FLOER HOMOLOGY, RELATIVE BRAID CLASSES, AND LOW DIMENSIONAL DYNAMICS

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Dedicated to Paul H. Rabinowitz on the occasion of his 70th birthday

Floer homology is a powerful variational technique used in Symplectic Geometry to derive a Morse type theory for the Hamiltonian action functional. In two and three dimensional dynamics the topological structures of braids and links can used to distinguish between various types of periodic orbits. Various classes of braids are introduced and Floer type invariants are defined. The definition and basic properties of the different braid invariants that are discussed in this article are obtained in joined work with R.W. Ghrist, J.B. van den Berg and W. Wójcik. In the second part of this article results concerning the relation between the different braid class invariantsare discussed.

Keywords: Floer homology, braid classes, Cauchy-Riemann and parabolic dynamics, Morse theory.

1. Prelude

Flows of non-autonomous vector fields on two dimensional phase spaces may behave in a very complicated way. When regarded on the (three dimensional) extended phase space, the flow lines of these systems may display various knotting and linking patterns. The topological structure of knots and links, which only exists in dimension three, can be used to develop forcing relations (like Morse theory) for certain types of orbits of such flows. In this article we are particularly interested in knotting and linking of periodic orbits. Moreover, we will restrict to non-autonomous Hamiltonian vector fields, in which case we assume that the time-dependence of the vector field is 1-periodic. Intimately related to Hamiltonian flows are its time-1 maps, or Hamiltonian diffeomorphisms. Our theory will therefore be applicable in both settings.

We will start with defining braid classes in various settings and define braid class invariants accordingly. These invariants are computable in some situations. One of

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the goals is to show that all these invariants are strongly related and essentially the same, which results in a statement that all invariants are computable.

2. Relative braid classes

In this section we start with a hands on definition of relative braid classes in three different contexts. All three settings are closely related and we will point out their relations.

2.1. *Braids on the 2-disc*

Consider the standard 2-disc $\mathbb{D}^2$ in the plane[a] and the cylinder $C = [0,1] \times \mathbb{D}^2$. An unordered collection of continuous functions $\mathrm{x}_0 = \{x^1(t), \cdots, x^m(t)\}$, $x^k : [0,1] \to \mathbb{D}^2$ — called strands —, is called a braid on the 2-disc $\mathbb{D}^2$ if: (i) $x^k(t+1) = x^{\sigma(k)}(t)$ for some permutation $\sigma \in S_m$, and (ii) $x^k(t) \neq x^{k'}(t)$ for all $k \neq k'$ and all $t \in [0,1]$. The set of all braids on $\mathbb{D}^2$ containing x_0 is denoted by $[\mathrm{x}_0]$ and is called a braid class. The collection of all braid classes on $\mathbb{D}^2$ with m strands is denoted by Ω^m.

A way to visualize a braid is to consider a so-called braid diagram in the plane. The latter is obtained by projecting onto a plane of the form $[0,1] \times L$, where is $L \subset \mathbb{D}^2$ is a diameter in $\mathbb{D}^2$. If we denote the projection by $\pi : \mathbb{D}^2 \to L$, then two strands $x^k(t)$ and $x^{k'}(t)$ have a positive crossing at $\pi x^k(t_0) = \pi x^{k'}(t_0)$ if $x^k - x^{k'}$ rotates counter clock wise rotation about the origin, for small interval of times t around t_0. A negative crossing corresponds to a clock wise rotation.

Now consider special collections of the form $\{x(t), x^1(t), \cdots, x^m(t)\}$, with $\mathrm{x} = \{x(t)\}$ a periodic function on $[0,1]$, with values in $\mathbb{D}^2$ and $\mathrm{x}_0 = \{x^1(t), \cdots, x^m(t)\}$ as above. Denote such collections by x rel x_0 and assume that they are braids with $m+1$ strands. Since we singled out two braid components we denote the braid class containing x rel x_0 by $[\mathrm{x}$ rel $\mathrm{x}_0]$, which will be called a relative braid class. The component x_0 is called the skeleton of the relative braid class. If we take the skeleton x_0 to be fixed, then the set of periodic functions x for which x rel x_0 is a braid is denoted by $[\mathrm{x}]$ rel x_0 and is called a relative braid class fiber. The topology on $[\mathrm{x}]$ rel x_0 is the C^0-topology. The space $[\mathrm{x}$ rel $\mathrm{x}_0]$ is a fibered space over $[\mathrm{x}_0]$ and the relative braid class $[\mathrm{x}]$ rel x_0 is a fiber in $[\mathrm{x}$ rel $\mathrm{x}_0]$. A relative braid class is called proper is x can not be deformed, or 'collapsed', onto any of the strands x^k in x_0 — non-collapsing —, or onto the boundary $\partial\mathbb{D}^2$.[3] We can easily generalize the notion of relative braid class with x consisting on n strand.[1,2] However, in this article we restrict to the case $n = 1$.

2.2. *Braid diagrams on the unit interval*

In the special case that strands $x(t)$ are of the form $x(t) = (q_t(t), q(t))$, the projection onto the q-coordinate provides a representation of a braid in terms of graphs. The

[a]For points in the plane we use standard coordinates $x = (p,q)$ and positive orientation.

range of $q(t)$ is the interval $[-1,1]$. Such strands satisfy the property that they lay in the kernel of $\theta = dq - pdt$: Legendrian property. An unordered collection of functions $\mathrm{Q}_0 = \{q^1(t), \cdots q^m(t)\}$ is called a braid diagram if: (i) $q^k(t+1) = q^{\sigma(k)}(t)$ for some permutation $\sigma \in S_m$, and (ii) all graphs $q^k(t)$ intersect transversally. The set of all braid diagrams containing Q_0 is denoted by $[\mathrm{Q}_0]_L$. As before we also consider collections of the form Q rel $\mathrm{Q}_0 = \{q(t), q^1(t), \cdots q^m(t)\}$ and the associated relative braid classes $[\mathrm{Q} \text{ rel } \mathrm{Q}_0]_L$ and $[\mathrm{Q}]_L$ rel Q_0 (fibers). Obviously these Legendrian braid classes are subsets of the braid classes on $\mathbb{D}^2$. A class of braid diagrams is called proper if Q cannot be collapsed onto any of the strands q^k, or cannot be identically equal to ± 1. By the Legendrian constraint all crossings of strands are positive!

2.3. *Discrete braid diagrams*

Yet another simplification is obtained by considering piecewise linear functions functions connecting the points $q_i = q(i/d)$, $i = 0, \cdots, d$. We represent such piecewise linear functions by sequences $\mathbf{q} = \{q_i\}$ and the range of the values q_i is the interval $[-1,1]$. Both the sequences and their piecewise linear extension will be denoted by the same symbol $\mathbf{q}$. An unordered collection of sequences $\mathbf{q}_0 = \{\mathbf{q}^1, \cdots, \mathbf{q}^m\} = \big\{\{q_i^1\}, \cdots \{q_i^m\}\big\}$ is called a discrete, or piecewise linear braid diagram if: (i) $q_{i+1}^k = q_i^{\sigma(k)}$ for some permutation $\sigma \in S_m$, and (ii) all graphs $q^k(t)$ intersect transversally.[b] The set of all braid diagrams containing $\mathbf{q}_0$ is denoted by $[\mathbf{q}_0]_D$. Crossings in this setting are also marked as positive. Collections of the form $\mathbf{q}$ rel $\mathbf{q}_0 = \{\mathbf{q}, \mathbf{q}^1, \cdots \mathbf{q}^m\}$ and the associated discrete relative braid classes $[\mathbf{q} \text{ rel } \mathbf{q}_0]_D$ and $[\mathbf{q}]_D$ rel $\mathbf{q}_0$ (fibers). A class of discrete braid diagrams is called proper if $\mathbf{q}$ cannot be collapsed onto any of the sequences $\mathbf{q}^k = \{q_i^k\}$, or cannot be identically equal to ± 1.

Remark 2.1. The properness condition is a topological condition that descents from braids on $\mathbb{D}^2$ to discrete braids, i.e. properness of $[\mathrm{X} \text{ rel } \mathrm{X}_0]$ implies

$$[\mathrm{X} \text{ rel } \mathrm{X}_0] \implies [\mathrm{Q} \text{ rel } \mathrm{Q}_0]_L \implies [\mathbf{q} \text{ rel } \mathbf{q}_0]_D.$$

The implications do not necessarily go in the opposite direction.

3. An invariant for discrete relative braid classes

A simple example of a discrete relative braid class is given in Fig. 3 below. The the skeleton consists of four strands and the resulting braid class as given in figure is proper. In Fig. 3 there is also co-orientation given of the co-dimension 1 faces of the boundary. This is based in the following principle: the co-orientation of a co-dimension 1 face is positive (arrow pointing outward) if for the associated neighboring braid class the total number of intersections of $\mathbf{q}$ with $\mathbf{q}_0$ is decreased by 2.

[b]An intersection is called transverse if $(q_{i-1}^k - q_{i-1}^{k'})(q_{i+1}^k - q_{i+1}^{k'}) > 0$, whenever $q_i^k = q_i^{k'}$.

The co-orientation is negative if the total number of intersections of $\mathbf{q}$ with $\mathbf{q}_0$ is increased by 2. In Fig. 3 we denote by $N = \mathrm{cl}\big([\mathbf{q}]_D \text{ rel } \mathbf{q}_0\big)$ the compact configuration space, and by N^- the closure of the union of all positively co-oriented co-dimension 1 faces. The latter will also be referred to as 'exit set'. For this example we define

$$HC_k([\mathbf{q} \text{ rel } \mathbf{q}_0]_D) := H_k(N, N^-),$$

which turns out to be an invariant, i.e. if we choose a different fiber $[\mathbf{q}']_D \text{ rel } \mathbf{q}_0'$, and thus a different pair (N', N'^-), then $H_k(N, N^-) \cong H_k(N', N'^-)$. This justifies the statement that HC_* is an invariant for $[\mathbf{q} \text{ rel } \mathbf{q}_0]$. In this example we have that $HC_1([\mathbf{q} \text{ rel } \mathbf{q}_0]_D) \cong \mathbb{Z}$ and $HC_k([\mathbf{q} \text{ rel } \mathbf{q}_0]_D) \cong 0$ for $k \neq 1$. To prove that HC_* is an invariant relies on the fact that (N, N^-) also has meaning for a natural class of dynamical systems on N and uses Conley index theory.

Fig. 1. The braid of Example 1 [left] and the associated conguration space, or relative braid class fiber $[\mathbf{q}]_D$ rel $\mathbf{q}_0$ [middle]. On the right is an expanded view of the adjacent braid classes and the fixed points that correspond to the four fixed strands in the skeleton $\mathbf{q}_0$. The adjacent braid classes are not proper.

In general we define HC_* for proper discrete relative braid classes as follows. Let $[\mathbf{q} \text{ rel } \mathbf{q}_0]_D$ be a proper discrete relative braid class and $[\mathbf{q}]_D \text{ rel } \mathbf{q}_0$ a fiber. Then, by the above rule for co-orienting the co-dimension 1 faces of the boundary of $N = \mathrm{cl}\big([\mathbf{q}]_D \text{ rel } \mathbf{q}_0\big)$. As before this yields a pair (N, N^-) and the homology $H_k(N, N^-)$ is well-defined. The following theorem holds.

Theorem 3.1 (Ghrist, VandenBerg and Vandervorst[1]). *Let $[\mathbf{q} \text{ rel } \mathbf{q}_0]_D$ be a proper discrete relative braid class. Then, for any to fibers $[\mathbf{q}]_D \text{ rel } \mathbf{q}_0$ and $[\mathbf{q}']_D \text{ rel } \mathbf{q}_0'$, with $N = \mathrm{cl}\big([\mathbf{q}]_D \text{ rel } \mathbf{q}_0\big)$, $N' = \mathrm{cl}\big([\mathbf{q}']_D \text{ rel } \mathbf{q}_0'\big)$ and N^-, N'^- accordingly, it holds that $H_*(N, N^-) \cong H_*(N', N'^-)$. This justifies the definition*

$$HC_k([\mathbf{q} \text{ rel } \mathbf{q}_0]_D) := H_k(N, N^-), \quad \forall k \geq 0, \tag{1}$$

which makes HC_* an invariant for the braid class $[\mathbf{q} \text{ rel } \mathbf{q}_0]_D$.

A second property of the invariant HC_* is a stability property with respect to the number of discretization points and is a first step towards a connection between invariants for discrete class and classes of braid diagram. Define the extension operator:

$$(\mathbb{E}q)_i = \begin{cases} q_i, & \text{for } \ i = 0, \cdots, d \\ q_d, & \text{for } \ i = d+1. \end{cases}$$

If $[\mathbf{q} \text{ rel } \mathbf{q}_0]$ is proper, then also $[\mathbb{E}\mathbf{q} \text{ rel } \mathbb{E}\mathbf{q}_0]_D$ is proper.

Theorem 3.2 (Ghrist, VandenBerg and Vandervorst[1]). *Let $[\mathbf{q} \text{ rel } \mathbf{q}_0]_D$ be a proper discrete relative braid class and a fiber $[\mathbf{q}]_D \text{ rel } \mathbf{q}_0$. Then, for $N =$*

$\mathrm{cl}\big([\mathbf{q}]_D \text{ rel } \mathbf{q}_0\big)$ *it holds that* $H_k(N, N^-) \cong H_k(\mathbb{E}N, (\mathbb{E}N)^-))$. *Therefore,*

$$HC_k([\mathbb{E}\mathbf{q} \text{ rel } \mathbb{E}\mathbf{q}_0]_D) \cong HC_k([\mathbf{q} \text{ rel } \mathbf{q}_0]_D), \quad \forall k \geq 0, \tag{2}$$

which shows that the invariant HC_* *is stable under the action of* $\mathbb{E}$.

4. Parabolic dynamics

Consider a class of differential equations and vector fields of the form

$$\tfrac{d}{ds} u_i = \mathcal{R}_i(u_{i-1}, u_i, u_{i+1}), \tag{3}$$

with the vector field $\mathcal{R}$ satisfying the hypotheses:

(p1) $\partial_1 \mathcal{R}_i > 0$ and $\partial_3 \mathcal{R}_i > 0$;
(p2) $\mathcal{R}_{i+d} = \mathcal{R}_i$ for all i;
(p3) $\mathcal{R}_i(-1,-1,-1) = \mathcal{R}_i(1,1,1) = 0$ for all i.

Vector fields satisfying these hypotheses are called parabolic.[c] If we restrict the range of the variables u_i to the interval $[-1,1]$, then Eq. (3) generates a flow ψ^s. This flow will be referred to as a parabolic flow on sequences. Note that by Hypothesis (p3) that the constant sequences ± 1 are stationary for the flow ψ^s. Parabolic flows have a crucial property with respect intersections of sequences. Consider two flow lines $u_i(s)$ and $u_i'(s)$ and assume that $u_{i_0}(s_0) = u_{i_0}(s_0)$ for some i_0 and s_0. Generically, $\big(u_{i_0-1}(s_0) - u'_{i_0-1}(s_0)\big)\big(u_{i_0+1}(s_0) - u'_{i_0+1}(s_0)\big) > 0$ and then by Hypothesis (p1) it follows that $\frac{d}{ds}(u_{i_0} - u'_{i_0})(s_0) > 0$, which implies that the number of intersections decreases by 2. This principle is also true in the non-generic case.[4–6] Assume that $\mathcal{R}$ is a parabolic vector field such that $\mathcal{R}(\mathbf{q}_0) = 0$, i.e. the strands $\{q_i^k\}$ are stationary solutions. This principle explains that the direction of the flow is determined by the co-orientation given in the previous section, see Fig. 4. As a consequence we have the following lemma.

Fig. 2. A parabolic ow on a discretized braid class is transverse to the boundary faces. The local linking of strands decreases strictly along the boundary.

Lemma 4.1. *Let* $[\mathbf{q} \text{ rel } \mathbf{q}_0]_D$ *be a proper discrete relative braid class with fiber* $[\mathbf{q}]_D$ rel $\mathbf{q}_0$ *and let* $\mathcal{R}(\mathbf{q}_0) = 0$. *Then,* $N = \mathrm{cl}\big([\mathbf{q}]_D \text{ rel } \mathbf{q}_0\big)$ *is an isolating neighborhood with respect* ψ^s *and* (N, N^-) *is an index pair in the sense of Conley.*[16]

Since N is essentially a cube-complex the Conley index of (N, ψ^s) is given by the relative homology $H_*(N, N^-)$. The proof of Theorem 3.1 is based on the fact that the invariant HC_* can be regarded as the Conley index of a parabolic flow on the braid class. The invariance properties of the Conley index prove the theorem.[1]

[c]Compare with discretizing the second derivative: $u_{i-1} - 2u_i + u_{i+1}$.

As for Theorem 3.2 we use singular perturbation theory in combination with the Conley index.[1,7]

The connection with the Conley index has another important feature. Non-triviallity of the invariant HC_*, and thus the Conley index, implies non-triviallity of the maximal invariant set $\textsc{Inv}(N, \psi^s) \subset N$. For example if $\mathcal{R}$ is a gradient vector field that then the number of terms of $P_t(HC_*)$[d] provides a lower bound on the number of zeroes of $\mathcal{R}$ and thus a lower bound on the number of non-trivial stationary braids in a braid class fiber.

5. Braids and dynamics

For discrete braids and parabolic dynamics the discrete lap-number principle reveals an intimate relation between parabolic dynamics and the topology of piecewise linear braid diagrams. The question is if similar principles holds for more general classes of braids as discussed in Sect. 2.

5.1. *The Cauchy-Riemann equations*

Consider the equations, called the non-linear Cauchy-Riemann equations, or Floer equations

$$u_s - J(t,u)\big[u_t - X_H(t,u)\big] = 0, \tag{4}$$

where $u(s,t)$ takes values in $\mathbb{D}^2$, $s \in \mathbb{R}$ and $t \in \mathbb{R}/\mathbb{Z}$. The parameters J and H are called an almost complex structure and a Hamiltonian respectively. An almost complex structure $J : \mathbb{R}/\mathbb{Z} \times \mathbb{D}^2 \to \mathrm{End}(T\mathbb{D}^2)$, with $J^2 = -\mathrm{id}$, ω_0-invariant[e] and $\omega_0(\cdot, J\cdot) = \langle \cdot, \cdot \rangle$. The vector field X_H, called the Hamilton vector field, is defined by the relation $\omega_0(X_H, \cdot) = -dH$. The smooth function $H : \mathbb{R}/\mathbb{Z} \times \mathbb{D}^2 \to \mathbb{R}$ is called the Hamiltonian and satisfies the properties

(h1) $H \in C^\infty(\mathbb{R} \times \mathbb{D}^2; \mathbb{R})$;
(h2) $H(t+1, x) = H(t, x)$ for all $(t,x) \in \mathbb{R} \times \mathbb{D}^2$;
(h3) $H(t,x) = 0$ for all $x \in \partial\mathbb{D}^2$.

For a braid x the total crossing number $\textsc{Cross}(\mathrm{x})$ is define as the number of positive minus the number of negative crossings in x. For relative braids this number is denoted by $\textsc{Cross}(\mathrm{x} \text{ rel } \mathrm{x}_0)$. For two local solutions $u(s,t)$ and $u'(s,t)$ of Eq. (4) denote its local winding number is given by $w(u,u')|_s = W(\Gamma^s, 0)$, where Γ^s is the closed curve in $\mathbb{D}^2$ obtained by the parametrization $u(s,t) - u'(s,t)$, $t \in \mathbb{R}/\mathbb{Z}$. The local winding number $w(u,u')|_s$ is defined for all s for which $\Gamma^s \subset \mathbb{R}^2 \backslash \{(0,0)\}$. If $u(s_0,t_0) = u'(s_0,t_0)$ for some pair $(s_0,t_0) \in \mathbb{R}/\mathbb{Z} \times \mathbb{R}$, then there exists an $\epsilon > 0$ such that

$$w(u,u')|_{s_0-\epsilon} > w(u,u')|_{s_0+\epsilon}.$$

[d] The Poincaré polynomial of HC_* is defined as $P_t(HC_*) = \sum_k \big(\dim HC_k\big) t^k$.
[e] The 2-form $\omega_0 = dp \wedge dq$ is the standard 2-form on $\mathbb{R}^2$.

As in the discrete case this property is the analogue of the lap-number property for discrete parabolic equations and it states that along flow-lines of the non-linear Cauchy-Riemann equations positive crossings can transform into negative crossings but not vice versa. Let $[\mathrm{x}]$ rel x_0 be a relative braid class fiber with skeleton x_0, then we can choose Hamiltonians H such that the skeletal strands are solutions of the Hamilton equations $x_t = X_H(t,x)$. Let $\mathrm{u}(s)$ rel x_0 and $\mathrm{u}'(s)$ rel x_0 denote local solutions (in s) of the Cauchy-Riemann equations, then $\mathrm{Cross}(\mathrm{u} \text{ rel } \mathrm{x}_0)|_{s_0-\epsilon} > \mathrm{Cross}(\mathrm{u} \text{ rel } \mathrm{x}_0)|_{s_0+\epsilon}$, whenever $u(s_0,t_0) = u^k(s_0,t_0)$ for some k. In Analogy to Lemma 4.1 we have the following result. Denote the set of bounded solutions of Eq. (4) in a braid class fiber $[\mathrm{x}]$ rel x_0, that exist for all $s \in \mathbb{R}$, by $\mathcal{M}([\mathrm{x}] \text{ rel } \mathrm{x}_0)$. The image under the mapping $u \mapsto u(0,\cdot)$ is denoted by $\mathcal{S}([\mathrm{x}] \text{ rel } \mathrm{x}_0) \subset C^\infty(\mathbb{R}/\mathbb{Z};\mathbb{D}^2)$.

Lemma 5.1. *Let* $[\mathrm{x} \text{ rel } \mathrm{x}_0]_D$ *be a proper relative braid class with fiber* $[\mathrm{x}]_D$ rel x_0 *and let the strands in* x_0 *be solutions of the Hamilton equations* $x_t = X_H(t,x)$ *with Hamiltonian* H*. Then,* $\mathcal{M}([\mathrm{x}] \text{ rel } \mathrm{x}_0)$ *is compact with respect to convergence on compact sets in* $\mathbb{R}\times\mathbb{R}/\mathbb{Z}$ *(with derivatives up to any order). Consequently,* $\mathcal{S}([\mathrm{x}] \text{ rel } \mathrm{x}_0)$ *is compact and contained in* $[\mathrm{x}]$ rel x_0.

The set $\mathcal{S}$ plays the role of maximal invariant set $[\mathrm{x}]$ rel x_0 and which is compact. This reflects the same situations as in Lemma 4.1. However, Conley theory is not applicable. Firstly, since the non-linear Cauchy-Riemann equations do not generated a semi-flow on $C^\infty(\mathbb{R}/\mathbb{Z};\mathbb{D}^2)$, and secondly the Morse (co)-index of critical points of the Hamilton action is infinite. In the next section we discuss the approach of A. Floer[8] which allows us to define invariants in this case as in the case of discrete braids.

5.2. *The Heat flow*

Consider the scalar parabolic equation, or Heat flow equation

$$u_s - u_{tt} - g(t,u) = 0, \tag{5}$$

where $u(s,t)$ takes values in the interval $[-1,1]$. For the function g we assume the following hypotheses:

(g1) $g \in C^\infty(\mathbb{R}\times\mathbb{R};\mathbb{R})$;
(g2) $g(t+1,q) = g(t,q)$ for all $(t,q) \in \mathbb{R}\times\mathbb{R}$;
(g3) $g(t,-1) = g(t,1) = 0$ for $t \in \mathbb{R}$.

This equation is closely related to the discrete parabolic equations and generates a local semi-flow ψ^s on periodic functions in $C^\infty(\mathbb{R}/\mathbb{Z};\mathbb{R})$. For a braid diagram q we define the intersection number $\mathrm{I}(\mathrm{q})$ as the total number of intersections, and since all intersections in a Legendrian braid of this type correspond to positive crossings, the total intersection number is equal to the crossing number defined above. The classical lap-number property[9–11] of non-linear scalar heat equations states that the number of intersections between two graphs can only decrease as time $s \to \infty$. As before let $[\text{Q}]_L$ rel Q_0 be a relative braid class fiber with skeleton Q_0, then we can choose a non-linearity g such that the skeletal strands in Q_0 are solutions of the equation $q_{tt} + g(t,q) = 0$. Let $\text{U}(s)$ rel Q_0 and $\text{U}'(s)$ rel Q_0 denote local solutions (in s) of the Heat flow equation, then $\text{I}(\text{U rel Q}_0)|_{s_0-\epsilon} > \text{I}(\text{U rel Q}_0)|_{s_0+\epsilon}$, whenever $u(s_0,t_0) = u^k(s_0,t_0)$ for some k. One approach for this equation is to use infinite dimensional versions of the Conley index.[12–14] However, the same approach as for the non-linear Cauchy-Riemann equations can be used. As before we define the sets of all bounded solutions in $[\text{Q}]_L$ rel Q_0, which we denote by $\mathcal{M}([\text{Q}]_L \text{ rel Q}_0)$. Similarly, $\mathcal{S}([\text{Q}]_L \text{ rel Q}_0) \subset C^\infty(\mathbb{R}/\mathbb{Z};\mathbb{R})$ denotes the image under the map $u \mapsto u(0,\cdot)$.

Lemma 5.2. *Let* $[\text{Q rel Q}_0]_L$ *be a proper relative braid class with fiber* $[\text{Q}]_L$ rel Q_0 *and let the strands in* Q_0 *be solutions of* $q_{tt} + g(t,q) = 0$ *for some non-linearity* g*. Then,* $\mathcal{M}([\text{Q}]_L \text{ rel Q}_0)$ *is compact with respect to convergence on compact sets in* $\mathbb{R}\times\mathbb{R}/\mathbb{Z}$ *(with derivatives up to any order). Consequently,* $\mathcal{S}([\text{Q}]_L \text{ rel Q}_0)$ *is compact and contained in* $[\text{Q}]_L$ rel Q_0.

The set $\mathcal{S}([\text{Q}] \text{ rel Q}_0)$ is exactly the maximal invariant set of the Heat (semi)-flow ψ^s generated by Eq. (5). In the next section we will carry out the same procedure for defining invariants for all three equations.

5.3. *Discrete parabolic equations*

In this case the natural dynamics are the discrete parabolic equations of the form

$$(u_i)_s - \mathcal{R}_i(u_{i-1}, u_i, u_{i+1}) = 0,$$

which are discussed in Sect. 4. The maximal invariant set $\text{Inv}(N, \psi^s)$ corresponds to $\mathcal{S}([\mathbf{q}]_D \text{ rel } \mathbf{q}_0)$ and an invariant can be defined in two different ways. The first way was described in Sect.'s 3 and 4. A second approach is discussed in the next section and provides the same invariants for discrete parabolic equations.

6. Invariants

In the previous section we linked the three types of braid classes to natural dynamical systems associated with these braid classes. They all share the property that proper braid classes yield isolating sets for the dynamics. In the case of discrete parabolic equations (Sect. 4) the Conley index is a natural tool to define an invariant and draw conclusion about the dynamics. In the previous section we also indicated that the Conley index approach does not work. Therefore we will use Floer's approach towards an analogue of the Conley index in the two remaining cases. In Floer's approach for solving the Arnold Conjecture he develops a Morse type theory for Hamilton action: $\mathcal{A}_H(x) = \int_0^1 pq_t - \int_0^1 H(t,x)$. The variational structure for the Heat flow equation is given by the action $\mathcal{A}_g(q) = \frac{1}{2}\int_0^1 |q_t|^2 - \int_0^1 G(t,q)$, where $G' = g$. Finally, a variational principle for discrete parabolic equations is given by the action $\mathcal{A}(\{q_i\}) = \sum_i W_i(q_i, q_{i+1})$, where W_i are smooth functions on $[-1,1]\times[-1,1]$ with the property that $\partial_1\partial_2 W_i > 0$. In this case $\mathcal{R}_i = \partial_2 W_{i-1} + \partial_1 W_i$. All equations introduced above are now gradient flow equations and we can carry out Floer's procedure.

6.1. *Basic ingredients*

Let us explain the basic ingredients of Floer theory for the Cauchy-Riemann equations. The same applies the other two cases.

- *Compactness.* Consider a proper relative braid class and the set $\mathcal{M}$ of bounded solutions of the Cauchy-Riemann equations in the braid class. Regularity and properness guarantee that the spaces $\mathcal{M}$ and $\mathcal{S}$ are compact with respect to the appropriate topologies, see Sect. 5. Compactness holds in all cases.
- *Genericity of critical points.* For a generic choice of Hamiltonians H, satisfying (h1)-(h3) and for which the skeletal strands in x_0 are solutions of the associated Hamilton equations, the critical points of $\mathcal{A}_H$ in $[\mathrm{x}]$ rel x_0 are non-degenerate.[2] We should stress that the strands in x_0 need not be non-degenerate. The same result holds for the other cases.
- *Genericity of connecting orbits.* If H is generic then by the compactness there are only finitely many critical points in $\mathrm{Crit}([\mathrm{x}]$ rel $\mathrm{x}_0)$ — the critical point set. By the gradient structure of the Cauchy-Riemann equations this implies that $\mathcal{M}$ is the union of space of connecting orbits:

  $$\mathcal{M}([\mathrm{x}]\ \mathrm{rel}\ \mathrm{x}_0) = \bigcup_{\mathrm{x}^-,\mathrm{x}^+\in\mathrm{Crit}} \mathcal{M}_{\mathrm{x}^-,\mathrm{x}^+}([\mathrm{x}]\ \mathrm{rel}\ \mathrm{x}_0),$$

  where $\mathcal{M}_{\mathrm{x}^-,\mathrm{x}^+}$ is the space of bounded solutions of Eq. (4) with limits x^- and x^+ at $s = \pm\infty$ respectively. For a generic choice of (J, H)[f] the spaces of connecting orbits are smooth manifolds without boundary. A generic pair (J, H) for which H is chosen such that x_0 consists of solutions of the associated Hamilton equations, is called an admissible pair.
- *Index function.* One can establish a grading $\mu(\mathrm{x})$ on the elements in $\mathrm{Crit}([\mathrm{x}]$ rel $\mathrm{x}_0)$ such that the dimension of $\mathcal{M}_{\mathrm{x}^-,\mathrm{x}^+}([\mathrm{x}]$ rel $\mathrm{x}_0)$ is given by the formula

  $$\dim \mathcal{M}_{\mathrm{x}^-,\mathrm{x}^+}([\mathrm{x}]\ \mathrm{rel}\ \mathrm{x}_0) = \mu(\mathrm{x}^-) - \mu(\mathrm{x}^+).$$

  This is based on the theory of Fredholm operators and holds in all cases. For the Cauchy-Riemann equation we choose μ to be the Conley-Zehnder index, for the Heat flow the classical morse index, and the same for the discrete parabolic equations.

[f] If we choose $J = J_0$ — the standard symplectic matrix — genericity can be obtained by choosing a generic Hamiltonian.

Obtaining these properties is possible in all three cases and is based on standard analytical techniques. However, working out all details is very tedious. With these requirements at hand we can build a chain complex.

6.2. *Floer homology, Morse homology and the Conley index*

The construction of the chain complex and thus the Floer homology is a standard procedure. By the compactness and genericity $\mathrm{Crit}([\mathrm{x}] \text{ rel } \mathrm{x}_0)$ is finite and we define the chain groups $C_k([\mathrm{x}] \text{ rel } \mathrm{x}_0)$ as formal sum $\sum_j \alpha_j \mathrm{x}_j$, with coefficients $\alpha_j \in \mathbb{Z}_2$. A boundary operator $\partial_k : C_k \to C_{k-1}$ is defined by the formula

$$\partial_k \mathrm{x} = \sum_{\mu(\mathrm{x}')=k-1} n(\mathrm{x}, \mathrm{x}')\mathrm{x}',$$

where $n(\mathrm{x}, \mathrm{x}')$ is the number of elements in $\mathcal{M}_{\mathrm{x}^-,\mathrm{x}^+}([\mathrm{x}] \text{ rel } \mathrm{x}_0)$, with $\mu(\mathrm{x}^-) - \mu(\mathrm{x}^+) = 1$. This number is finite by compactness and genericity.[g] To prove that ∂_k is a boundary operator requires showing that $\partial_{k-1}\partial_k = 0$. The composition counts the number of broken trajectories, i.e. the number of elements in the set

$$\bigcup_{\mu(\mathrm{x}')=k-1} \Big(\mathcal{M}_{\mathrm{x}^-,\mathrm{x}'}([\mathrm{x}] \text{ rel } \mathrm{x}_0) \times \mathcal{M}_{\mathrm{x}',\mathrm{x}^+}([\mathrm{x}] \text{ rel } \mathrm{x}_0)\Big).$$

The space $\mathcal{M}_{\mathrm{x}^-,\mathrm{x}^+}([\mathrm{x}] \text{ rel } \mathrm{x}_0)/\mathbb{R}$, with $\mu(\mathrm{x}^-) - \mu(\mathrm{x}^+) = 2$, is manifold (without boundary) of dimension 1, and a detailed analysis — Floer's gluing construction — reveals that if $\mathcal{M}_{\mathrm{x}^-,\mathrm{x}^+}([\mathrm{x}] \text{ rel } \mathrm{x}_0)/\mathbb{R}$ is not compact, then the manifolds can be compactified to manifolds with boundary diffeomorphic to $[0, 1]$ by adding broken trajectories in $\bigcup_{\mu(\mathrm{x}')=k-1}\big(\mathcal{M}_{\mathrm{x}^-,\mathrm{x}'} \times \mathcal{M}_{\mathrm{x}',\mathrm{x}^+}\big)$. The gluing construction also reveals that the procedure is surjective and thus the number of broken trajectories is even, and thus $\partial_{k-1}\partial_k = 0$. Summaring (C_*, ∂_*) is a chain complex and its homology is well-defined and finite.

Definition 6.1. Let $[\mathrm{x} \text{ rel } \mathrm{x}_0]$ be a proper relative braid class with fiber $[\mathrm{x}] \text{ rel } \mathrm{x}_0$. Then,

$$HF_k([\mathrm{x}] \text{ rel } \mathrm{x}_0; J, H) := H_k(C, \partial),$$

where (J, H) is an admissible pair, is called the Floer homology of $[\mathrm{x}] \text{ rel } \mathrm{x}_0$.

The same constructions can be carried out for the Heat flow equation and the discrete parabolic equations leading to the homologies

$$HM_*([\mathrm{Q}]_L \text{ rel } \mathrm{Q}_0; g), \quad \text{and} \quad HC_*([\mathrm{Q}]_D \text{ rel } \mathrm{Q}_0; W),$$

where the latter is isomorphic with the Conley index. This statement can be proved by using the arguments that relate ∂_k to Conley's connection matrix.[15] The former will be referred to as the Morse homology of $[\mathrm{Q}]_D$ rel Q_0 and the latter as the homological Conley index of $[\mathrm{Q}]_D$ rel Q_0.

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[g] When $\mu(\mathrm{x}^-) - \mu(\mathrm{x}^+) = 1$, then set is 1-dimensional and this $\mathcal{S}$ is finite.

Another important virtue of Floer homology is that Conley's continuation invariance remains valid. In the setting of braid class invariants we can use this to show that the invariants independent of (J, H), g and W respectively, but also are independent of the choice of the fiber.

Theorem 6.1 (Ghrist, VandenBerg, Vandervorst and Wójcik[2]). *Let* $[\mathrm{x} \text{ rel } \mathrm{x}_0]$ *be a proper relative braid class and let* $[\mathrm{x}] \text{ rel } \mathrm{x}_0$ *and* $[\mathrm{x}'] \text{ rel } \mathrm{x}'_0$ *be fibers. Then,*

$$HF_k([\mathrm{x}] \text{ rel } \mathrm{x}_0; J, H) \cong HF_k([\mathrm{x}'] \text{ rel } \mathrm{x}'_0; J', H'),$$

for any choice of admissible pairs (J, H) *and* (J', H').

By the above theorem we can define the invariant

$$HF_k([\mathrm{x} \text{ rel } \mathrm{x}_0]) := HF_k([\mathrm{x}] \text{ rel } \mathrm{x}_0; J, H), \tag{6}$$

for any fiber $[\mathrm{x}] \text{ rel } \mathrm{x}_0$ and any admissible pair (J, H).

The proof of Theorem is based on the continuation principle which was first proved by Floer.[8] In proof two chain complexes are compares by considering a non-autonomous version of the Cauchy-Riemann equations (4). This analysis is standard by now, but is very involved in all its details.

For the remaining two cases we obtain the same result which yields the invariants $HM_*([\mathrm{Q} \text{ rel } \mathrm{Q}_0]_L)$ and $HC_*([\mathrm{Q} \text{ rel } \mathrm{Q}_0]_D)$, where the latter corresponds to the invariant defined in Theorem 3.1.

6.3. *Basic properties*

As pointed out before, braids on $\mathbb{D}^2$ may have positive and negative crossings. However, Legendrian and piecewise linear braid only have positive crossings. The next theorem gives a relation between crossings and homology shifts. Consider the mapping $x(t) \mapsto (Sx)(t) := \exp(2\pi J_0 t)x(t)$, which adds a full twist to x, i.e.

$$W(S\Gamma, 0) = W(\Gamma, 0) + 1,$$

where Γ is the curve traced out by x and $S\Gamma$ the curve traced out by Sx. The same applies to arbitrary integer powers of S. The map $x \mapsto Sx$ can be applied to a braid by applying it to all strands at the same time. In particular the mapping $\mathrm{x} \text{ rel } \mathrm{x}_0 \mapsto S^\ell \mathrm{x} \text{ rel } S^\ell \mathrm{x}_0$ is well-defined. If $[\mathrm{x} \text{ rel } \mathrm{x}_0]$ is proper, then so is $[S^\ell \mathrm{x} \text{ rel } S^\ell \mathrm{x}_0]$ for all $\ell \in \mathbb{Z}$.

Theorem 6.2 (Ghrist, VandenBerg, Vandervorst and Wójcik[2]). *Let* $[\mathrm{x} \text{ rel } \mathrm{x}_0]$ *be a proper relative braid class. Then*

$$HF_k([S^\ell \mathrm{x} \text{ rel } S^\ell \mathrm{x}_0]) \cong HF_{k-2\ell}([\mathrm{x} \text{ rel } \mathrm{x}_0]), \quad \forall k \in \mathbb{Z},$$

and for any $\ell \in \mathbb{Z}$.

If we choose $\ell \geq 0$ sufficiently large, then the braid class $[S^\ell \mathrm{X} \text{ rel } S^\ell \mathrm{X}_0]$ is positive, i.e. there exist representatives $\mathrm{X}^+ \text{ rel } \mathrm{X}_0^+$ in $[S^\ell \mathrm{X} \text{ rel } S^\ell \mathrm{X}_0]$, which are positive braids and thus $[S^\ell \mathrm{X} \text{ rel } S^\ell \mathrm{X}_0] = [\mathrm{X}^+ \text{ rel } \mathrm{X}_0^+]$. For a generic choice of $\mathrm{X}^+ \text{ rel } \mathrm{X}_0^+ \in [S^\ell \mathrm{X} \text{ rel } S^\ell \mathrm{X}_0]$ the q-projection is a braid diagram $\mathrm{Q}^+ \text{ rel } \mathrm{Q}_0^+$ and under the embedding $q \mapsto (q_t, q)$ the associated braid $\mathrm{X}^L \text{ rel } \mathrm{X}_0^L$ is contained in $[S^\ell \mathrm{X} \text{ rel } S^\ell \mathrm{X}_0] = [\mathrm{X}^+ \text{ rel } \mathrm{X}_0^+]$. From $\mathrm{Q}^+ \text{ rel } \mathrm{Q}_0^+$ we can discretize by $q_i = q(i/m+2)$ where we chose $d = m+2$ (or larger), and which yields $\mathbf{q}^+ \text{ rel } \mathbf{q}_0^+$. By regarding the sequences as piecewise linear functions we obtain braid diagrams $\mathrm{Q}^D \text{ rel } \mathrm{Q}_0^D$ contained in $[\mathrm{Q}^+ \text{ rel } \mathrm{Q}_0^+]$. The following diagram gives an overview of the relations:

$$\begin{array}{ccccccc}
\mathrm{X} \text{ rel } \mathrm{X}_0 & \Longrightarrow & \mathrm{X}^+ \text{ rel } \mathrm{X}_0^+ & \Longrightarrow & \mathrm{Q}^+ \text{ rel } \mathrm{Q}_0^+ & \Longrightarrow & \mathbf{q}^+ \text{ rel } \mathbf{q}_0^+ \\
 & & \downarrow & & \downarrow & & \downarrow \\
 & & [S^\ell \mathrm{X} \text{ rel } S^\ell \mathrm{X}_0] & \longleftarrow & \mathrm{X}^L \text{ rel } \mathrm{X}_0^L & & \mathrm{Q}^D \text{ rel } \mathrm{Q}_0^D \\
 & & & & \downarrow & & \downarrow \\
 & & & & [S^\ell \mathrm{X} \text{ rel } S^\ell \mathrm{X}_0] & \longleftarrow & [\mathrm{Q}^+ \text{ rel } \mathrm{Q}_0^+]
\end{array}$$

Theorem 6.3. *Let* $[\mathrm{X} \text{ rel } \mathrm{X}_0]$ *be a proper relative braid class, and let* $\mathrm{X}^+ \text{ rel } \mathrm{X}_0^+$, $\mathrm{Q}^+ \text{ rel } \mathrm{Q}_0^+$, $\mathbf{q}^+ \text{ rel } \mathbf{q}_0^+$ *as described above. Then,*

$$HF_{*-2\ell}([\mathrm{X} \text{ rel } \mathrm{X}_0]) \cong HM_*([\mathrm{Q}^+ \text{ rel } \mathrm{Q}_0^+]) \cong HC_*([\mathbf{q}^+ \text{ rel } \mathbf{q}_0^+]), \tag{7}$$

where $\ell \geq 0$ *is chosen such that* $[S^\ell \mathrm{X} \text{ rel } S^\ell \mathrm{X}_0]$ *is a positive braid class (see above).*

This theorem is fundamental for computing the Floer and Morse homology via the discrete invariants.[1] The latter are computable via cubical homology.

7. An example

Consider a skeleton X_0 consisting of two braid components $\mathrm{X}_0 = \{\mathrm{X}_0^1, \mathrm{X}_0^2\}$, with X_0^1 and X_0^2 defined by

$$\mathrm{X}_0^1 = \left\{ r_1 e^{\frac{2\pi n}{m} it}, \cdots, r_1 e^{\frac{2\pi n}{m} i(t-m+1)} \right\}, \quad \mathrm{X}_0^2 = \left\{ r_2 e^{\frac{2\pi n'}{m'} it}, \cdots, r_2 e^{\frac{2\pi n'}{m'} i(t-m'+1)} \right\}.$$

where $0 < r_1 < r_2 \leq 1$, and (n, m) and (n', m') are relatively prime integer pairs with $n \neq 0$, $m \geq 2$, and $m' > 0$. A free strand is given by $\mathrm{X} = \{x(t)\}$, with $x(t) = re^{2\pi \ell it}$, for $r_1 < r < r_2$ and some $\ell \in \mathbb{Z}$, with either $n/m < \ell < n'/m'$ or $n/m > \ell > n'/m'$, depending on the ratios of n/m and n'/m'. A relative braid class $[\mathrm{X} \text{ rel } \mathrm{X}_0]$ is defined via the representative $\mathrm{X} \text{ rel } \mathrm{X}_0$. The associated braid class is proper and the Floer homology is given by

$$FH_k([\mathrm{X} \text{ rel } \mathrm{X}_0]) = \begin{cases} \mathbb{Z}_2 & \text{for } k = 2\ell, 2\ell \pm 1 \\ 0 & \text{otherwise.} \end{cases}$$

The choice of either $2\ell - 1$ or $2\ell + 1$ depends on the cases $n/m < \ell < n'/m'$ or $n/m > \ell > n'/m'$ respectively. From this one derives the existence of non-trivial solutions for any Hamiltonian system for which X_0 are periodic solutions. We can also apply these ideas to diffeomorphisms of the 2-disc. An invariant set A for f, i.e.

$f(A) = A$, can be related to a braid class x_0 via its mapping class. The following result is taken from,[2] not as a novel result, but more as an example.

Theorem 7.1 (Ghrist, VandenBerg, Vandervorst and Wójcik[2]). *Let $f : \mathbb{D}^2 \to \mathbb{D}^2$ be an area-preserving diffeomorphism with invariant set $A \subset \mathbb{D}^2$ having as braid class representative x_0, where $[\mathrm{x}_0]$ is as described above, with $\frac{n}{m} \neq \frac{n'}{m'}$ relatively prime. Then, for for each $l \in \mathbb{Z}$ and $k \in \mathbb{N}$, satisfying*

$$\frac{n}{m} < \frac{l}{k} < \frac{n'}{m'}, \quad \text{or} \quad \frac{n}{m} > \frac{l}{k} > \frac{n'}{m'},$$

there exists a distinct period k orbit of f. In particular, f has infinitely many distinct periodic orbits.

Proof. Since f is area-preserving on $\mathbb{D}^2$, there exists a Hamiltonian H such that $f = \psi_{1,H}$, where $\psi_{t,H}$ the Hamiltonian flow generated by the Hamiltonian system $x_t = X_H(t,x)$ on $(\mathbb{D}^2, \omega_0)$. Up to full twists Δ^2, the invariant set A generates a braid $\psi_{t,H}(A)$ of braid class $[\psi_{t,H}(A)] = [\mathrm{x}_0] \bmod \Delta^2$ with

$$\psi_{t,H}(A) = \tilde{\mathrm{x}}_0 = \{\tilde{\mathrm{x}}_0^1, \tilde{\mathrm{x}}_0^2\}.$$

When $k = 1$, there exists an integer N such that the number of turns in $\tilde{\mathrm{x}}_0^1$ and $\tilde{\mathrm{x}}_0^2$ are related to x_0 as follows: $\frac{\tilde{n}}{m} = \frac{n}{m} + N$ and $\frac{\tilde{n}'}{m'} = \frac{n'}{m'} + N$ respectively. Consider a free strand $\tilde{\mathrm{x}}$ such that $\tilde{\mathrm{x}}$ rel $\tilde{\mathrm{x}}_0 \sim (\mathrm{x}$ rel $\mathrm{x}_0) \cdot \Delta^{2N}$, with $[\mathrm{x}$ rel $\mathrm{x}_0]$ as before and with l satisfying the inequalities above. Geometrically $\tilde{\mathrm{x}}$ turns $l + N$ times around $\tilde{\mathrm{x}}_0^1$ and each strand in $\tilde{\mathrm{x}}_0^2$ turns $\frac{\tilde{n}'}{m'}$ times around $\tilde{\mathrm{x}}$. The Floer homology of $[\mathrm{x}$ rel $\mathrm{x}_0]$ is given above and is non-trivial and by Theorem 6.2, $FH_k([\tilde{\mathrm{x}} \text{ rel } \tilde{\mathrm{x}}_0]) \cong FH_{k-2nN}([\mathrm{x} \text{ rel } \mathrm{x}_0])$. Therefore the Floer homology of $[\tilde{\mathrm{x}}$ rel $\tilde{\mathrm{x}}_0]$ is non-trivial, which implies the existence of a stationary relative braid $\tilde{\mathrm{x}}$ and therefore a fixed point for f.

For the case $k > 1$, consider the Hamiltonain kH; the time-1 map associated with Hamiltonian system $x_t = X_{kH}$ is equal to f^k. The fixed point implied by the proof above descends to a k-periodic point of f. □

A more interesting application is to combine Theorem 6.3 with the discrete calculation of the braid class invariants for the braid class in Fig. 1. For any map $f : \mathbb{D}^2 \to \mathbb{D}^2$ or any Hamiltonian system for which the strands x_0 in Fig. 1 are stationary the Floer homology is given by

$$FH_k([\mathrm{x} \text{ rel } \mathrm{x}_0]) = \begin{cases} \mathbb{Z}_2 & \text{for } k = \#\text{ middle crossings} \\ 0 & \text{otherwise.} \end{cases}$$

In Fig. 1 the crossing at $i = 1$ is a 'middle crossing'. This becomes a choice between three different crossings in a concatenated diagram. As before, by considering iterates f^n of f we obtain an exponential explosion of periodic points (or, periodic solutions for the Hamiltonian system), showing that the topological entropy of f is positive whenever the mapping class of f rel A (invariant set A) is represented by $[\mathrm{x}_0]$.[2]

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