiplier of Λ_+ equals (see, e.g., [26, p. 198])

$$\exp\left(\oint_{\Lambda_+} \nabla \cdot w\right) = \exp\left(\oint_{\Lambda_+} -(3s+5)Q\right).$$

Since Q is uniformly positive on Λ_+, this Floquet multiplier is in $(0,1)$. Close to the periodic orbit Λ_+ we choose ϕ, θ, X_5, and H_0 as coordinates on the Poincaré cylinder Π, where H_0 given by (4.16). Since $H = H_0 X_5^{-4\lambda(s+1)}$ is a conserved quantity on Π, it follows from (4.10) that

$$\dot{H}_0 = -4(s+1)Q\,H_0.$$

Together with (4.10) this implies that the other Floquet multipliers are

$$\exp\left(\oint_{\Lambda_+} -4(s+1)Q\right) \quad \text{and} \quad \exp\left(\oint_{\Lambda_+} -\frac{1}{\lambda}Q\right),$$

which are in $(0,1)$ as before. Thus Λ_+ is exponentially stable. The statement for Λ_- is obtained by time reversal. □

LEMMA 4.6. *Every orbit (other than $\Lambda_\pm$) on the sphere Σ has Λ_- as α-limit set and Λ_+ as ω-limit set.*

Proof. We have already dealt with the flow on the torus T in Lemma 4.3. Orbits of the flow on the complement $\Sigma\backslash T$ of the torus T on the sphere Σ correspond to solutions with nonzero Hamiltonian H. Since X_5 does not appear in (4.10), the motion on Σ is independent of X_5. Let $X_5 \neq 0$, then the dynamics of X_5 are governed by (4.11), and the motion takes place in the part of the Poincaré cylinder Π that corresponds to the finite part of phase space in the x-variables. In other words, orbits of the flow on the set $\Sigma \setminus T$ correspond to solutions of (4.2) with nonzero Hamiltonian.

Since $H = H_0 X_5^{-4\lambda(s+1)}$ and H_0 is bounded on Σ (because Σ is compact), it follows that for such orbits X_5 remains bounded, i.e., in x-variables the solution stays away from the origin. Thus orbits in $\Sigma \setminus T$ are bounded in the X-variables and hence have nonempty invariant α- and ω-limit sets. We have to show that these limit sets can only be the two periodic orbits Λ_- and Λ_+ provided by Lemma 4.3. To this end it suffices to show that all solutions of (4.1) with $H \neq 0$ are unbounded in forward and backward time, i.e., $X_5 \to 0$ along a sequence of points in forward and backward time.

Postponing the proof of the unboundedness of solutions with $H \neq 0$, we first show how unboundedness in backward and forward time implies that Λ_- and Λ_+ are the α- and ω-limit sets. By (4.17) $X_5 \to 0$ implies that also $H_0 \to 0$. An unbounded orbit thus comes arbitrary close to the torus T. We choose an open tubular neighborhood Λ_-^ε of Λ_- in T, such that $Q < 0$ in Λ_-^ε. Clearly all orbits starting in $T \setminus \Lambda_-^\varepsilon$ tend to

Λ_+ in forward time. Note that $T_0 \cup T_+ \subset T \setminus \Lambda_-^\varepsilon$. By compactness of T and since Λ_+ is asymptotically stable (see Lemma 4.5), there exists an open neighborhood T^ε of $T \setminus \Lambda_-^\varepsilon$ in Π such that all orbits starting in T^ε tend to Λ_+ in forward time. Since an orbit which comes close to $X_5 = 0$ (and thus close to T) can only do so with nonnegative Q, it enters T^ε and hence tends to Λ_+. The statement for Λ_- follows by time reversal.

We still have to prove that any solution of (4.1) with nonzero Hamiltonian is unbounded in forward and backward time. We recall that solutions with $H \neq 0$ stay away from the origin. If an orbit would be bounded in backward or forward time, then its (nonempty) α- or ω-limit set would consist of bounded orbits, i.e., orbits which are bounded for all time. However, this is not possible, because it has been proved in [19] that (4.1) admits no bounded solutions except $u \equiv 0$. Here we present a different proof of the fact that (4.1) admits no bounded solutions except $u \equiv 0$, because we need to extend this result to more general situations (see Remark 4.7).

Assume, by contradiction, that $u \not\equiv 0$ is a bounded solution of (4.1). First observe that if u tends to a limit as $t \to \pm\infty$, then this limit can only be 0. It follows that u attains at least one positive maximum or one negative minimum. Switching from u to $-u$ if necessary, we may suppose that u attains a positive maximum at t_0:

$$u(t_0) > 0, \quad u'(t_0) = 0, \quad u''(t_0) \leq 0.$$

Changing from t to $-t$ if necessary, we may assume that $u'''(t_0) \leq 0$ and apply an oscillation argument from [29] which we repeat here for the sake of completeness. There exists a $t^* > t_0$ such that $u'''(t) < 0$ for $t_0 < t < t^*$ and $u'''(t^*) = 0$. Using the fact that

$$H = -u'u''' + \frac{1}{2}u''^2 - \frac{1}{s+1}|u|^{s+1}$$

is constant, it follows that $u(t^*) < -u(t_0)$ and that the next minimum must occur at $t_1 > t^*$ with $u(t_1) < u(t^*) < -u(t_0)$ and both $u''(t_1)$ and $u'''(t_1)$ positive. Repeating this argument we obtain a sequence $t_1 < t_2 < t_3 < \ldots$, in which $u(t)$ has nondegenerate extrema with $|u(t_1)| < |u(t_2)| < |u(t_3)| < \ldots$. By assumption these extrema remain bounded, say $\lim_{i\to\infty} |u(t_i)| = a \in \mathbb{R}^+$, and the derivatives are bounded as well. A compactness argument now shows that there must be a solution $\tilde{u}$ of (4.2) in the ω-limit set of u with

$$\tilde{u}(t_0) = a, \quad \tilde{u}'(t_0) = 0, \quad \tilde{u}''(t_0) < 0, \quad \text{and} \quad \tilde{u}'''(t_0) \leq 0 \qquad \text{at some } t_0 \in \mathbb{R},$$

and such that $|\tilde{u}(t)| \leq a$ for all $t \in \mathbb{R}$. However, when we apply the above argument to $\tilde{u}$ we obtain that $\tilde{u} < -a$ at the first minimum to the right of t_0, a contradiction. This completes the proof of Lemma 4.6. ∎

Remark 4.7. The oscillation argument above will be applied several times in this paper to differential equations that differ from the present one. It holds that any solution of (1.3) with $c = 0$ and $\beta \geq 0$ which does not have its range contained in

$$\{u \in \mathbb{R} \,|\, F(u) \geq F(-a)\}$$

oscillates toward infinity either in forward or in backward time in exactly the way described above (the additional second order term does not cause any difficulties). For more details we refer to [29].

4.3. The reduced system in the linear limit. We have shown in the previous section that for any $s \geq 1$ the flow of (4.1) is basically governed by two periodic orbits at infinity. For the linear equation ($s = 1$) this was already observed (in a broader setting) by Palis [21]. The analysis thus shows that the behavior for all $s > 1$ is largely analogous to the linear equation. In this section we make some observations about the limit $s \downarrow 1$.

Let us rewrite this system as

$$\dot{X} = V(X; s), \quad X = (X_1, X_2, X_3, X_4). \tag{4.28}$$

Then the vector field $V(\cdot, s)$ is continuously differentiable for every $s \geq 1$ and the first order partial derivatives are bounded on compact sets, uniformly in $s \geq 1$. We do not have that $V(\cdot, s) \to V(\cdot, 1)$ in C^1_{loc} because of the term X_1^s appearing in V, but we do have that $V(\cdot, s) \to V(\cdot, 1)$ uniformly on compact sets. Therefore the orbits of (4.28) with $s > 1$, which are bounded uniformly in s in view of (4.12), converge to orbits of (4.28) with $s = 1$ as $s \to 1$. More precisely, the solution map

$$(\tau, \xi, s) \to X(\tau; \xi, s),$$

where $X(\tau; \xi, s)$ is the solution $X(\tau)$ of (4.28) with $X(0) = \xi$, is continuous on $\mathbb{R} \times \mathbb{R}^4 \times [1, \infty)$. In particular, this implies that the two periodic orbits Λ_- and Λ_+ depend continuously on s for $s \in [1, \infty)$.

In the limit case $s = 1$ the two periodic orbits on

$$T = \{(\xi_1, \xi_2, \xi_3, \xi_4) \,|\, \xi_1^2 + \xi_2^2 = \xi_3^2 + \xi_4^2 = C_0\}$$

are given by

$$\xi_1\xi_3 - \xi_2\xi_4 = 0, \tag{4.29}$$

or in terms of (4.22), by $\phi + \theta = \pm\frac{\pi}{2}$. This can be seen from a second conservation law that exists in the linear case: multiplying $u'''' + u = 0$ by u''' we have that $\frac{1}{2}u'''^2 + uu'' - \frac{1}{2}u'^2$ is constant. In particular, after transforming to the X-variables,

$$\frac{1}{2}X_4^2 + X_1X_3 - \frac{1}{2}X_2^2 = 0$$

is invariant, whence (4.29), which defines two circles on the torus T.

4.4. Small solutions. We observed in section 4.1 that the role of $X_5 = 0$ and $X_5 = \infty$ may be reversed. This is a direct consequence of the scaling invariance of (4.1). Thus we may also use (4.4) for the analysis of small solutions to (4.1). The situation is depicted schematically in Figure 4.2. We simply apply (4.4) with a negative λ so that $X_5 \to 0$ corresponds to $u \to 0$. This changes only the sign in (4.10) for X_5 and means that the orbit Λ_+ now lies in the part of $X_5 = 0$ which repels solutions with $X_5 > 0$. Hence the stable manifold of Λ_+ is contained in $\Pi \cap \{X_5 = 0\}$. The unstable manifold of Λ_+ is given by the direct product $\Lambda_+ \times \{X_5 \,|\, X_5 > 0\}$ and has dimension 2. In the original variables it is the unstable manifold of $u = 0$ if $s = 1$ and the center-unstable manifold if $s > 1$. Likewise, the stable manifold of Λ_- is $\Lambda_- \times \{X_5 \,|\, X_5 > 0\}$, i.e., the direct product of Λ_- and the positive X_5-axis. As we have seen in section 4.3, the limit $s \to 1$ is well behaved in the X-variables.

We will use this analysis of the behavior near the equilibrium point $u = 0$ in section 5 to perform a continuous deformation of the stable manifold for $s = 1$ to the center-stable manifold for $s > 1$. We remark that, based on the similarity of the linear and nonlinear problem, the equilibrium point $u = 0$ of (4.1) for $s > 1$ can be considered as the nonlinear equivalent of a saddle-focus.

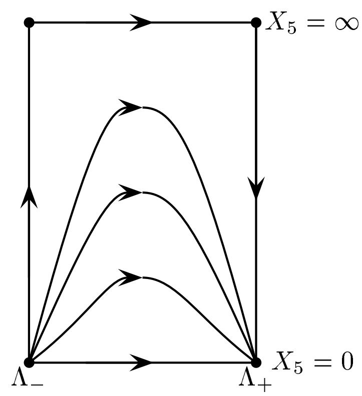

FIG. 4.2. *A schematic view of the flow on the Poincaré cylinder* Π *for the equation* $u'''' + u^s = 0$. *The role of* $X_5 = 0$ *and* $X_5 = \infty$ *is reversed when* λ *is negative.*

4.5. The full system. Applying the Poincaré transformation (4.4) with exponents (4.6) to the differential equation (1.3), or, more generally, to

$$x_1' = x_2; \quad x_2' = x_3; \quad x_3' = x_4; \quad x_4' = \Phi(x_1, x_2, x_3, x_4),$$

we arrive at

$$\dot{X}_1 = X_2 P - 4X_1 Q\,; \tag{4.30a}$$

$$\dot{X}_2 = X_3 P - (3+s)X_2 Q\,; \tag{4.30b}$$

$$\dot{X}_3 = X_4 P - (2+2s)X_3 Q\,; \tag{4.30c}$$

$$\dot{X}_4 = \Psi P - (1+3s)X_4 Q\,; \tag{4.30d}$$

$$\dot{X}_5 = -\frac{1}{\lambda} X_5 Q\,, \tag{4.30e}$$

where

$$Q = X_1^s X_2 + X_4 \Psi + X_3(X_2 + X_4) \tag{4.31}$$

and

$$\Psi = X_5^{4\lambda s} \Phi\left(\frac{X_1}{X_5^{4\lambda}}, \frac{X_2}{X_5^{(3+s)\lambda}}, \frac{X_3}{X_5^{(2+2s)\lambda}}, \frac{X_4}{X_5^{(1+3s)\lambda}} \right). \tag{4.32}$$

In the case of (1.3) we have

$$\Phi(x_1, x_2, x_3) = \alpha x_3 - c x_2 + f(x_1),$$

where $f(x_1) = -x_1^3 + g(x_1)$ with $g(x_1)$ compactly supported. With $s = 3$ and $\lambda = \frac{1}{2}$ we thus obtain

$$\Psi = -X_1^3 + \alpha X_3 X_5^2 - c X_2 X_5^3 + g\left(\frac{X_1}{X_5^2}\right) X_5^6. \tag{4.33}$$

The last term in (4.33) is C^2 and has its derivatives up to second order vanishing in $X_5 = 0$. The extra terms are thus at least quadratic in X_5 for small X_5. Therefore the local analysis near $X_5 = 0$ and in particular the Floquet multipliers of $\Lambda_\pm$ in the previous section are completely unaffected. The flow on the sphere Σ (at infinity) is identical to the flow for the reduced equation (4.2). Only the flow on $\Pi \setminus \Sigma$ is different. Note that in this analysis it is essential that the exponent s is larger than 1. We have the following theorem (compare Lemmas 4.3, 4.5 and 4.6).

THEOREM 4.8. *Let f satisfy hypothesis* (3.1) *and let $\alpha, c \in \mathbb{R}$.*

(a) *The stable periodic orbit Λ_+ of* (4.11) *is an asymptotically stable periodic orbit of* (4.30) *with nontrivial Floquet multipliers in* $(0, 1)$. *Every solution of* (1.3) *which is unbounded in forward time corresponds to a solution of* (4.30) *having Λ_+ as ω-limit set. A similar statement holds for solutions unbounded in backward time and Λ_-.*

(b) *Unbounded solutions of* (1.3) *blow up oscillatorily in finite time.*

(c) *If $c \neq 0$, the energy $\mathcal{E}$ also blows up.*

Proof. By Lemma 4.6 all solutions of (4.30) which lie in the invariant set $\Pi \cap \{X_5 = 0\} \setminus \Lambda_- \subset \Pi$ tend to Λ_+ in forward time. Reminiscent of the proof of Lemma 4.6 we choose a small negatively invariant open tubular neighborhood Λ_-^ε of Λ_- in Π. By compactness of $\Pi \cap \{X_5 = 0\}$ there exists an open neighborhood Σ^ε of $\Pi \cap \{X_5 = 0\} \setminus \Lambda_-^\varepsilon$ in Π such that all orbits with starting point in Σ^ε tend to Λ_+ in forward time. Clearly every unbounded solution of (1.3) enters Σ^ε and thus tends to Λ_+.

For part (b) we observe that the exponent b in (4.8) is smaller than 1 so that in the old time variable X_5 can only go to zero in finite time. Finally we have that the energy $\mathcal{E}$ can remain bounded only if its derivative is integrable. For $c \neq 0$ this implies that u' is square integrable (see (1.5)) and thus u itself is (locally) bounded, which prohibits finite time blow-up, a contradiction. $\square$

Remark 4.9. Theorem 4.8 establishes that large solutions of (1.3) are really described by oscillating solutions of $u'''' + u^3 = 0$. Thus large solutions do not "see" the other terms in (1.3) as they oscillate away to infinity. This is not only true for perturbations of the form $-u^3 + g(u)$ with g compactly supported and smooth, but also for global lower order perturbations. For such lower order perturbations Theorem 4.8 applies as well.

5. The winding number for small speeds. In this section we prove part (a) of Proposition 3.3. Before we can prove this theorem we first need a description of the global behavior of $W^s(1)$ for $c = 0$. In the following lemma we show that for $\alpha > \frac{1}{\sqrt{\sigma(f)}}$ all orbits in the stable manifold $W^s(1)$ are unbounded, and, after transforming to the X-variables in section 4, they all have Λ_- as α-limit set. Because all the nontrivial Floquet multipliers of Λ_- lie in $(1, \infty)$ (see Theorem 4.8(a)), this remains true for $c > 0$ sufficiently small.

LEMMA 5.1. *Let f satisfy hypothesis* (3.1), *let $\alpha > \frac{1}{\sqrt{\sigma(f)}}$, and let $c = 0$. Then $W^s(1)$ consists of unbounded orbits only, all of which connect Λ_- to $u = 1$.*

Proof. The proof is a combination of arguments also used in [25]. Any bounded solution must have its range in the set

$$V = \{u \in \mathbb{R} \,|\, F(u) \geq F(-a)\}$$

because a solution reaching outside this interval oscillates away toward infinity, as mentioned in Remark 4.7. Besides, any bounded solution must have at least one minimum below the line $u = -a$, again basically by the same oscillation argument as in the proof of Lemma 4.5. We now assume, arguing by contradiction, that u is a bounded orbit in $W^s(1)$. We will show that the range of u is not contained in V, so

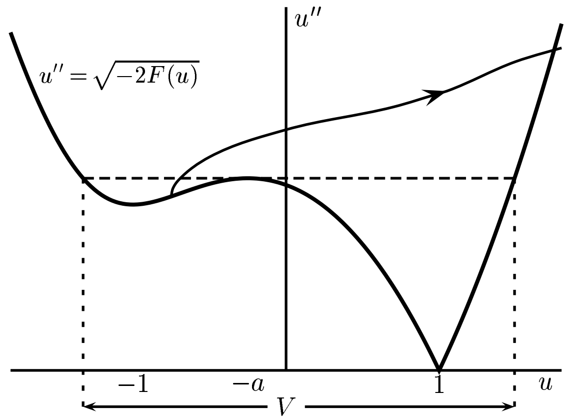

FIG. 5.1. *The (u, u'')-plane with the curve $u'' = \sqrt{-2F(u)}$. We have sketched the orbit of u for $t \geq t_0$, which is discussed in the proof of Lemma* 5.1. *We have also indicated the set V, in which every bounded solution has its range.*

that u is in fact unbounded. It then follows from Theorem 4.8 that u tends to Λ_- as $t \to -\infty$.

Thus, suppose that u is a bounded solution in $W^s(1)$. Changing from t to $-t$ if necessary we have that in such a minimum (using the fact that $\mathcal{E}(u, u', u'', u''') = 0$)

$$u(t_0) \leq -a, \quad u'(t_0) = 0, \quad u''(t_0) = \sqrt{-2F(u(t_0))} > 0, \quad u'''(t_0) \geq 0. \tag{5.1}$$

We will show that $u(t)$ increases to a value outside V for $t > t_0$, which immediately leads to a contradiction.

Define an auxiliary function

$$G(t) \stackrel{\text{def}}{=} u''(t) - \sqrt{-2F(u(t))}.$$

The following line of reasoning is depicted in Figure 5.1. First, $G(t_0) = 0$ and we show that $G(t) > 0$ in a right neighborhood of t_0. It is seen from the condition on α and the observation that $f(u) > 0$ on $(-\infty, -1) \cup (-a, 1)$ that

$$f(u) > -\sqrt{-\frac{\alpha^2}{2}F(u)} \qquad \text{for } u < 1. \tag{5.2}$$

If $u'''(t_0) > 0$, then clearly $G'(t_0) > 0$, whereas when $u'''(t_0) = 0$, then $G'(t_0) = 0$, and (since $u'(t_0) = 0$)

$$G''(t_0) = u''''(t_0) + \frac{f(u(t_0))}{\sqrt{-2F(u(t_0))}}u''(t_0) = \alpha\sqrt{-2F(u(t_0))} + 2f(u(t_0)) > 0$$

by the differential equation, and (5.1) and (5.2). Thus $G(t) > 0$ in a right neighborhood of t_0.

Second, we show that $G(t) > 0$ as long as $u(t) < 1$. We define $t_1 > t_0$ as the first maximum of $u(t)$ and $t_2 > t_0$ as the first point where $G(t_2) = 0$ (a priori, both t_1 and t_2 may be ∞). Then $t_2 < t_1$ since $u''(t) > 0$ as long as $G(t) > 0$. It now follows from the expression (1.4) for the energy and by (5.2) that

$$G'(t) = u'''(t) + \frac{f(u(t))}{\sqrt{-2F(u(t))}}u'(t)$$

$$= \frac{\frac{1}{2}u''^2(t) + F(u(t))}{u'(t)} + \left(\frac{\alpha}{2} + \frac{f(u(t))}{\sqrt{-2F(u(t))}}\right) u'(t)$$
$$> 0$$

as long as $G(t) > 0$ and $u(t) < 1$. Since $G(t) > 0$ in a right neighborhood of t_0 this implies that $G(t) > 0$ and $G'(t) > 0$ as long as $u(t) < 1$, and thus $u(t_2) \geq 1$.

Finally, we define $t_3 > t_0$ as the first point where $u(t) = -a$. It is easily seen that $t_3 < t_2$. By the energy expression we have that $u'''(t) > 0$ as long as $G(t) > 0$, thus $u''(t_2) > u''(t_3) > \sqrt{-2F(-a)}$. Combining the inequalities $u(t_2) \geq 1$ and $F(u(t_2)) = -\frac{1}{2}u''^2(t_2) < F(-a)$, we infer that $u(t_2)$ lies outside V, so that u is unbounded. By Theorem 4.8 all these unbounded orbits converge to Λ_-. □

Remark 5.2. Because all the nontrivial Floquet multipliers of Λ_- lie in $(1, \infty)$ (see Theorem 4.8(a)), Lemma 5.1 remains true for $c > 0$ sufficiently small.

The following theorem is equivalent to Proposition 3.3(a). We recall that K is defined in (3.3), and that its boundary δK is a level set of the energy.

THEOREM 5.3. *Let f satisfy hypothesis* (3.1) *and let $\alpha > \frac{1}{\sqrt{\sigma(f)}}$. For $F(-a) < E_0 < F(-1)$ let K be defined by* (3.3) *and let $W^s(1)$ be the stable manifold of the equilibrium $u = 1$. Then, provided $c > 0$ is sufficiently small, $W^s(1) \cap \delta K$ is a topological circle. Its projection Γ on the (u, u''')-plane winds exactly once around a disk containing both closed curves defined by $E_0 - F(u) + \frac{1}{2\alpha}u'''^2 = 0$ (see also Figure* 3.2*), i.e., $n(\Gamma, -1) = n(\Gamma, 1) = 1$.*

Proof. Our strategy is to deform $f(u)$ in several steps to the pure cubic $-u^3$ and let α go to zero. We have to do this in such a way that for each intermediate f the conclusion of Lemma 5.1 remains valid. All orbits in the stable manifold $W^s(1)$ thus tend to Λ_- in backward time, and this remains true during the entire deformation process. At the end of the deformation process we arrive at the reduced equation $u'''' + u^3 = 0$. We then use the analysis performed in section 4 to find a precise description of the orbits in $W^s(1)$. Finally, we obtain the results of Theorem 5.3 for the original equation (1.3) via continuation arguments.

Recall that $f(u) = -u^3 + g(u)$ with g having compact support, say, $g(u) = 0$ for all $|u| \geq C_0$. Taking C_0 sufficiently large, define the cut-off function $\phi \in C_0^\infty$ with $0 \leq \phi \leq 1$, $\phi(y) = 1$ for $|y| \leq C_0$, and $\phi(y) = 0$ for $|y| > C_0 + 1$.

Step 1. First deform $f(u)$ to a function which changes sign at $u = 1$ only. Let

$$f_\lambda(u) = f(u) - \lambda(u-1)\phi(u).$$

For λ large enough, say, $\lambda > \lambda_0$, the function $f_\lambda(u)$ has a zero at $u = 1$ only.

LEMMA 5.4. *Let $\alpha > \frac{1}{\sqrt{\sigma(f)}}$ and replace $f(u)$ by $f_\lambda(u)$. Then for all $\lambda \in [0, \lambda_0]$ the stable manifold $W^s(1)$ consists of unbounded orbits only, all of which connect Λ_- to $u = 1$.*

Proof. Let $\lambda_1 = \inf\{\lambda \mid f_\lambda(u) > 0 \text{ for all } u < 1\}$. For any $\lambda < \lambda_1$ the argument is exactly the same as in the proof of Lemma 5.1, where we use the following generalized definition of σ:

$$\sigma(f_\lambda) = \min\left\{\frac{-F(u)}{2f(u)^2} \;\middle|\; u < 1 \text{ and } f(u) < 0\right\}.$$

Note that $\sigma(f_\lambda) \leq \sigma(f_0)$ for $0 < \lambda < \lambda_1$ since $f_\lambda(u)$ and $-F_\lambda(u)$ are increasing in λ for all $u < 1$. For $\lambda \geq \lambda_1$ the result also holds, but by a different and less restrictive oscillation argument, which applies to any $f(u)$ with a single zero at which it goes from positive to negative, and all $\alpha \geq 0$. We already used this in the proof of Lemma 4.6; the argument showing that every solution $u \not\equiv 1$ oscillates toward infinity is almost identical (for $\alpha \geq 0$ the second order term does not cause any difficulties). This completes the proof of the lemma. ◻

Continuing with the proof of Theorem 5.3, we change f to $f^1 \stackrel{\text{def}}{=} f_{\lambda_0}$ by letting λ go from 0 to λ_0. This leaves the local structure near $X_5 = 0$, and in particular near Λ_-, unaffected (see section 4.5).

Step 2. We change $f^1(u) = -u^3 + g^1(u)$ with $g^1(u) = g(u) - \lambda_0(u-1)\phi$ to $f^2(u) \stackrel{\text{def}}{=} -u^3(1-\phi) - (u-1)\phi$. Using the deformation functions

$$f_\lambda(u) = -u^3(1-\phi(u)) + (1-\lambda)(-u^3\phi(u) + g^1(u)) - \lambda(u-1)\phi(u),$$

we let λ go from 0 to 1, thus continuously deforming f^1 into f^2. All orbits in $W^s(1)$ are still unbounded and tend to Λ_- as $t \to -\infty$ during this deformation, since $f_\lambda(u)$ has a single zero at which it goes from positive to negative (see the proof of Lemma 5.4).

Step 3. It is now easy to shift the zero to the origin. Define

$$f_\lambda(u) = -u^3(1-\phi(u)) - (u-(1-\lambda))\phi(u).$$

Letting λ change from 0 to 1 deforms f^2 into $f^3 \stackrel{\text{def}}{=} -u^3(1-\phi) - u\phi$. Since we have shifted the origin we now have $W^s(0)$ instead of $W^s(1)$. All orbits in $W^s(0)$ are still unbounded and tend to Λ_- as $t \to -\infty$.

Step 4. Next we let α go to zero. The stable manifold $W^s(0)$ changes smoothly and the local structure near Λ_- again remains unaffected because α only appears in terms quadratic in X_5. For $\alpha = 0$ we have arrived at the equation

$$u'''' - f^3(u) = 0, \quad \text{with } f^3(u) = -u^3(1-\phi) - u\phi.$$

Step 5. We change f^3 using a family of functions

$$f_s(u) = -u^3(1-\phi) - u^s\phi.$$

Letting s increase from $s = 1$ to $s = 3$ we obtain a function $f^4(u) \stackrel{\text{def}}{=} u^3$. We note (see section 4.4) that for $s > 1$ the manifold W is the center-stable manifold of 0. Here we use section 4.3 to conclude that in this process W changes continuously, with the orbits in manifold $W = W^{cs}(0)$ still tending to Λ_- in backward time.

By sections 4.1 and 4.4 we have that, after going through Steps 1–5, W is the product of Λ_- and the X_5-axis. In view of the nontrivial Floquet multipliers of Λ_- being in $(1,\infty)$, it holds that for any small $\varepsilon > 0$ there exists a negatively invariant tubular neighborhood Λ_-^ε of Λ_- in Π with

$$\Lambda_-^\varepsilon \subset \{X = (X_1, X_2, X_3, X_4, X_5) \in \Pi \,|\, d(X, \Lambda_-) < \varepsilon\}.$$

We can choose this neighborhood such that

$$\overline{\Lambda_-^\varepsilon} \cap \{X_5 = \varepsilon\} \;=\; \{(X_1, X_2, X_3, X_4) \in \Lambda_-,\, X_5 = \varepsilon\}. \tag{5.3}$$

Besides, we can choose Λ_ε such that the flow for our final equation $u'''' + u^3 = 0$ is transversal to $\delta\Lambda_-^\varepsilon$. Moreover, for $\varepsilon > 0$ sufficiently small, we can choose Λ_ε such the flow is transversal to $\delta\Lambda_-^\varepsilon$ for every intermediate $f(u)$ and α in the deformation process of Steps 1–5 above, hence also for the original equation (1.3) with $c = 0$.

For any given $r > 0$ we can choose $\varepsilon > 0$ so small that the projection Γ_ε of $W \cap \delta\Lambda_-^\varepsilon$ on the (x_1, x_4)-plane (or, equivalently, on the (u, u''')-plane) is a curve with minimal distance to the origin at least r. To see this, we observe that the solution of (4.1) represented by Λ_- cannot have a point where $u = u''' = 0$, for in such a point also $u'' = 0$ in view of the energy $\mathcal{E}$ being zero. This would contradict the fact that $Q < 0$ on Λ_-. Thus in the X-variables Λ_- is uniformly bounded away from $(X_1, X_4) = (0, 0)$, so that for any $r > 0$ we can find an $\varepsilon > 0$ such that the projection of Λ_-^ε on the (u, u''')-plane has a distance larger than r from the origin. Therefore, the winding numbers around $u = \pm 1$ of the projection Γ_ε of $W \cap \delta\Lambda_-^\varepsilon$ on the (u, u''')-plane are well defined for ε sufficiently small.

It follows from (5.3) that for our final equation $u'''' + u^3 = 0$ we have

$$W \cap \delta\Lambda_-^\varepsilon = \{(X_1, X_2, X_3, X_4, X_5) \,|\, (X_1, X_2, X_3, X_4) \in \Lambda_-,\ X_5 = \varepsilon\},$$

so that, choosing r large, $n(\Gamma_\varepsilon, -1) = n(\Gamma_\varepsilon, 1) = 1$. By continuity the winding numbers of Γ_ε do not change if we reverse Steps 1–5, and again by continuity arguments and Remark 5.2 this remains true for $c > 0$ sufficiently small.

Finally, for our original equation (1.3) we know that, tracing back orbits in $W^s(1)$ until they hit $\delta\Lambda_-^\varepsilon$, their energy $\mathcal{E}$ remains close to 0, provided we keep $c > 0$ sufficiently small. Thus $W^s(1) \cap \delta K$ is contained in Λ_-^ε for small $c > 0$. Following $W^s(1) \cap \delta\Lambda_-^\varepsilon$ backwards along the flow to $W^s(1) \cap \delta K$ (which is a transversal intersection for $c > 0$), we see that the winding numbers $n(\Gamma, \pm 1)$ of the projection of $W^s(1) \cap \delta K$ are also 1. This completes the proof of Theorem 5.3. ∎

6. The winding number for large speeds. In this section we proof part (b) of Proposition 3.3.

THEOREM 6.1. *Let f satisfy hypothesis* (3.1) *and let $\alpha \in \mathbb{R}$. For $c > 0$ sufficiently large the intersection of the stable manifold $W^s(1)$ of $u = 1$ and the boundary δK of K is a smooth simple closed curve which projects on a closed curve Γ in the (u, z)-plane with $n(\Gamma, -1) = 0$ and $n(\Gamma, 1) = 1$.*

Proof. We first prove the theorem for a deformation of $f(u)$. We choose the nonlinearity $\tilde{f}(u)$ to satisfy

$$\tilde{f}(u) = f'(1)(u-1) \quad \text{in a neighborhood } B_\varepsilon(1) \text{ of } u = 1.$$

For this deformed nonlinearity $\tilde{f}$ we compute the energy $\tilde{\mathcal{E}}$ on a closed curve in $\tilde{W} = W^s(1)$ winding once around $u = 1$ with u-values contained in $B_\varepsilon(1)$. The equation is now linear near $u = 1$, and the characteristic equation

$$-\mu^4 + \alpha\mu^2 + f'(1) = c\mu$$

has two eigenvalues $-\mu_1$ and $-\mu_2$ with negative real part (recall that $f'(1) < 0$). For $c > 0$ large enough μ_1 and μ_2 are real, and asymptotically

$$\mu_1 \sim c^{\frac{1}{3}} \quad \text{and} \quad \mu_2 \sim \frac{-f'(1)}{c} \qquad \text{as } c \to \infty. \tag{6.1}$$

Since the equation is linear, $\tilde{W}$ is given by (for c large enough)

$$\tilde{W} = \{(u, v, w, z) \,|\, u = u(t) = 1 + A_1 e^{-\mu_1 t} + A_2 e^{-\mu_2 t},\ v = u'(t),\ w = u''(t),\ z = u'''(t)\}. \tag{6.2}$$

We may choose a curve $S_1 \subset \tilde{W}$ around $u = 1$ parametrized by $\phi \in [0, 2\pi)$ by taking $t = 0$ and $A_1 = r\cos\phi$, $A_2 = r\sin\phi$ in (6.2) for some fixed $r > 0$. The projection of S_1 on the (u, u''')-plane is given by

$$\{(u,z)\,|\,u = 1 + r(\cos\phi + \sin\phi),\ z = -r(\mu_1^3\cos\phi + \mu_2^3\sin\phi),\ 0 \le \phi < 2\pi\}.$$

The energy on S_1 is given by

$$\begin{aligned} -\mathcal{E} &= \int_0^\infty cu'(t)^2 dt = c\int_0^\infty (A_1\mu_1 e^{-\mu_1 t} + A_2\mu_2 e^{-\mu_2 t})^2 dt \\ &= c\left(\frac{A_1^2\mu_1}{2} + \frac{2A_1A_2\mu_1\mu_2}{\mu_1+\mu_2} + \frac{A_2^2\mu_2}{2}\right) = c\mu_2\left(\frac{A_1^2\mu_1}{2\mu_2} + \frac{2A_1A_2\mu_1}{\mu_1+\mu_2} + \frac{A_2^2}{2}\right). \end{aligned} \tag{6.3}$$

Using (6.1) and estimating (6.3) from below we have, for c sufficiently large,

$$\mathcal{E} \le \frac{f'(1)}{4}r^2 < 0 \quad \text{on } S_1.$$

Thus, choosing an energy level $0 > \tilde{E}_0 > \frac{f'(1)}{4}r^2$, we have that S_1 lies in the complement of K. Let $\tilde{S} = \tilde{W} \cap \delta\tilde{K}$. Then $\tilde{S}$ lies inside S_1 and is obtained by tracing solutions in (6.2) of the linear equation forwards in time until they enter $\tilde{K}$. It follows that $\tilde{S}$ winds around $u = 1$ in $\tilde{W}$ exactly once and therefore its projection $\tilde{\Gamma}$ on the (u,z)-plane winds once around $(u,z) = (1,0)$.

The calculations above involve only u-values between $1 - r\sqrt{2}$ and $1 + r\sqrt{2}$ so we may change the definition of $\tilde{f}(u)$ outside this range. In particular, taking r small, we may choose $\tilde{f}(u)$ such that $\tilde{F}(u)$ has a minimum $\tilde{F}(-a) < \tilde{E}_0$ and a maximum $\tilde{F}(-1) \in (\tilde{E}_0, \tilde{F}(1))$, with $-1 < -a < 1 - r\sqrt{2}$. Clearly $\tilde{\Gamma}$ does not wind around the point $(u,z) = (-1,0)$.

We continue $\tilde{f}$ to f and $\tilde{E}_0$ to E_0, taking c large enough as to stay within a class of nonlinearities for which there does not exist a connection between $u = -1$ and $u = 1$ (see Lemma 2.1). By continuity we still have that $n(\Gamma, -1) = 0$ and $n(\Gamma, 1) = 1$. ◻

7. Travelling waves connecting an unstable to a stable state. In this section we focus on travelling waves that connect the unstable state $u = -a$ to one of the two stable states $u = \pm 1$. As in the proof of Theorem 1.1 in section 3 we begin by reducing to nonlinearities f which satisfy (3.1).

To obtain the necessary bound for $\alpha > 0$ we fix $c > 0$ and simply follow the argument in the proof of Lemma 2.2 with $F(-1)$ replaced by $F(-a)$ (for connections from $-a$ to $+1$), or by $F(-1) - F(-a)$ (for connections from $-a$ to -1).

By different methods it is also possible to prove a priori bounds in the case that $\alpha \le 0$. Applying a result by Gallay [14] to the present context we obtain the following. Let f satisfy (H_1), i.e., $\lim_{|u|\to\infty} \frac{f(u)}{u} = -\infty$. Then for any $\alpha \in \mathbb{R}$ there exists a constant C_0 such that any travelling wave solution $u(t,x) = U(x+ct)$ of (1.1) satisfies $\|u\|_\infty \le C_0$. The constant C_0 will thus depend only on α and $m \stackrel{\text{def}}{=} \sup\bigl\{|u| : \frac{f(u)}{u} \ge -D_\alpha\bigr\}$, where $D_\alpha > 0$ is a constant which depends on α only.

The idea is to consider $\Phi_y(t) = \int_{-\infty}^{\infty} h_y(x)u^2(t,x)dx$, where $h_y(x) = \frac{1}{1+(x-y)^2}$. Using the differential equation (1.1) one obtains an estimate of the form $\frac{d\Phi_y}{dt} \le A_0 - \Phi_y$ for some constant A_0 independent of y and t, hence $\Phi_y(t) \le A_0 + \Phi_y(0)e^{-t}$. Defining $\Psi(t) = \sup_{y\in\mathbb{R}} \Phi_y(t)$ one derives that for travelling waves, Ψ is independent of t, hence

$\Psi \le A_0$. Combining this with the fact that $\int_{-\infty}^{\infty}(\frac{du}{dx})^2 dx = \frac{F(\pm 1)-F(-a)}{c}$, one then obtains an L^∞-bound on u.

Thus, for every $c > 0$ there exists a constant $C_0 > 0$ such that any solution of (1.3) connecting $-a$ to ± 1 satisfies $\|u\| < C_0$. This a priori estimate implies that we may replace f by $\tilde{f}(u) = \phi(u)f(u) - u^3(1-\phi(u))$, where the cut-off function $\phi \in C_0^\infty$ is such that $0 \le \phi \le 1$, $\phi(y) = 1$ for $|y| \le C_0$, and $\phi(y) = 0$ for $|y| > C_0 + 1$. As in section 3 it holds that u is a travelling wave solution with speed c for nonlinearity $f(u)$ if and only if u is a travelling wave solution with speed c for nonlinearity $\tilde{f}(u)$.

The above argument shows that, looking for travelling waves, we may as well assume that f satisfies (3.1). The next theorem thus proves Theorem 1.2.

THEOREM 7.1. *Let f satisfy hypothesis* (3.1) *and let $\alpha \in \mathbb{R}$. For every $c > 0$ there exists a solution of* (1.3) *connecting $u = -a$ to $u = -1$.*

Proof. For all $c > 0$ we have that the three equilibria are hyperbolic and

$$\dim W^s(\pm 1) = \dim W^u(\pm 1) = 2, \quad \dim W^u(-a) = 3, \quad \dim W^s(-a) = 1.$$

Travelling wave solutions connecting $u = -a$ and $u = -1$ correspond to a nonempty intersection of $W^u(-a)$ and $W^s(-1)$. Recall that

$$\mathcal{E}(u, u', u'', u''') = -u'u''' + \frac{1}{2}u''^2 + \frac{\alpha}{2}u'^2 + F(u), \quad \text{where} \quad F(u) = \int_1^u f(s)ds,$$

satisfies (1.5). We take $F(-1) < E_1 < F(1)$ and consider the set

$$\tilde{K} = \left\{ (u, v, w, z) \,|\, E(u, v, w, z) = -vz + \frac{1}{2}w^2 + \frac{\alpha}{2}v^2 + F(u) \le E_1 \right\}.$$

Now suppose that for some $c > 0$ the theorem is false. Then all orbits in $W^u(-a)$ have to leave $\tilde{K}$ through $\delta\tilde{K}$, because an orbit with bounded energy has no other choice than to converge to an equilibrium (see the proof Lemma 3.2) and $u = -1$, the only equilibrium in $\tilde{K}$ with energy larger than $E(-a)$, is excluded by assumption. Thus we have that the intersection of $W^u(-a)$ and $\delta\tilde{K}$ is homeomorphic to a 2-sphere S^2.

For the moment we consider the case that $\alpha > 0$. Since $\delta\tilde{K}$ is given by

$$\alpha\left(v - \frac{z}{\alpha}\right)^2 + w^2 = 2E_1 - 2F(u) + \frac{z^2}{\alpha}, \tag{7.1}$$

we may deform it smoothly into

$$\{(u, v, w, z) \,|\, u^2 + z^2 = 1 + v^2 + w^2\},$$

which defines a 3-manifold homeomorphic to $\mathbb{R}^2 \times S^1$. As deformations we use

$$(\lambda\alpha + 1 - \lambda)\left(v - \lambda\frac{z}{\alpha}\right)^2 + w^2 = G(u, \lambda) + \left(1 - \lambda + \frac{\lambda}{\alpha}\right)z^2,$$

with λ running from 1 to 0, and $G(u, 1) = 2E_1 - 2F(u)$ and $G(u, 0) = -1 + u^2$. Singularities can only appear in points on these manifolds where $G_u = v = w = z = 0$ and can thus be avoided by the choice of E_1.

It follows that $\delta\tilde{K}$ is homeomorphic to $\mathbb{R}^2 \times S^1$, or, equivalently, to the open solid torus. The intersection $W^u(-a) \cap \delta\tilde{K}$, being homeomorphic to S^2, divides $\delta\tilde{K}$ into two components, one bounded and homeomorphic to an open ball in $\mathbb{R}^3$, the other unbounded. This division is in fact not completely straightforward. One needs to lift (a neighborhood of) $W^u(-a) \cap \delta\tilde{K}$ to the universal covering space $\mathbb{R}^3$ of $\tilde{K}$ and show that the unbounded part of the complement of the countable union of lifts is path-connected. Using the fact that the intersection $W^u(-a) \cap \delta\tilde{K}$ is induced by a flow, one can invoke the generalized Schoenflies theorem (see [5, Theorem 19.11]) to conclude that a lift of $W^u(-a) \cap \delta\tilde{K}$ divides $\mathbb{R}^3$ into an unbounded and a bounded component, which is homeomorphic to an open ball, in $\mathbb{R}^3$. Besides, the bounded components of the countable infinity of lifts can be contracted to points. The unbounded component (the complement of the countable union of bounded components) is thus homeomorphic to $\mathbb{R}^3 \setminus \mathbb{Z}$, hence path-connected.[5]

Now consider the piecewise smooth 3-manifold formed by the disjoint union of the point $(-a, 0, 0, 0)$, $W^u(-a) \cap \tilde{K}$ and the bounded component of $\delta\tilde{K} \setminus (W^u(-a) \cap \delta\tilde{K})$. This 3-manifold is homeomorphic to two closed balls in $\mathbb{R}^3$ sharing an S^2, namely, $W^u(-a) \cap \delta\tilde{K}$, as boundary and is therefore homeomorphic to an S^3. By the Jordan–Brouwer theorem this 3-manifold divides $\mathbb{R}^4$ to two components, one bounded, the other unbounded. We notice that the bounded component is negatively invariant. Clearly both components contain exactly one of the two orbits which together form $W^s(-a)$. Now consider the orbit in $W^s(-a)$ contained in the bounded component (which is negatively invariant). Since its energy is bounded we may, again by the argument in the proof of Lemma 3.2, conclude that, tracing it backwards, it must go to an equilibrium with energy less than the energy of $u = -a$. Since such an equilibrium does not exist, we have arrived at a contradiction.

The cases $\alpha < 0$ and $\alpha = 0$ are similar, the only changes being that we deform $\delta\tilde{K}$, given by (7.1), to $u^2 + v^2 = 1 + z^2 + w^2$ if $\alpha < 0$, and that for $\alpha = 0$ we rewrite $\delta\tilde{K}$ as $-2vz + w^2 = 2E_1 - 2F(u)$, which deforms into $-2vz + w^2 = -1 + u^2$ or $\frac{1}{2}(v+z)^2 + u^2 = \frac{1}{2}(v-z)^2 + w^2 + 1$. This completes the proof of the theorem. □

Remark 7.2. In the proof of Theorem 7.1 above we have used the nondegeneracy of the equilibrium point $u = -a$, while $u = -1$ may degenerate (i.e., $f'(-1) = 0$). The theorem also holds when $u = -a$ is degenerate but $u = -1$ is nondegenerate; in this case the argument in the proof of Theorem 7.3 below can be used. If $F(-1) = F(1)$ one also applies the proof of Theorem 7.3; see Remark 7.4.

Next we prove Theorem 1.3. Let

$$c^* \stackrel{\text{def}}{=} \inf\{\tilde{c} > 0 \,|\, \text{there is no connection from } -1 \text{ to } +1 \text{ for } c > \tilde{c}\}.$$

From Lemma 2.1 we see that c^* is well defined, and $c^* > 0$ for $\alpha > \frac{1}{\sqrt{\sigma(f)}}$ by Theorem 1.1. The argument at the beginning of this section shows that, in order to prove Theorem 1.3, we may restrict to nonlinearities f which satisfy (3.1). If $c_* > 0$, then it follows from Lemma 3.2 that for $c = c^*$ there exists a solution of (1.3) which connects -1 to $+1$. The following theorem thus proves both Theorem 1.3 and Corollary 1.4.

THEOREM 7.3. *Let f satisfy hypothesis* (3.1) *and let $\alpha \in \mathbb{R}$. For every $c > c^*$ there exists a solution of* (1.3) *connecting $u = -a$ to $u = 1$.*

Proof. We consider the stable manifold $W = W^s(1)$ of $u = 1$. We have shown in Theorem 6.1 that for $c > 0$ large enough the intersection of the stable manifold W of $u = -1$ and the boundary δK of K (defined in (3.3)) is a smooth simple closed

[5] We gratefully acknowledge several discussions with H. Geiges. He showed us that, via the Jordan–Brouwer separation theorem and an inductive Mayer–Vietoris argument, the division of $\delta\tilde{K}$ into two components can also be derived without using the extra information provided by the flow.

curve which projects on a closed curve Γ in the (u,z)-plane with $n(\Gamma,-1)=0$ and $n(\Gamma,1)=1$. It follows from the definition of c^* and Lemma 3.2 that, by continuity, this remains true for all $c>c_*$. Now fix $c>c^*$.

Let us assume by contradiction that there is no connection between $u=-a$ and $u=1$. The intersection between W and δK depends continuously on the energy level E as long as we do not encounter an equilibrium point. Assuming there is no connection between $u=-a$ and $u=1$, we let E decrease from $F(-1)>E_0>F(-a)$ to $E_2<F(-a)$. The projection Γ in the (u,z)-plane then depends continuously on E, as do the winding numbers, so that $n(\Gamma,-1)=0$ and $n(\Gamma,1)=1$ for all $E_0\le E\le E_2$. However, for the energy level E_2 we have that $(-1,0)$ and $(1,0)$ lie in the same component of the complement of the projection of δK onto the (u,z)-plane. Therefore $n(\Gamma,-1)=n(\Gamma,1)$, a contradiction. ◻

Remark 7.4. When $F(-1)=F(+1)$, then the same method shows that there exist travelling waves connecting $u=-a$ to $u=\pm1$ for all $c>0$ and all $\alpha\in\mathbb{R}$. Besides, as already noted in Remark 7.2, the method in the proof of Theorem 7.3 can be used to obtain an alternative proof of Theorem 7.1.

Finally, we prove Theorem 1.5 which deals with nonlinearities with two zeros (and a different behavior for $u\to\pm\infty$).

THEOREM 7.5. *Let $\alpha\in\mathbb{R}$ and let f satisfy hypothesis* (H$_2$). *For every $c>0$ there exists a solution of* (1.3) *connecting $u=0$ to $u=1$.*

Proof. Since the shape of the nonlinearity differs significantly from the one considered so far, we cannot invoke Lemma 3.2 directly. Besides, we find a priori bounds via a slightly different method.

Let $D\stackrel{\text{def}}{=}\sup\{\tilde u<0\,|\,F(u)>0 \text{ on } (-\infty,\tilde u)\}$. Travelling wave solutions connecting 0 to 1 satisfy $u\ge D$, since it follows from (1.4) and (1.5) that u can have no extremum in the range $u<D$ (at an extremum one would have $\mathcal{E}>F(1)$, which is impossible). Therefore, we may without loss of generality replace f by any function f_1 for which $f_1(u)=f(u)$ for $u\ge D$, and $f_1(u)<0$ for $u<D$. We choose f_1 such that $f_1(u)=u$ for $u<D-1$.

Now that we have a bound from below, we can also obtain a bound from above. A connecting solution of (1.3) is also a solution of (1.3) with f_1 replaced by any f_2 for which $f_2(u)=f_1(u)$ for all $u\ge D-1$. We choose $f_2(u)=-u^3$ for $u<D-2$, and argue as at the beginning of this section to conclude that there exists a uniform bound $\|u\|_\infty\le C_0$ on all travelling wave solutions. We may thus replace f_1 by a function f_3 for which $f_3(u)=f_1(u)$ for $u\le C_0$ and $f_3(u)=-u^3$ for $u\ge C_0+1$. We conclude that u is a travelling wave solution with speed c for nonlinearity $f(u)$ if and only if u is a travelling wave solution with speed c for nonlinearity $f_3(u)$.

In the following we therefore assume, without loss of generality, that $f(u)=u$ for $u\le D-1$, and $f(u)=-u^3$ for $u\ge C_0+1$.

We now follow the argument in the proof of Lemma 3.2. However, we cannot use Lemma 3.1 to show that orbits in $W^s(1)$, which are completely contained in K, are bounded. Instead, we argue as follows. Suppose, by contradiction, that an orbit $u(t)$ in $W^s(1)$ is completely contained in K and is unbounded. As in the proof of Lemma 3.2 it follows from (3.4) that $u(t)$ exists for all $t\in\mathbb{R}$. There are now two possibilities: either $u(t)\ge D-1$ for all $t\in\mathbb{R}$ or there exists some $t_0\in\mathbb{R}$ such that $u(t_0)<D-1$. First we deal with the latter case.

Since (see above) $u(t)$ cannot attain an extremum in the range $u<D$, it follows that $u(t)$ is increasing for $t\le t_0$. Hence $u(t)$ obeys, for $t\le t_0$, the linear equation $cu'=-u''''+\alpha u''+u$. Since u is unbounded as $t\to-\infty$, it follows that $u=$

$-a_0e^{-a_1t} + o(1)$ for some $a_0, a_1 > 0$ as $t \to -\infty$. By substituting this into (3.4) a contradiction is reached.

Next we deal with the case where $u(t) \geq D - 1$ for all $t \in \mathbb{R}$. Clearly $u(t)$ is a solution of (1.3) with f replaced by any function $\tilde{f}$ for which $\tilde{f}(u) = f(u)$ for all $u \geq D-1$. We choose $\tilde{f}(u) = -u^3$ for $u < D-2$, and it follows from Lemma 3.1 that u blows up in finite time, a contradiction.

Having circumvented the problem in the proof of Lemma 3.2 we conclude that for $F(0) < E_0 < F(-1)$ the intersection of the stable manifold W of $u = -1$ and the boundary δK of K (defined in (3.3)) is a smooth simple closed curve which projects on a closed curve Γ in the (u, z)-plane with $n(\Gamma, 1) = 1$.

The rest of the argument is analogous to the proof of Theorem 7.3. Assuming that there is no connection between $u = 0$ and $u = 1$, the final contradiction is now obtained by the fact that $n(\Gamma, 1) = 0$ for $E_2 < F(0)$. $\square$

8. Concluding remarks. The most apparent open problem concerns the range of α-values for which a travelling wave connecting -1 to $+1$ exists. For some examples it can be shown that such a travelling wave does not exist for all $\alpha \in \mathbb{R}$. The more general question whether for any nonlinearity satisfying (H_1) a bound α_* exists such that there are no travelling waves for $\alpha < \alpha_*$ remains open.

Regarding the uniqueness of the various travelling wave solutions not much is known. For large α (i.e., $\gamma \approx 0$) the travelling wave connecting -1 to $+1$ may be expected to be unique (analogous to the limiting second order case). The results in [6] show that uniqueness does not hold for $f_a(u) = (u + a)(1 - u^2)$ with a small when $\alpha < \sqrt{8}$. Equation (1.1) with $f(u) = u - u^3$ admits an abundance of standing wave solutions for $0 \leq \alpha < \sqrt{8}$. It has been proved in [6] that these solutions can be perturbed to travelling waves for $f_a(u)$ with small a and small $c = c(a)$. Since this can be done for any standing wave, an infinite family of curves in the (a, c)-plane passing through the origin is thus obtained.

The method used in this paper does not give any information about the shape of the solution. For example, we would like to know for which values of α the solution is monotone. Since we do not know the value of c for which a traveling wave occurs, we do not in general even know whether the connected equilibrium points are approached monotonically or in an oscillatory manner.

Finally, the question arises to what extent the travelling wave solution is of importance to the dynamics of the PDE. It might be a limit profile for a broad class of initial conditions as is the case for the second order equation [13]. Since travelling waves connecting $u = -a$ to $u = \pm 1$ exist for large ranges of c, it would be interesting to know which of these waves is generally encountered. In [9, 12] the wave selection mechanism has been investigated for a propagating front which is formed from localized initial data (i.e., $u + a$ is localized). Using the physically motivated assumption that the linearized equation (around $u = -a$) drives the system, it is argued that for $\alpha > \sqrt{12f'(-a)}$ one of the travelling waves is selected (and the wave speed is calculated), while for $\alpha < \sqrt{12f'(-a)}$ the propagating front is argued not to have a fixed profile. However, the only rigorous stability result that we know of is of a perturbative nature [27] (i.e., α very large) and moreover it does not answer the question of the selection of the wave speed.

REFERENCES

[1] N.N. Akhmediev, A.V. Buryak, and M. Karlsson, *Radiationless optical solitons with oscil-*

lating tails, Opt. Commun., 110 (1994), pp. 540–544.
[2] M.E. Akveld and J. Hulshof, *Travelling wave solutions of a fourth-order semilinear diffusion equation*, Appl. Math. Lett., 11 (1998), pp. 115–120.
[3] C.J. Amick and J.F. Toland, *Homoclinic orbits in the dynamic phase space analogy of an elastic strut*, European J. Appl. Math., 3 (1992), pp. 97–114.
[4] D.G. Aronson and H.F. Weinberger, *Nonlinear diffusion in population genetics, combustion, and nerve pulse propagation*, in Partial Differential Equations and Related Topics, J.A. Goldstein, ed., Lecture Notes in Math. 446, Springer-Verlag, Berlin, 1975.
[5] G.E. Bredon, *Topology and Geometry*, Springer-Verlag, New York, 1993.
[6] B. Buffoni, *Shooting methods and topological transversality*, Proc. Roy. Soc. Edinburgh Sect. A, 129 (1999), pp. 1137–1155.
[7] B. Buffoni, A.R. Champneys, and J.F. Toland, *Bifurcation and coalescence of multi-modal homoclinic orbits for a Hamiltonian system*, J. Dynam. Differential Equations, 8 (1996), pp. 221–281.
[8] C. Conley, *Isolated invariant sets and the Morse index*, CBMS Reg. Conf. Ser. Math. 38, AMS, Providence, RI, 1978.
[9] G.T. Dee and W. van Saarloos, *Bistable systems with propagating fronts leading to pattern formation*, Phys. Rev. Lett., 60 (1988), pp. 2641–2644.
[10] F. Dumortier, *Singularities of vector fields on the plane*, J. Differential Equations, 23 (1977), pp. 53–106.
[11] F. Dumortier, *Local study of planar vector fields: Singularities and their unfoldings*, in Structures in Dynamics, Finite Dimensional Deterministic Studies, H.W. Broer et al., eds., Stud. Math. Phys., Vol. 2, North-Holland, Amsterdam, 1991, pp. 161–241.
[12] U. Ebert and W. van Saarloos, *Universal algebraic relaxation of fronts propagating into an unstable state and implications for moving boundary approximations*, Phys. Rev. Lett., 80 (1998), pp. 1650–1653.
[13] P.C. Fife and J.B. McLeod, *The approach of solutions of nonlinear diffusion equations to travelling front solutions*, Arch. Rational Mech. Anal., 65 (1977), pp. 335–361.
[14] T. Gallay, *private communication*.
[15] R.A. Gardner and C.K.R.T. Jones, *Traveling waves of a perturbed diffusion equation arising in a phase field model*, Indiana Univ. Math. J., 39 (1990), pp. 1197–1222.
[16] W.D. Kalies, J. Kwapisz, J.B. van den Berg, and R.C.A.M. Vandervorst, *Homotopy classes for stable periodic and chaotic patterns in fourth-order Hamiltonian systems*, Comm. Math. Phys., 214 (2000), pp. 573–592.
[17] W.D. Kalies, J. Kwapisz, and R.C.A.M. Vandervorst, *Homotopy classes for stable connections between Hamiltonian saddle-focus equilibria*, Comm. Math. Phys., 193 (1998), pp. 337–371.
[18] W.D. Kalies and R.C.A.M. Vandervorst, *Multitransition homoclinic and heteroclinic solutions to the extended Fisher-Kolmogorov equation*, J. Differential Equations, 131 (1996), pp. 209–228.
[19] J. Kwapisz, *Uniqueness of the stationary wave for the extended Fisher-Kolmogorov equation*, J. Differential Equations, 165 (2000), pp. 235–253.
[20] P.J. Olver, *Applications of Lie Groups to Differential Equations*, Springer-Verlag, New York, 1993.
[21] G. Palis, *Linearly induced vector fields and $\mathbb{R}^2$-actions on spheres*, J. Differential Equations, 23 (1977), pp. 53–106.
[22] L.A. Peletier, A.I. Rotariu-Bruma, and W.C. Troy, *Pulse-like spatial patterns described by higher-order model equations*, J. Differential Equations, 150 (1998), pp. 124–187.
[23] L.A. Peletier and W.C. Troy, *Spatial patterns described by the extended Fisher-Kolmogorov (EFK) equation: Kinks*, Differential Integral Equations, 8 (1995), pp. 1279–1304.
[24] L.A. Peletier and W.C. Troy, *A topological shooting method and the existence of kinks of the extended Fisher-Kolmogorov equation*, Topol. Methods Nonlinear Anal., 6 (1995), pp. 331–355.
[25] M.A. Peletier, *Non-existence and uniqueness results for fourth-order Hamiltonian systems*, Nonlinearity, 12 (1999), pp. 1555–1570.
[26] L. Perko, *Differential Equations and Dynamical Systems*, Springer-Verlag, New York, 1991.
[27] V. Rottschäfer and C.E. Wayne, *Existence and stability of traveling fronts in the extended Fisher-Kolmogorov equation*, J. Differential Equations, to appear.
[28] J.B. van den Berg, *Uniqueness of solutions for the extended Fisher-Kolmogorov equation*, C. R. Acad. Sci. Paris Sér. I Math., 326 (1998), pp. 447–452.
[29] J.B. van den Berg, *The phase-plane picture for some fourth-order conservative differential equations*, J. Differential Equations, 161 (2000), pp. 110–153.