TOPOLOGICAL METHODS FOR NONLINEAR DIFFERENTIAL EQUATIONS

— FROM DEGREE THEORY TO FLOER HOMOLOGY —

R.C.A.M. Vandervorst
This is a self contained set of lecture notes. The notes were written by Rob Vandervorst. These notes are based on the class entitled ‘Topological Methods for Nonlinear Differential Equations’ at the Vrije Universiteit in Amsterdam in the springs of 2005, 2006 and 2008.

This document was produced in \LaTeX{} and the pdf-file of these notes is available on the following website

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**CONTENTS**

I. **Smooth degree theory**

1. Notation
2. The $C^1$-mapping degree
   a. Regular values
   b. Homotopy invariance
   c. The degree for arbitrary values
3. The general homotopy principle
   a. Variations in domains
   b. The index of isolated zeroes
4. The integral representation of the $C^1$-mapping degree
   a. Regular integrals
   b. A general representation
5. Proper mappings
   a. Local and global degree
   b. Proper mappings on open subsets

II. **The Brouwer degree and the axioms of degree theory**

6. The Brouwer degree
7. Properties and axioms for the Brouwer degree
8. Boundary dependence of the degree
   a. Generalized winding numbers
   b. Winding numbers in the plane
9. Mappings between smooth manifolds and the mapping degree
   a. Topological and smooth manifolds
   b. The $C^1$-mapping degree for mappings between manifolds
   c. Local degree and proper mappings
10. The homological definition of the Brouwer degree

III. **Applications of finite dimensional degree theory**

11. The Brouwer fixed point theorem
12. The mapping degree for holomorphic functions 44
13. Linking numbers 47

IV. Extensions of the degree and elementary homotopy theory 50
14. Homotopy types and Hopf’s Theorem 50
15. The extension problem for mappings on a ball 55
16. The general extension problem 56
17. Framed cobordisms 59
18. Pontryagin manifolds 61
18a. Pontryagin manifolds of bounded domains 62
18b. Pontryagin manifolds of smooth boundaries 64
18c. Homotopy types 65
19. Framed cobordism classes and homotopy types 66
19a. Framed cobordism classes as Pontryagin manifolds 66
19b. Pontryagin manifolds and homotopy types 67
19c. The degree isomorphism for n-framed submanifolds 68
19d. The group structure of framed cobordism classes and cohomotopy groups 70

V. The Leray-Schauder degree 71
20. Notation 71
20a. Continuity 72
20b. Differentiability 73
20c. Fredholm mappings and proper mappings 74
21. Compact and finite rank maps 74
22. Definition of the Leray-Schauder degree 75
23. Properties of the Leray-Schauder degree 78
24. Compact homotopies 80
25. Stable cohomotopy 82
26. Semi-linear elliptic equations and a priori estimates 82

VI. Minimax methods 85
27. Palais-Smale functions and compactness 85
28. The deformation lemma 86
29. The linking theorem and minimax characterizations 88
30. Ljusternik-Schnirelmann category and index theory 91
31. Variational principles and critical points 95
32. Existence of solutions 97

VII. Morse theory 101
33. Deformations and homotopy types 101
34. Morse inequalities 104
35. Solutions via Morse Theory 105
36. Multiplicity results for critical points 108
37. Functions lacking compactness 109

VIII. Conley theory 114

IX. Morse-Floer homology 115

X. Appendix 116
   Appendix A. Differentiable mappings 116
      1a. Approximation 116
      1b. The theorem’s of Tietze, Sard and Smale 117
   Appendix B. Basic Nemytskii maps 118
   Appendix C. Sobolev Spaces 121
      3a. Weak derivatives and Sobolov spaces 121
      3b. Sobolev inequalities 123
      3c. Continuous and compact embeddings 126
   Appendix D. Partitions of unity 130
   Appendix E. Homology and cohomology 134
      5a. Simplicial homology 135
      5b. Simplicial cohomology 138
      5c. Definition of De Rham cohomology 138
      5d. Homotopy invariance of cohomology 139
   Index 143
   References 145
I. Smooth degree theory

The mapping degree is a topological tool that can be used to find zeroes of functions defined on a compact domain in \( \mathbb{R}^n \) with values in \( \mathbb{R}^n \). To give an idea consider the functions \( f_1(x, \lambda) = x^3 - 2x^2 + 1 - \lambda \) and \( f_2(x, \lambda) = x^3 - x - \lambda \). In both cases, for \( \lambda = 0 \), the functions have only non-degenerate zeroes. Assign either \( \pm 1 \) to each root depending on the sign of derivative of the function at a zero, and define the degree to be the sum of the \( +1 \)'s and \( -1 \)'s. For \( f_1 \) the degree is equal to zero and for \( f_2 \) the degree is equal to 1. By varying the parameter \( \lambda \), the degree can be computed in most cases, i.e. when the zeroes are all non-degenerate. Notice that for \( f_1 \) the answer is always 0 and for \( f_2 \) the answer is always 1. In the latter case there is always at least one zero, while \( f_1 \) does not need to have zeroes at all. In Section 2 this idea is formalized for \( C^1 \)-functions on \( \mathbb{R}^n \).

1. Notation

Let \( \Omega \subset \mathbb{R}^n \) be a bounded, open subset of \( \mathbb{R}^n \), which will be referred to as a bounded domain. Its closure is denoted by \( \overline{\Omega} \) and the boundary is defined as \( \partial \Omega = \overline{\Omega} \setminus \Omega \). The closure \( \overline{\Omega} \) is a compact set. Points \( x \in \Omega \) are represented in coordinates as follows; \( x = (x_1, \cdots, x_n) \). Super-indices will be used to label points in \( \mathbb{R}^n \).

The class of functions \( f : \overline{\Omega} \subset \mathbb{R}^n \rightarrow \mathbb{R}^n \) that are continuous on \( \overline{\Omega} \) is denoted by \( C^0(\overline{\Omega}; \mathbb{R}^n) \), or \( C^0(\Omega) \) for short. Functions that are continuous on \( \Omega \) are denoted by \( C^0(\Omega; \mathbb{R}^n) \). If \( f : \Omega \subset \mathbb{R}^n \rightarrow \mathbb{R}^n \) is uniformly continuous, then \( f \) can be extended to a continuous function on \( \overline{\Omega} \). Therefore \( C^0(\overline{\Omega}) \subset C^0(\Omega) \), which is also referred to as the subspace of uniformly continuous functions on \( \overline{\Omega} \). A function \( f \) is said to be \( k \)-times continuously differentiable on \( \Omega \) if \( f \) and all its derivatives up to order \( k \) are continuous on \( \Omega \). This class is denoted by \( C^k(\Omega; \mathbb{R}^n) \). A function \( f \) is \( k \)-times continuously differentiable on \( \overline{\Omega} \) if \( f \) and all derivatives up to order \( k \) are uniformly continuous, and thus extend continuously to \( \overline{\Omega} \). The class \( k \)-times continuously differentiable on \( \overline{\Omega} \) is denoted by \( C^k(\overline{\Omega}; \mathbb{R}^n) \).

In order to extend degree theory to unbounded domains an appropriate class of admissible mappings is needed. Let \( \Omega \subset \mathbb{R}^n \) be an unbounded domain. A continuous mapping \( f : \overline{\Omega} \subset \mathbb{R}^n \rightarrow \mathbb{R}^n \) is said to be proper if \( f^{-1}(K) = \{ x \in \overline{\Omega} \mid f(x) \in K \} \) is compact for any compact set \( K \subset \mathbb{R}^n \). Proper mappings are closed, i.e. a mappings \( f \) is called a closed mapping if it maps closed sets \( A \subset \overline{\Omega} \) to closed sets \( f(A) \subset \mathbb{R}^n \).

\[ \text{1.1 Exercise. Show that a proper mapping is a closed mapping.} \]

If \( \Omega \) is a bounded domain, then \( f : \overline{\Omega} \subset \mathbb{R}^n \rightarrow \mathbb{R}^n \) is a proper mapping since \( f^{-1}(K) \subset \overline{\Omega} \) is a closed subset and thus compact. Proper mappings on non-compact domains are therefore a natural extension of continuous mappings on compact domains.
The Jacobian of \( f \in C^1(\Omega) \) at a point \( x \in \Omega \) is defined by \( J_f(x) = \det(f'(x)) \), where \( f'(x) \) is the \( n \times n \) matrix of partial derivatives, i.e. if \( f = (f_1, \cdots, f_n) \), then

\[
f'(x) = \begin{pmatrix}
\frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\
\vdots & \ddots & \vdots \\
\frac{\partial f_n}{\partial x_1} & \cdots & \frac{\partial f_n}{\partial x_n}
\end{pmatrix}.
\]

It is sometimes useful to measure distance in \( \mathbb{R}^n \) using the so-called \( p \)-norms, which are defined as follows; for \( x = (x_1, \cdots, x_n) \in \mathbb{R}^n \),

\[
|x|_p = \left( \sum_i |x_i|^p \right)^{1/p}, \quad 1 \leq p < \infty, \quad \text{and} \quad |x|_\infty = \max_i \{ |x_i| \}.
\]

The latter is also referred to as the supremum norm. It is easy to show, using the fact that \( \mathbb{R}^n \) is finite dimensional, that all these norms are equivalent. Therefore a \( p \)-norm is used which is most convenient, or which is most natural to the setting.

\begin{exercise}
Prove that the \( p \)-norms defined above are all equivalent norms on \( \mathbb{R}^n \). ▶
\end{exercise}

In the case that no subscript is given, \( | \cdot | \) indicates the 2-norm, or Euclidean norm. The 2-norm can be associated to an inner product. For \( x, y \in \mathbb{R}^n \), define \( \langle x, y \rangle = \sum_i x_i y_i \), and \( |x|^2 = \langle x, x \rangle \). The norms given above can also be used to define the notion of distance. For any two points \( x, y \in \mathbb{R}^n \) define the distance to be \( d_p(x, y) = |x - y|_p \). The distance is also referred to as a metric, and \( \mathbb{R}^n \) is a metric space. The distance between a set \( \Omega \) and a point \( x \) is defined by \( d_p(x, \Omega) = \inf_{y \in \Omega} d_p(x, y) \), and more generally, the distance between two sets \( \Omega \) and \( \Omega' \) is then given by \( d_p(\Omega, \Omega') = \inf_{x \in \Omega} \sup_{y \in \Omega'} d_p(x, y) \). The distance is symmetric in \( \Omega \) and \( \Omega' \). If no subscript is indicated, \( d(x, y) \) is the distance associated to the standard Euclidean norm. An open ball in \( \mathbb{R}^n \) of radius \( r \) and center \( x \) is denoted by \( B_r(x) = \{ y \in \mathbb{R}^n \mid |x - y| < r \} \).

Compact subsets \( K \subset \mathbb{R}^n \) in general have special metric properties as a space. The set of compact subsets \( K \subset \mathbb{R}^n \) of \( \mathbb{R}^n \) is denoted by \( \mathcal{H}_{\mathbb{R}^n} \) and for any two sets \( K, K' \in \mathcal{H}_{\mathbb{R}^n} \) the Hausdorff distance is defined by

\[
h(K, K') = \max \{ h_s(K, K') , h^*(K, K') \},
\]

with \( h_s(K, K') = \sup_{\epsilon \in K} \inf_{\epsilon' \in K'} d_p(x, x') \) and \( h^*(K, K') = \sup_{\epsilon \in K} \inf_{\epsilon' \in K'} d_p(x, x') \), the lower and upper semi-metrics respectively. The Hausdorff distance defines a metric on \( \mathcal{H}_{\mathbb{R}^n} \) and \( (\mathcal{H}_{\mathbb{R}^n}, h) \) inherits the metric properties of \( \mathbb{R}^n \). In particular, \( (\mathcal{H}_{\mathbb{R}^n}, h) \) is a complete metric space.

\begin{exercise}
Show that \( h_s(K, K') = \inf \{ \epsilon > 0 \mid K \subset B_\epsilon(K') \} \) and \( h^*(K, K') = \inf \{ \epsilon > 0 \mid K' \subset B_\epsilon(K) \} \), where \( B_\epsilon(K) = \{ x' \mid |x' - x| < \epsilon, \, x \in K \} \). ▶
\end{exercise}

The linear spaces of \( C^k \)-functions can be regarded as a normed space. For \( k = 0 \) the norm is given by

\[
\| f \|_{C^0} = \max_{x \in \Omega} | f(x) |.
\]

and for functions \( f \in C^1 \) the norm \( \| f \|_{C^1} = \| f \|_{C^0} + \max_{1 \leq i \leq n} \| \partial_x f \|_{C^0} \), where \( \partial_x f \) denotes the partial derivative with respect to the \( i \)-th coordinate. The norms for
$k \geq 2$ are defined similarly by considering the higher derivatives in the supremum norm. On these normed linear spaces the norm can be used to define a distance, or metric as explained above for $\mathbb{R}^n$. Since $\overline{\Omega}$ is compact the spaces $C^k(\overline{\Omega})$, equipped with the norms described above are complete and are therefore Banach spaces. For function $f \in C^k(\overline{\Omega})$ the support is defined as the closed set

$$\text{supp}(f) = \{ x \in \overline{\Omega} \mid f(x) \neq 0 \}.$$  

Functions whose support is contained in $\Omega$ are denoted by $C^k_0(\Omega) = \{ f \in C^k(\overline{\Omega}) \mid \text{supp}(f) \subset \Omega \}$, and form a linear subspace of $C^k(\overline{\Omega})$. As matter of fact $C^k_0(\Omega)$ is a closed subspace and therefore again a Banach space with respect to the norm of $C^k(\overline{\Omega})$.

A value $p = f(x)$ is called a regular value of $f$ if $J_f(x) \neq 0$ for all $x \in f^{-1}(p) = \{ y \in \overline{\Omega} \mid f(y) = p \}$, and $p$ is called a critical value if $J_f(x) = 0$ for some $x \in f^{-1}(p)$. The points $x \in f^{-1}(p)$ for which $J_f(x) \neq 0$ are called regular points, and those for which $J_f(x) = 0$ are called critical points. The set of all critical points of $f$, i.e. all points $x \in \overline{\Omega}$ for which $J_f(x) = 0$, is denoted by $\text{Crit}_f(\overline{\Omega})$, or $\text{Crit}_f$ for short.

\[\blacktriangleright\textbf{1.4 Remark.}\] The notions of regular and singular values can also be defined for functions $f : \mathbb{R}^n \to \mathbb{R}^m, n,m \geq 1$. In that case $f'(x)$ replaces the role of the Jacobian, i.e. $p$ is regular if $f'(x)$ is of maximal rank for all $x \in f^{-1}(p)$ and singular if $f'(x)$ is not of maximal rank for some $x \in f^{-1}(p)$. A regular point is therefore a point for which $f'(x)$ is of maximal rank and a singular point is a point for which $f'(x)$ is not of maximal rank. In the special case of functions $f : \mathbb{R}^n \to \mathbb{R}$, the critical points are those points for which $f'(x) = 0$.

\[\blacktriangleright\text{rmk:cldeg-r1}\]

2. The $\mathcal{C}^1$-mapping degree

The definition of the $\mathcal{C}^1$-mapping degree is carried out in two steps. The first step is to define the degree in the generic case — regular values —, and secondly the extension to singular values, using the homotopy invariance of the degree. In Section 4 a direct definition of the $\mathcal{C}^1$-mapping degree is given via an integral representation that does not require a distinction between regular and singular values. Because both approaches are common these two equivalent definitions are explained here.

\[\text{subsec:reg}\]

2a. Regular values. Let $f : \overline{\Omega} \subset \mathbb{R}^n \to \mathbb{R}^n$ be a differentiable mapping, i.e. $f \in \mathcal{C}^1(\overline{\Omega})$, and let $p \in \mathbb{R}^n$ be a regular value, i.e. $f^{-1}(p) \cap \text{Crit}_f = \emptyset$. Since $\overline{\Omega}$ is compact, and $J_f(x)$ is non-zero for all $x \in f^{-1}(p)$, the Inverse Function Theorem implies that $f^{-1}(p)$ is a finite set.

\[\blacktriangleright\textbf{2.1 Exercise.}\] Let $p \notin f(\partial \Omega)$. Show, using the Inverse Function Theorem, that $f^{-1}(p) \subset \Omega$ consists of finitely many isolated points whenever $p$ is a regular value.

\[\blacktriangleright\text{exer:IFT1}\]
2.2 Definition. For a regular value $p \notin f(\partial \Omega)$, define the $C^1$-mapping degree by

$$\deg(f, \Omega, p) := \sum_{x \in f^{-1}(p)} \text{sign}(J_f(x)),$$

which takes values in $\mathbb{Z}$.

2.3 Exercise. Explain that when $p \in f(\partial \Omega)$ the degree is not stable under small perturbations.

2.4 Exercise. (Local continuity/stability of the degree in $p$) Show that if $p$ is regular with $p \notin f(\partial \Omega)$, there exists an $\varepsilon > 0$ such that all $p' \in B_\varepsilon(p)$ are regular values for $f$. Use this to prove that $\deg(f, \Omega, p') = \deg(f, \Omega, p)$ for all $p' \in B_\varepsilon(p)$ with $\varepsilon > 0$ is small enough so that $p' \notin f(\partial \Omega)$ for all $p' \in B_\varepsilon(p)$.

2.5 Exercise. (Local continuity of the degree in $f$) Let $p$ be a regular value for $f$ with $p \notin f(\partial \Omega)$. Show that there exists an $\varepsilon > 0$ such that all for $g \in C^1(\overline{\Omega})$, with $\|f - g\|_{C^1} < \varepsilon$, $p$ is a regular value for $g$. Use this to prove that $\deg(f, \Omega, p) = \deg(g, \Omega, p)$ for all $\|f - g\|_{C^1} < \varepsilon$ with $0 < \varepsilon \leq \frac{1}{2}d(p, f(\partial \Omega))$ small enough so that $p$ is a regular value for all such $g$.

Definition 2.2 of degree was used in the prelude to this chapter and gives a convenient way of computing the mapping degree in the case of regular values $p$. The condition $p \notin f(\partial \Omega)$ is an isolation condition, and makes $\overline{\Omega}$ a set that strictly contains solutions of $f(x) = p$ on $\Omega$, i.e. $\overline{\Omega}$ isolates the solution set $f^{-1}(p)$. This isolation requirement in the definition of degree equips the mapping degree with various robustness properties, see e.g. Exercise 2.3 - 2.5.

The definition yields a number of crucial properties. For the identity map $f = \text{Id}$ the degree is easily computed, i.e. if $p \in \Omega$, then

$$\deg(\text{Id}, \Omega, p) = 1,$$

and for $p \notin \overline{\Omega}$, $\deg(\text{Id}, \Omega, p) = 0$. Another important property that follows immediately from the definition is that the equations $f(x) = p$ and $f(x) - p = 0$ have the same solution set, and $J_f = J_{f - p}$. Therefore

$$\deg(f, \Omega, p) = \deg(f - p, \Omega, 0).$$
If $\Omega^1, \Omega^2 \subset \Omega$ are two disjoint, open subsets, such that $p \not\in f(\Omega \setminus (\Omega^1 \cup \Omega^2))$, then
\begin{equation}
\deg(f, \Omega, p) = \deg(f, \Omega^1, p) + \deg(f, \Omega^2, p)
\end{equation}

\textbf{2.7 Example.} Consider the mapping $f : \mathbb{D}^2 \subset \mathbb{R}^2 \to \mathbb{R}^2$ defined by $f(x_1, x_2) = (2x_1^2 - 1, 2x_1x_2)$. This mapping gives a 2-fold covering of the disc. Figure 2.2 shows that the boundary $\partial \mathbb{D}^2 = S^1$ winds around the origin twice under the image of the map $f$. For the value $(0, 0)$, the pre-image consists of the points $x^1 = (-\frac{1}{2}\sqrt{2}, 0)$ and $x^2 = (\frac{1}{2}\sqrt{2}, 0)$, and
\begin{align*}
f'(x^1) &= \begin{pmatrix} -2\sqrt{2} & 0 \\ 0 & -\sqrt{2} \end{pmatrix}, & f'(x^2) &= \begin{pmatrix} 2\sqrt{2} & 0 \\ 0 & \sqrt{2} \end{pmatrix}.
\end{align*}
Therefore $(0, 0)$ is a regular value for $f$, and since $J_f(x^1) = J_f(x^2) = +1$, the degree is given by $\deg(f, \mathbb{D}^2, 0) = 2$.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{fig1deg2}
\caption{Winding $S^1$ twice around the origin.}
\label{fig:fig1deg2}
\end{figure}

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{fig1deg3}
\caption{Two different orientations with respect to the point $p^1$ and $p^2$.}
\label{fig:fig1deg3}
\end{figure}

\textbf{2.9 Example.} Consider the mapping $f(x_1, x_2) = (2x_1x_2, x_1)$ on $\Omega = \mathbb{D}^2$, and the image points $p^1 = (0, -1/2)$, and $p^2 = (0, 1/2)$. Then, as in Example 2.7, $\deg(f, \mathbb{D}^2, p^1) = -\deg(f, \mathbb{D}^2, p^2) = 1$. The positive degree corresponds to a counter clockwise rotation around $p^1$, and the negative degree corresponds to a clockwise rotation around $p^2$, see Figure 2.3.
fig:figc1deg3

Basically, for regular values $p$, the degree is a count of the elements in $f^{-1}(p)$ with orientation, i.e., a point $x^i \in f^{-1}(p)$ is counted with either $+1$ or $-1$ whenever $f$ is locally orientation preserving or reversing respectively. The degree counts how many times the image $f(\Omega)$ covers $p$ counted with multiplicity. This is a purely local but stable property for regular values, see also Section 5. Of course, whether $p$ is a regular value of a given function $f$ or not is not always straightforward to decide. Sard’s Theorem (see Appendix 1b) claims that a value $p$ is regular with ‘probability’ 1. This fact can be used to extend the definition of degree to arbitrary values $p$ (Chapter II).

\begin{itemize}
  \item 2.11 Remark. A rougher version of degree is the so-called mod-2 degree and is defined as follows: $\deg_2(f, \Omega, p) = \#(f^{-1}(p)) \mod 2$. This degree contains less information than the degree defined in Definition 2.2, but will be of importance for mappings between non-orientable spaces. See [12].
\end{itemize}

rmk:mod2

subsec:ht1

2b. Homotopy invariance. A crucial property of the $C^1$-mapping degree is the homotopy invariance with respect to $f$. Large perturbations $f$ which do not destroy the isolation along the homotopy leave the degree unchanged.

\begin{itemize}
  \item 2.12 Lemma. Let $t \mapsto f_t$, $t \in [0, 1]$ be a continuous path in $C^1(\overline{\Omega})$, with $p \not\in f_t(\partial\Omega)$ for all $t \in [0, 1]$ and let $p$ be a regular value for both $f_0$ and $f_1$. Then $\deg(f_0, \Omega, p) = \deg(f_1, \Omega, p)$.
\end{itemize}

lem:pert1a

Proof: Let $F(t, \cdot) = f_t$ and consider the equation $F(t, x) = p$. By assumption $p$ is a regular value for both $f_0$ and $f_1$. From Theorem A.3 it follows that $F$ can be approximated arbitrarily close in $C^0$ by a function $\tilde{F} \in C^\infty([0, 1] \times \overline{\Omega})$ such that $\tilde{f}_t = F(t, \cdot)$ is arbitrary close to $f_t$ in $C^1$, uniformly in $t \in [0, 1]$. By Sard’s Theorem (see Theorem A.5) we can choose value $p'$ arbitrary close to $p$ which is regular for both $F$ and $\tilde{F}$. By the local stability of the degree (Exercise 2.4) there exists an $\varepsilon > 0$ such that $\deg(f_0, \Omega, p') = \deg(f_0, \Omega, p)$ and $\deg(f_1, \Omega, p') = \deg(f_1, \Omega, p)$ for all $p' \in B_\varepsilon(p)$. Using the local continuity of the degree, see Exercise 2.5, there exists a $\delta > 0$ such that $\deg(f_0, \Omega, p') = \deg(f_0, \Omega, p)$ and $\deg(f_1, \Omega, p') = \deg(f_1, \Omega, p)$ for all $p' \in B_\delta(p)$, and all $\max_{t \in [0, 1]} \|\tilde{f}_t - f_t\|_{C^1} < \delta$. For a regular value $p'$ the solution set $\tilde{F}^{-1}(p')$ of the equation

\[ \tilde{F}(t, x) = p', \]

is a smooth 1-dimensional manifold with boundary given by $\partial \tilde{F}^{-1}(p') = [\tilde{f}_0^{-1}(p') \times \{0\}] \cup [\tilde{f}_1^{-1}(p') \times \{1\}]$ (see Appendix 1b). Since $p \not\in f_t(\partial\Omega)$ it holds that $p' \not\in \tilde{f}_1(\partial\Omega)$, consequently $\tilde{F}^{-1}(p') \subset [0, 1] \times \Omega$. Therefore, the 1-dimensional components diffeomorphic to $[0, 1]$ are curves connecting elements in $\partial \tilde{F}^{-1}(p')$ and components diffeomorphic to $S^1$ are contained in $(0, 1) \times \Omega$, since $p'$ is a regular value for both $\tilde{f}_0$ and $\tilde{f}_1$. It’s worth mentioning that by the Transversality Theorem (see Appendix 1b) $p'$ is a regular value for $\tilde{f}_t$ for almost every $t \in [0, 1]$. 

The manifold $\tilde{F}^{-1}(p')$ can be given a canonical orientation as follows. Since $p'$ is regular the matrix

$$F'(t,x) = \begin{pmatrix} \frac{\partial}{\partial t} F_1 & \frac{\partial}{\partial x_1} F_1 & \cdots & \frac{\partial}{\partial x_n} F_1 \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial}{\partial t} F_n & \frac{\partial}{\partial x_1} F_n & \cdots & \frac{\partial}{\partial x_n} F_n \end{pmatrix},$$

has maximal rank. The rows define the column vectors $\xi_j, j = 1, \ldots, n$ and the tangent space $T\tilde{F}^{-1}(p')$ is spanned by $X = X(t,x)$ The vector $X$ satisfies $F'(t,x)X = 0$ and $X \perp \xi$. Consider the $(n+1)$-form $dx = dt \wedge dx_1 \wedge \cdots \wedge dx_n$. Then the 1-form $\alpha = dx(\xi_1, \ldots, \xi_n)$ defines an orientation on $\tilde{F}^{-1}(p')$. The vector $X(t,x)$ can be identified with $\alpha = X_0 dt + X_1 dx_1 + \cdots X_n dx_n$, and $\alpha(X) = |X|^2$. The component $X_0$ in the $t$-direction is given by

$$X_0(t,x) = \alpha(e_0) = f(t).$$

**Figure 2.4.** The opposite points connected by a curve have the same sign of the Jacobian, and the points at $t = 0$, or $t = 1$ connected by a curve have opposite sign of the Jacobian. The same holds for any regular section.

then $J_{\tilde{f}_0}(x)$ and $J_{\tilde{f}_1}(x')$ have opposite signs by the induced orientation $\alpha$, and do not contribute to the sum $\sum_{x \in \tilde{f}_0^{-1}} \text{sign}(J_{\tilde{f}_0}(x))$. For two points $x \in \tilde{f}_0^{-1}(p')$ and $x' \in \tilde{f}_1^{-1}(p')$ connected by a curve in $\tilde{F}^{-1}(p')$, it holds that $J_{\tilde{f}_0}(x)$ and $J_{\tilde{f}_1}(x')$ have the same sign by the induced orientation $\alpha$. Since all points in $\partial \tilde{F}^{-1}(p')$ are connected, the contributing terms in $\tilde{f}_0^{-1}$ and $\tilde{f}_1^{-1}$ are in one-to-one correspondence and the Jacobians have the same signs. It immediately follows now that

$$\sum_{x \in \tilde{f}_0^{-1}} \text{sign}(J_{\tilde{f}_0}(x)) = \sum_{x' \in \tilde{f}_1^{-1}} \text{sign}(J_{\tilde{f}_1}(x')),$$

which proves the homotopy property.
2.14 Remark. If the assumption \( p \not\in f_t(\partial \Omega) \), for all \( t \in [0,1] \), is removed the above proof may fail at a number of points. Most important to mention in this context is that without the isolation property the points in \( \partial \tilde{F}^{-1}(p') \) are not necessarily connected by a curve and the contributing terms in \( \tilde{f}_0^{-1} \) and \( \tilde{f}_1^{-1} \) are not necessarily in one-to-one correspondence, see also Exercise 2.3.

Let \( D \subset \mathbb{R}^n \backslash f(\partial \Omega) \) be any connected component,\(^1\) then the degree \( \deg(f, \Omega, p) \) is independent of \( p \in D \). This easily follows from the homotopy principle.

2.15 Lemma. For any curve \( t \mapsto p_t \in D, t \in [0,1] \), with \( p_0 \) and \( p_1 \) regular values, it holds that \( \deg(f, \Omega, p_0) = \deg(f, \Omega, p_1) \).

Proof: From Equation (2.2) it follows that \( \deg(f, \Omega, p_0) = \deg(f - p_0, \Omega, 0) \), and \( \deg(f, \Omega, p_1) = \deg(f - p_0, \Omega, 0) \). It holds that \( p_t \in D \) if and only if \( p_t \not\in f(\partial \Omega) \). The homotopy \( f_t = f - p_t \) therefore satisfies the requirements of Lemma 2.12, and

\[
\deg(f, \Omega, p_0) = \deg(f - p_0, \Omega, 0) = \deg(f - p_0, \Omega, 0) = \deg(f, \Omega, p_1),
\]

which proves the statement.

2.16 Example. Consider the mapping \( f(x, y) = (x^2, y) \) on the the standard 2-disc \( D^2 \) in the plane. The image of \( D^2 \) under \( f \) is the ‘folded pancake’ \( f(D^2) = \{ p = (p_1, p_2) \in \mathbb{R}^2 \mid p_1 + p_2^2 = 1, p_1 \geq 0 \} \). The image of the boundary \( S^1 = \partial D^2 \) is homeomorphic to a semi-circle and \( \mathbb{R}^2 \backslash f(D^2) \) is connected. Note that \( f(\partial D^2) \neq \partial f(D^2) \). By the homotopy invariance the degree can be evaluated by choosing any \( p \in \mathbb{R}^2 \backslash f(\partial D^2) \). Since \( D^2 \) is compact, so is the image. We can therefore choose a value \( p^1 \in \mathbb{R}^2 \backslash f(\partial D^2) \) which does not lie in \( f(D^2) \). This implies that \( \deg(f, D^2, p) = 0 \). If we choose \( p^2 = (1/4, 0) \), then \( f^{-1}(p^2) = \{ (\pm 1/2, 0) \} \), which gives a positive and a negative determinant. The sum is zero which confirms the previous calculation.

\[
\begin{array}{c}
\text{Figure 2.5. The disc is folded to the right half plane and the} \\
\text{boundary of the image is not given by } f(\partial D^2).
\end{array}
\]

\(^1\)Open subsets of \( \mathbb{R}^n \) are connected if and only if they are path-connected.
If we choose a path \( t \mapsto p_t \), connecting the regular values \( p_1 \) and \( p_2 \) and which lies in \( \mathbb{R}^2 \backslash f(\mathbb{D}^2) \), then \( p_t \) crosses the boundary \( \partial f(\mathbb{D}^2) \) in the vertical. However, \( p_t \notin f(\mathbb{D}^2) \) for all \( t \in [0,1] \) and the pre-image \( f^{-1}(p_t) \in \mathbb{D}^2 \) for all \( t \in [0,1] \). The values in \( f(\mathbb{D}^2) \) on the vertical are necessarily singular. This again shows that the boundary of the image should not be considered as a restriction on \( p \). In the next subsection we show that the degree is defined for all \( p \) in \( \mathbb{R}^2 \backslash f(\mathbb{D}^2) \).

The previous example yields the following property of the mapping degree.

\[2.18 \text{Lemma.} \] Suppose that \( \mathbb{R}^n \backslash f(\partial \Omega) \) is connected, then for any regular value \( p \in \mathbb{R}^n \backslash f(\partial \Omega) \) holds that \( \deg(f, \Omega, p) = 0 \).

Proof: See Example 2.16.

\[2.19 \text{Definition.} \] Let \( p \in D \), with \( D \) a connected component of \( \mathbb{R}^n \backslash f(\partial \Omega) \). Then

\[ \deg(f, \Omega, p) := \deg(f, \Omega, p'), \]

for any regular value \( p' \in D \) and thus \( \deg(f, \Omega, p) = \deg(f, \Omega, D) \).

By Sard’s Theorem (Appendix 1b, Theorem A.5) the regular values in \( D \) lie dense in \( D \). By Lemma 2.15 the choice of regular value \( p' \) does not matter and therefore the extension of the degree as given by Definition 2.19 is well-defined. The properties of the generic degree listed in Equations (2.1) - (2.3) and Lemma 2.12 also hold for the general \( C^1 \)-mapping degree and are the fundamental axioms that define a degree theory, see Section 7.

\[2.20 \text{Theorem.} \] The degree function \( \deg(f, \Omega, p) \) in Definition 2.19 satisfies the following axioms:

(A1) if \( p \in \Omega \), then \( \deg(\text{Id}, \Omega, p) = 1 \);
(A2) for \( \Omega^1, \Omega^2 \subset \Omega \), disjoint open subsets of \( \Omega \) and \( p \notin f(\Omega^1 \cup \Omega^2) \), it holds that \( \deg(f, \Omega, p) = \deg(f, \Omega^1, p) + \deg(f, \Omega^2, p) \);
(A3) for any continuous path \( t \mapsto f_t, f_t \in C^1(\Omega) \), with \( p \not\in f_t(\partial \Omega) \), it holds that \( \deg(f_t, \Omega, p) \) is independent of \( t \in [0,1] \);
(A4) \( \deg(f, \Omega, p) = \deg(f - p, \Omega, 0) \).

The application \( (f, \Omega, p) \mapsto \deg(f, \Omega, p) \) is called a \( C^1 \)-degree theory.

Proof: Axiom (A1) follows immediately from Equation (2.1). As for Axiom (A2), by assumption, \( f^{-1}(p) \subset \Omega^1 \cup \Omega^2 \) and therefore \( f^{-1}(p') \subset \Omega^1 \cup \Omega^2 \) for any regular value \( p' \) sufficiently close to \( p \). Consequently,

\[
\deg(f, \Omega, p) = \deg(f, \Omega, p') = \deg(f, \Omega^1, p') + \deg(f, \Omega^2, p') = \deg(f, \Omega^1, p) + \deg(f, \Omega^2, p).
\]
Choose a value \( p' \) that is regular for both \( f_0 \) and \( f_1 \). If \( p' \) is chosen sufficiently close to \( p \), then \( p' \not\in f_i(\partial \Omega) \), and thus by Lemma 2.12 and Definition 2.19

\[
\deg(f_0, \Omega, p) = \deg(f_0, \Omega, p') = \deg(f_1, \Omega, p') = \deg(f_1, \Omega, p).
\]

By considering the homotopy \( t \mapsto f_{i_0} \) it follows that \( \deg(f_0, \Omega, p) = \deg(f_0, \Omega, p) \), for any \( t_0 \in [0, 1] \), which proves Axiom (A3). Finally, let \( p' \) be a regular value sufficiently close to \( p \), then by Equation (2.3), \( \deg(f, \Omega, p) = \deg(f, \Omega, p') = \deg(f - p', \Omega, 0) \). Consider the homotopy \( f_t = (1 - t)(f - p) + t(f - p') = f - (1 - t)p - tp' \). Since \( p' \) is close to \( p \), the line-segment \( \{(1 - t)p + tp'\}_{t \in [0, 1]} \) does not intersect \( f(\partial \Omega) \), and therefore \( 0 \not\in f_i(\partial \Omega) \). From Axiom (A3) it then follows that

\[
\deg(f, \Omega, p) = \deg(f, \Omega, p') = \deg(f - p', \Omega, 0) = \deg(f - p, \Omega, 0),
\]

which proves Axiom (A4), and thereby completing the proof.

\[\square\]

3. The general homotopy principle

The homotopy invariance established in the previous section allows for deformations in both \( f \) and \( p \). Using the axioms of a degree theory one can prove that the domain \( \Omega \) can also be varied. In Section 7 the homotopy principle will be derived from the axioms. In this section a direct proof using the definition will be given.

3a. Variations in domains. Let \( \Omega \subset \mathbb{R}^n \times [0, 1] \) be bounded and relatively open subset of \( \mathbb{R}^n \times [0, 1] \). Define the \( t \)-slices by

\[
\Omega_t = \{x \mid (x, t) \in \Omega\}, \quad t \in [0, 1]
\]

and their boundaries by \( (\partial \Omega)_t = \{x \mid (x, t) \in \partial \Omega\} = (\overline{\Omega})_t \setminus \Omega_t \).

Note that \( \overline{\Omega}_t \subset (\overline{\Omega})_t \), which is essential for the definition of \( (\partial \Omega)_t \).

\begin{definition}
Two triples \( (f, \Omega_f, p) \) and \( (g, \Omega_g, q) \) are said to be homotopic, or cobordant, if there exists a bounded and relatively open subset \( \Omega \subset \mathbb{R}^n \times [0, 1] \) and a continuous function \( F: \overline{\Omega} \to \mathbb{R}^n \), with \( f_t = F(\cdot, t) \in C^1(\overline{\Omega}_t) \), such that

(i) \( f_0 = f \), and \( f_1 = g \);
(ii) \( \Omega_0 = \Omega_f \), and \( \Omega_1 = \Omega_g \);
(iii) there exists a continuous path \( t \mapsto p_t \), such that \( p_t \not\in f_i(\partial \Omega)_t \) for all \( t \in [0, 1] \), and \( p_0 = p \) and \( p_1 = q \).

Notation \( (f, \Omega_f, p) \sim (g, \Omega_g, q) \). Triples \( (f, \Omega_f, D_f) \) and \( (g, \Omega_g, D_g) \), for which the above requirements are met, are also called homotopic.
\end{definition}
\begin{align*}
\textbf{3.2 Theorem.} \text{ Let } (f, \Omega_f, p) \text{ and } (g, \Omega_g, q) \text{ be homotopic triples, then}
\deg(f, \Omega_f, p) = \deg(g, \Omega_g, q).
\end{align*}

In particular, \( \deg(f_t, \Omega_t, p_t) \) is constant in \( t \in [0, 1] \), where \( f_t \) is a homotopy as defined in Definition 3.1.

\textbf{Proof:} From the definition of the degree we can choose values \( p' \) and \( q' \) which are regular values of \( f \) and \( g \) respectively. The values \( p' \) and \( q' \) can be chosen arbitrary close to \( p \) and \( q \). Then \( \deg(f, \Omega_f, p) = \deg(f, \Omega_f, p') \) and \( \deg(g, \Omega_g, q) = \deg(g, \Omega_g, q') \). Let \( p'_t \) be a continuous path in \( \Omega \) connecting \( p' \) and \( q' \) and which lies in an \( \varepsilon \)-neighborhood of the path \( p_t \). Therefore, if \( \varepsilon \) is chosen small enough, \( p'_t \notin f_t((\partial \Omega)_t) \), for all \( t \in [0, 1] \). By Axiom (A4) it holds that

\[ \deg(f_t, \Omega_t, p'_t) = \deg(f_t - p'_t, \Omega_t, 0), \]

for all \( t \in [0, 1] \). Now consider the equation

\[ G(t, x) = F(t, x) - p'_t. \]

The proof now follows along the same lines as the proof of Lemma 2.12. The only difference is the domain \( \Omega \). By assumption, the solution set \( G^{-1}(0) \) is contained in \( \Omega \), i.e. \( G^{-1}(0) \cap (\partial \Omega)_t = \emptyset \) for all \( t \in [0, 1] \). Choose a \( C^\infty \)-perturbation \( \tilde{F} \), which yields \( \tilde{G} = \tilde{F} - p'_t \). By Sard’s Theorem choose a regular value \( 0' \), arbitrary close to 0, and the solution set \( \tilde{G}^{-1}(0') \) can be described in exactly the same way as in the proof of Lemma 2.12. Figure 3.1 below shows the slightly different situation with Lemma 2.12. The fact that \( \deg(f_t, \Omega_t, p_t) \) is constant in \( t \) in the same way as Axiom (A3).
\[\textbf{3.4 Remark.}\] For \( t \in [0, 1],\) let \( D_t \) be the connected component of \( \mathbb{R}^n \setminus f_t((\partial \Omega)_t) \) containing \( p_t.\) Then the result of Theorem 3.2 can be reformulated as

\[
\begin{equation}
\deg(f_t, \Omega_t, D_t) = \text{const.},
\end{equation}
\]

which establishes continuity of the degree in \( f, \Omega \) and \( p.\)

\[\textbf{3b. The index of isolated zeroes.}\] It the case that a mapping has only isolated zeroes, and thus finitely many, Property (A3) gives the degree as a sum of the local degrees. More precisely, let \( x^i \in \Omega \) be the zeroes of \( f \) and let \( \Omega^i \subset \Omega \) be sufficiently small small neighborhoods of \( x^i \in \Omega^i, \) such that \( x^i \) the only solution of \( f(x) = p \) in \( \Omega^i \) for all \( i.\) Then \( \deg(f, \Omega, p) = \sum_i \deg(f, \Omega^i, p) \) and we define

\[
t(f, x^i, p) := \deg(f, \Omega^i, p),
\]

which is called the index of an isolated zero of \( f.\) The index for isolated zero does not depend on the domain \( \Omega^i.\) Indeed, if \( \Omega^i \) and \( \tilde{\Omega}^i \) are both neighborhoods of \( x^i \) for which \( x^i \) is the only zero of \( f(x) = p, \) then we can define a cobordism between \( (f, \Omega^i, p) \) and \( (f, \tilde{\Omega}^i, p) \) as follows. Let \( \Omega = \bigcup_{\epsilon \in [0,1]} \Omega_{\epsilon} \) with

\[
\Omega_{\epsilon} = \begin{cases} 
\Omega^i & \text{for } \epsilon < \frac{1}{2}, \\
\Omega^i \cap \tilde{\Omega}^i & \text{for } \epsilon = \frac{1}{2}, \\
\tilde{\Omega}^i & \text{for } \epsilon > \frac{1}{2},
\end{cases}
\]

and \( f_{\epsilon} = F(\cdot, \epsilon) = f, \) \( p_{\epsilon} = p.\) By Theorem 3.2 \( \deg(f, \Omega^i, p) = \deg(f, \tilde{\Omega}^i, p). \) The expression for the degree becomes

\[
\begin{equation}
\deg(f, \Omega, p) = \sum_{x \in f^{-1}(p)} t(f, x, p).
\end{equation}
\]

\[\textbf{3.5 Exercise.}\] Show that if \( x \in f^{-1}(p) \) is a non-degenerate zero of \( f, \) then \( t(f, x, p) = (-1)^{\beta}, \) where \( \beta = \#\{\text{negative real eigenvalues}\} \) (counted with multiplicity).

\[4. \text{The integral representation of the } C^1\text{-mapping degree}\]

The expression for the \( C^1\)-mapping degree for regular values points to an obvious integral definition of the degree which allows for a formulation of of the \( C^1\)-degree without distinguishing between regular and singular values. The integral formulation is also useful sometimes for establishing various properties.

\[\textbf{4.1a. Regular integrals.}\] Let \( \omega : \mathbb{R}^n \to \mathbb{R} \) be a continuous function with \( \text{supp}(\omega) = B_\varepsilon(p). \) Choose \( \varepsilon > 0 \) small enough such that \( \text{supp}(\omega) \subset \mathbb{R}^n \setminus f(\partial \Omega) \) and is a coordinate neighborhood of \( p \) with respect to the change of coordinates \( p = f(x), \) see Figure 2.1. The weight function can be normalized by

\[
\int_{\mathbb{R}^n} \omega(x)dx = 1.
\]

A function \( \omega \) that satisfies the above conditions is called a weight function, or test function. In the calculations that follow it is convenient to use the notation of
differential forms on \( \mathbb{R}^n \). Write \( dx = dx_1 \wedge \cdots \wedge dx_n \) as the standard \( n \)-form on \( \mathbb{R}^n \) and consider the differential \( n \)-forms
\[
\omega = \omega(y) dy, \quad \text{and} \quad f^* \omega = \omega(f(x)) J_f(x) dx.
\]
The latter is called the pullback under \( f \), where \( y = f(x) \). The \( n \)-form \( dx \) provides \( \mathbb{R}^n \) with a standard orientation. With this notation a lot of the calculations simplify considerably. The space of compactly supported continuous \( n \)-forms on \( \mathbb{R}^n \) is denoted by \( \Gamma^n_c(\mathbb{R}^n) \).

\[ \textbf{4.1 Lemma.} \] Let \( p \notin f(\partial \Omega) \) be a regular value and \( \omega \) a weight function as defined above. Then the integral \( I \) represents the \( C^1 \)-mapping degree;
\[
I = \int_\Omega f^* \omega = \deg(f, \Omega, p).
\]

**Proof:** As pointed out before \( f^{-1}(p) \) is a finite set strictly contained in \( \Omega \). Since \( J_f \) is non-zero at points in \( f^{-1}(p) = \{ x^1, \ldots, x^k \} \), the Inverse Function Theorem gives that \( f \) maps neighborhoods \( N_\epsilon(x^j) \) of points in \( x^j \in f^{-1}(p) \) diffeomorphically onto \( B_\epsilon(p) \), see Figure 2.1. Thus \( f \) is a local change of coordinates near every point in \( x^j \in f^{-1}(p) \). Indeed, the fact that \( J_f \) is non-zero at points in \( x^j \in f^{-1}(p) \), implies that \( J_f \) is also non-zero at the points in \( N_\epsilon(x^j) \), provided that \( \epsilon \) is small enough. The integral \( I \) splits in \( k \) local integrals
\[
\int_\Omega f^* \omega = \sum_j \int_{N_\epsilon(x^j)} f^* \omega = \sum_j \sign(J_f(x^j)) \int_{B_\epsilon(p)} \omega
\]
which proves that both \( I \) is independent of \( \omega \) and represents the degree defined in Definition 2.2. The above calculation uses that locally \( f \) is a coordinate transformation \( y = f(x) \) and \( \int \omega(f(x)) J_f(x) dx = \sign(J_f(x^j)) \int \omega(y) dy \).

\[ \textbf{4.2 Exercise.} \] Verify the above change of coordinates formula given by \( y = f(x) \).

\[ \textbf{4.3 Remark.} \] If in the above lemma we choose weight functions \( \omega \) with the property that \( \int_\Omega \omega \neq 0 \), then
\[
\deg(f, \Omega, p) \cdot \int_{\mathbb{R}^n} \omega = \int_\Omega f^* \omega.
\]
See also Remark 4.15.

4b. **A general representation.** The integral characterization of the degree in the generic case motivates a representation of the \( C^1 \)-degree in general, i.e. regardless whether \( p \) is regular or not. In order for the integral representation in (4.1) to serve as a definition of degree for general \( p \), the independence on \( \omega \) needs to be established. As before let \( \omega \) be a continuous weight function on \( \mathbb{R}^n \) with the properties
\[
\text{supp}(\omega) \subset D \subset \mathbb{R}^n \setminus f(\partial \Omega), \quad \text{and} \quad \int_{\mathbb{R}^n} \omega = 1,
\]
where \( D \) is the connected component of \( \mathbb{R}^n \setminus f(\partial \Omega) \) containing \( p \). The first property allows for a larger class of weight functions in the sense that \( \text{supp}(\omega) \) is not necessarily a local coordinate neighborhood of \( p \). The space of continuous \( n \)-forms \( \omega = \omega(x)dx, \) with \( \text{supp}(\omega) \subset D \), are denoted by \( \Gamma^n_0(D) \). For \( \omega \in \Gamma^n_0(D) \) and \( \int_D \omega = 1 \) we define the integral over \( \Omega \) by

\[
I(f, \Omega, D) := \int_\Omega f^* \omega.
\]

The notation is justified by the following lemmas which show that the integral does not depend on \( \omega \), but does depend on in which component \( D \) its support lies. Moreover, we establish that \( I \) is integer valued. For regular \( p \in D \), and \( \text{supp}(\omega) = B_r(p) \), a local coordinate neighborhood, the integral representation in \eqref{eqn:indef1} is retrieved.

\section*{4.4 Lemma.} Let \( \omega, \omega' \in \Gamma^n_0(D) \) be two compactly supported \( n \)-forms on \( D \), with \( \int_D \omega = \int_D \omega' = 1 \) and \( \text{supp}(\omega), \text{supp}(\omega') \subset K^n \subset D \), where \( K^n \) is an \( n \)-dimensional cube. Then

\[
\int_\Omega f^* \omega = \int_\Omega f^* \omega'.
\]

\section*{4.5 Lemma.} Let \( \mu \) be a compactly supported \( n \)-form on \( \mathbb{R}^n \) with \( \int_{\mathbb{R}^n} \mu = 0 \) and \( \text{supp}(\mu) \subset K^n \). Then there exists a compactly supported \( (n-1) \)-form \( \theta \) on \( \mathbb{R}^n \), with \( \text{supp}(\theta) \subset K^n \) such that \( \mu = d\theta \), where

\[
\theta = \sum_{i=1}^n (-1)^{i-1} \chi_i(x) dx_1 \wedge \cdots \wedge \hat{dx}_i \wedge \cdots \wedge dx_n,
\]

with \( \chi_i \in C^1_0(\mathbb{R}^n) \), and \( \text{supp}(\chi_i) \subset K^n \).

\section*{4.6 Lemma.} Establishing \( \mu = d\theta \) is equivalent to finding a vector field \( \chi \) such that \( \mu = \text{div}\chi \). Indeed, in terms of differential forms, if we set \( \mu' = \mu(x)dx \), then

\[
\theta = \sum_{i=1}^n (-1)^{i-1} \chi_i(x) dx_1 \wedge \cdots \wedge \hat{dx}_i \wedge \cdots \wedge dx_n.
\]

\begin{table}
\end{table}

\section*{5. Fig-support}

For \( n = 1 \) we take \( \chi(x) = \int^x \mu(s)ds \). Suppose the above statement is true in dimension \( n-1 \). Write \( x = (y, x_n) \), where \( y = (x_1, \ldots, x_{n-1}) \), and define \( \alpha(y) = \int_{\mathbb{R}} \mu(y, x_n) dx_n \). By the induction hypothesis \( \alpha \) is of divergence form, i.e. \( \alpha = \text{div}\xi \), for some vector field \( \xi \), with \( \text{supp}(\xi) \subset K^{n-1} \). Let \( \tau \in C^\infty(\mathbb{R}) \) with \( \text{supp}(\tau) \subset K \) and define

\[
\chi_n(y, x_n) = \int_{-\infty}^{x_n} (\mu(y, z) - \tau(z) \alpha(y)) dz.
\]

\footnote{As for a function on \( \mathbb{R}^n \) the support of a \( k \)-form \( \lambda \) is given as the set of points \( \text{supp}(\lambda) = \{ x \in \mathbb{R}^n | x \neq 0 \} \).}
By Construction $\text{supp}(\chi_n) \subset K^n$, and $\frac{\partial \chi_n}{\partial x_n} = \mu(x) - \tau(x_n)\alpha(y)$. Now let

$$\chi(x) = (\xi(y)\tau(x_n), \chi_n(y,x_n)),$$

then

$$\text{div}\chi(x) = \tau(x_n)\text{div}\xi(y) + \frac{\partial \chi_n}{\partial x_n}(x) = \tau(x_n)\alpha(y) + \mu(x) - \tau(x_n)\alpha(y) = \mu(x),$$

and $\text{supp}(\chi) \subset K^n$. \hfill \blackslug

Now apply Lemma 4.5 to the form $\mu = \omega' - \omega$ with support in $K^n \subset D$. Therefore, $\mu = \omega' - \omega = d\theta$ for some compactly supported $(n-1)$-form $\theta$. Moreover, the support of the form $\theta$ is contained in $K^n \subset D$.

**Independence.** The cube $K^n \subset D$ has a piecewise smooth boundary and therefore by Stokes’ Theorem

$$\int_{\Omega} f^*\omega' - \int_{\Omega} f^*\omega = \int_{\Omega} f^*(\omega' - \omega) = \int_{\Omega} f^*\mu = \int_{\Omega} f^*d\theta = \int_{f^{-1}(K^n)} f^*d\theta = \int_{\partial f^{-1}(K^n)} f^*\theta = 0,$$

since $\text{supp}(f^*\theta) \subset f^{-1}(K^n) \subset \Omega$. This proves the lemma. \hfill \blackslug

**4.7 Exercise.** Check, using differential forms calculus, that $f^*d\theta = d(f^*\theta)$ (Hint: show this first for $C^2$-functions).

**4.8 Remark.** The first step of the proof of Lemma 4.4 holds for any two $n$-forms $\omega$ and $\omega'$ with compact support. This is essentially the Poincaré Lemma as given by Lemma 4.5. The second step is Stokes’ Theorem and uses in an essential way that both $\omega$ and $\omega'$ have their supports in $D$. If the latter does not hold then the integrals $\int_{\Omega} f^*\omega$ and $\int_{\Omega} f^*\omega'$ are not necessarily equal. As a consequence it will become...
clear from the forthcoming discussion that \( p \notin f(\partial \Omega) \) is an essential condition in order for the integral definition of the degree to hold.

\[ \int f^* \omega = \deg(f, \Omega, p) \in \mathbb{Z}, \]

for any regular value \( p \in \text{supp}(\omega) \).

**Proof:** By Sard’s Theorem choose a regular value \( p \in \text{supp}(\omega) \) and choose a coordinate neighborhood \( B_\varepsilon(p) \subset K^n \). As before choose \( \omega' \) with \( \text{supp}(\omega') = B_\varepsilon(p) \).

From Lemma 4.1 it follows that \( \int f^* \omega' = \deg(f, \Omega, p) \) and from Lemma 4.4

\[ \int f^* \omega = \int f^* \omega' = \deg(f, \Omega, p), \]

which proves the lemma.

The next lemma shows that for any form \( \omega \in \Gamma_c^n(D) \), with support in some cube \( K^n \subset D \), the integrals are the same.

**Lemma 4.10** Let \( \omega, \omega' \in \Gamma_c^n(D) \) be two compactly supported \( n \)-forms on \( D \), with \( \int_{\partial \Omega} \omega = \int_{\partial \Omega} \omega' = 1 \), and \( \text{supp}(\omega) \subset K^n \subset D \) and \( \text{supp}(\omega') \subset K'' \subset D \). Then

\[ \int f^* \omega = \int f^* \omega', \]

and therefore \( \int f^* \omega \) does not depend on \( p \in D \), but only on the connected component \( D \).

**Proof:** Choose two balls \( B_\varepsilon(p) \subset \text{supp}(\omega) \) and \( B_\varepsilon(p') \subset \text{supp}(\omega') \) and a curve \( \gamma \subset D \) connecting \( p \) and \( p' \). Cover \( \gamma \) by finitely many small balls \( B_\varepsilon(p), \) with \( j = 1, \cdots, k \), such that for any two consecutive balls it holds that

\[ B_{\varepsilon_j}(p_j) \cup B_{\varepsilon_{j+1}}(p^{j+1}) \subset K_j \subset D, \]

for some cube \( K_j \). Let \( \omega^j \) be forms with \( \text{supp}(\omega^j) = B_\varepsilon(p^j) \). Then by Lemma 4.4

\[ \int f^* \omega^j = \int f^* \omega^{j+1}, \]

and therefore

\[ \int f^* \omega = \int f^* \omega^1 = \cdots = \int f^* \omega^k = \int f^* \omega', \]

which proves the lemma.

Lemma 4.10 justifies the notation \( I(f, \Omega, D) \) and by Lemma 4.9 the integral is integer valued. In particular, the above considerations prove that:
\begin{lemma}
For any regular values \( p, p' \in D \subset \mathbb{R}^n \setminus f(\partial \Omega) \) it holds that
\[
\deg(f, \Omega, p) = \deg(f, \Omega, p'),
\]
and \( I(f, \Omega, D) = \deg(f, \Omega, p) \) for any regular value \( p \in D \).
\end{lemma}

It is clear from the previous considerations that the degree is independent of \( p \in D \) and coincides with the definition of degree in the regular case; Definition 2.2.

The advantage of the integral representation is that a lot of properties of the degree can be obtained via fairly simple proofs. The final step is to show that one can use any compactly supported form \( \omega \in \Gamma_c^n(D) \) to represent the mapping degree.

\begin{lemma}
Let \( \omega \in \Gamma_c^n(D) \) with \( \int_{\mathbb{R}^n} \omega = 1 \). Then \( \int_{\Omega} f^* \omega = I(f, \Omega, D) \).
\end{lemma}

\textbf{Proof:}
Since \( \text{supp}(\omega) \subset D \) is compact there exists a finite covering of open balls \( U^j = B_{e_j}(p^j) \) with the additional property that \( U^j \subset K^j \subset D \) for all \( j \). Let \( \{ \eta^j \} \) be a partition of unity subordinate to \( \{ U^j \} \) and define the \( n \)-forms \( \omega^j = \eta^j \omega \) (see Appendix). It holds that \( \sum_j \omega^j = \omega \). \text{supp}(\omega^j) \subset U^j \). If \( \int_D \omega^j \neq 0 \), then by Lemma 4.12 and Remark 4.3
\[
(4.3) \quad I(f, \Omega, D) \cdot \int_D \omega^j = \deg(f, \Omega, p^j) \cdot \int_D \omega^j = \int_{\Omega} f^* \omega^j.
\]

If \( \int_D \omega^j = 0 \), then by Lemma 4.5, \( \omega^j = d\theta^j \) and
\[
\int_{\Omega} f^* \omega = \int_{\Omega} f^* d\theta = \int_{\Omega} df^* \theta = 0.
\]

Therefore, Equation (4.3) holds for all \( j \). Now sum over \( j \) in equation (4.3), which then proves the lemma.

This leads to the following alternative definition of the mapping degree for arbitrary values \( p \in D \).

\begin{definition}
Let \( p \in D \subset \mathbb{R}^n \setminus f(\partial \Omega) \) and \( \omega \in \Gamma_c^n(D) \). Define
\[
\deg(f, \Omega, p) := I(f, \Omega, D) = \int_{\Omega} f^* \omega,
\]
as the \( C^1 \)-mapping degree.
\end{definition}

\textbf{Exercise.} Prove that \( \deg(f, \Omega, p) = \int_{\Omega} f^* \omega / \int_D \omega \) for any \( p \in D \subset \mathbb{R}^n \setminus f(\partial \Omega) \) and any \( \omega \in \Gamma_c^n(D) \), with \( \int_D \omega \neq 0 \), i.e. \( \omega \) not exact.

\textbf{Remark.}
The definition of the \( C^1 \)-mapping degree can formulated in terms of compactly supported cohomology; \( H_c^n \). Consider \( f : \bar{\Omega} \to \mathbb{R}^n \). By considering a connected component \( D \subset \mathbb{R}^n \setminus f(\partial \Omega) \) the map \( f \) yields a homomorphism \( f^* \) in compactly supported cohomology via pull-back;
\[
f^* : H_c^n(D) \longrightarrow H_c^n(\Omega), \quad \{\omega\} \mapsto \{f^* \omega\},
\]
where $[\omega]$ is a non-trivial cohomology class in $H^k_c(D)$. Choose $k = n$, then following diagram is a commutative diagram

$$
\begin{array}{ccc}
H^k_c(D) & \xrightarrow{f^*} & H^k_c(\Omega) \\
\cong & \downarrow & \downarrow f_* \\
\mathbb{R} & \xrightarrow{\deg(f_*D)} & \mathbb{R}
\end{array}
$$

where the map $\int_\Omega : H^n_c(\Omega) \to \mathbb{R}$ is onto and the isomorphism $\int_D : H^n_c(D) \to \mathbb{R}$ is given by $[\omega] \to \int_D \omega$. In the case that $\Omega$ is connected, then $\int_\Omega$ is an isomorphism between $H^n_c(\Omega)$ and $\mathbb{R}$. The commutativity of the diagram gives the relation

$$\deg(f, \Omega, D) \int_D \omega = \int_\Omega f^* \omega,$$

which is exactly the definition of the $C^1$-mapping degree in Definition 4.13.

4c. Homotopy invariance. The degree $\deg(f, \Omega, p)$ is independent of $p \in D$, with $D \subset \mathbb{R}^{n-1} \setminus f(\Omega)$, a connected component. Therefore, for any curve $t \mapsto p_t$ in $D$, $\deg(f, \Omega, p_t)$ is a constant function of $t$; the degree is invariant under homotopies in $p$.

The integral representation of the degree can be used to establish homotopy invariance of the degree with respect to $f$. In particular, since in the definition of degree the domain $\Omega$ isolates the solution set $f^{-1}(p)$, the degree is stable under small perturbations of the map $f$, see Exercise 2.4. The general homotopy invariance of the degree will be proved in several steps. The key ingredient is the continuity of the integral representation with respect to $f$.

**4.16 Lemma.** The function $f \mapsto \int_\Omega f^* \omega = \int_\Omega \omega(f(x)) J_f(x) dx$ is continuous with respect to the $C^1$-topology.

**Proof:** By the continuity of $\omega(x)$, $\|f - g\|_{C^1} < \delta$, implies that $|\omega(f(x)) - \omega(g(x))| < \epsilon$ uniformly for $x \in \overline{\Omega}$. Similarly, since $J_f(x)$ is a polynomial term in $\frac{\partial f}{\partial x^i}$, $\|f - g\|_{C^1} < \delta$ implies that $|J_f(x) - J_g(x)| < \epsilon$, uniformly in $x \in \overline{\Omega}$. These estimates combined yield the continuity of the integral $\int_\Omega f^* \omega$ with respect to $f$.

**4.17 Lemma.** Let $t \mapsto f_t$ and $t \mapsto \omega_t$, $t \in [0, 1]$ be a continuous paths in and assume that $\text{supp}(\omega_t) \cap f_t(\partial \Omega) = \emptyset$ for all $t \in [0, 1]$, then $\int_\Omega f_t^* \omega_t = \text{const}$.

**Proof:** By assumption, for each $t \in [0, 1]$ the integral represents a degree, i.e. $\int_\Omega f_t^* \omega_t = \deg(f_t, \Omega, p_t)$ for some $p_t \in \text{supp}(\omega)$. Therefore the integral is integer valued. On the other hand by Lemma 4.16 the integral is a continuous function of $t$ and therefore constant.

We can use these lemmas to prove the general homotopy principle as given in Theorem 3.2.
4.18 Lemma. Let \( t \mapsto f_t \) and \( t \mapsto p_t, t \in [0, 1] \) be a continuous paths and assume that \( p_t \notin f_t(\partial \Omega) \) for all \( t \in [0, 1] \). Then, \( \deg(f_t, \Omega, p_t) \) is a continuous function of \( t \) and is therefore constant along \((f_t, \Omega, p_t)\).

Proof: Choose an \( \varepsilon > 0 \) small enough such that \( B_\varepsilon(p_t) \subset \mathbb{R}^n \setminus f_t(\partial \Omega) \). Define a form \( \omega = \omega(x)dx \) such that \( \text{supp}(\omega) = B_\varepsilon(0) \) and set \( \omega_t = \omega(x - p_t)dx \). Consequently \( t \mapsto \omega_t \) is a continuous path with \( \text{supp}(\omega_t) \cap f_t(\partial \Omega) = \emptyset \) for all \( t \in [0, 1] \) and \( \int_\Omega f_t^* \omega_t = \deg(f_t, \Omega, p_t) \). By Lemma 4.17 the integral \( \int_\Omega f_t^* \omega_t \) is constant, which proves the lemma.

5. Proper mappings

So far the mapping degree has been defined for mappings on bounded domains \( \Omega \). For unbounded domains \( \Omega \) the generic construction of the degree does not make sense in general due to the possible non-compactness of the set \( f^{-1}(p) \). However, if a mapping is proper the \( C^1 \)-degree can be defined in the usual manner. Let \( f: \overline{\Omega} \subset \mathbb{R}^n \to \mathbb{R}^n \) be a proper mappings and \( \Omega \) an unbounded domain. If \( p \in \mathbb{R}^n \) is a regular value then the degree \( \deg(f, \Omega, p) \) is given by Definition 2.2. The degree can be extended to arbitrary values \( p \) following the procedures in Section 2c.

5a. Local and global degree. The integral representation in Section 4 can be used to define the \( C^1 \)-mapping degree for proper mappings for arbitrary values \( p \) directly. Proper mappings are the natural morphisms that induce the homomorphisms on compactly supported cohomology, see e.g. [4]. Using the construction in Section 4b yields to the following definition. Consider triples \((f, \Omega, p)\), where \( \Omega \subset \mathbb{R}^n \) is open, \( f: \overline{\Omega} \subset \mathbb{R}^n \to \mathbb{R}^n \) proper, and \( p \notin \mathbb{R}^n \setminus f(\partial \Omega) \), and let \( \omega \in \Gamma^n(D) \), with \( \int_D \omega = 1 \), where \( D \subset \mathbb{R}^n \setminus f(\partial \Omega) \) a connected component containing \( p \). Then

\[
\deg(f, \Omega, p) := \int_\Omega f^* \omega.
\]

As before the degree is independent of \( p \in D \) and is therefore sometimes written as \( \deg(f, \Omega, D) \). If \( p \) is a regular value then the degree is given by Definition 2.2. In particular, for proper mappings from \( \mathbb{R}^n \) to \( \mathbb{R}^n \) the degree does not depend on \( p \) and is defined as \( \deg(f) \). The identity map on \( \mathbb{R}^n \) is an example of a proper map, and \( \deg(\text{Id}) = 1 \). The theory discussed in the remainder this chapter will mainly concern bounded sets \( \Omega \). However, if \( f: \Omega \subset \mathbb{R}^n \to \mathbb{R}^n \) is a proper mapping the degree \( \deg(f, \Omega, p) \) can be computed by choosing a bounded domain \( \Omega' \subset \Omega \) containing \( f^{-1}(p) \). In that case \( \deg(f, \Omega, p) = \deg(f, \Omega', p) \), which is useful for translating various properties of the degree for proper mappings. The degree \( (f, \Omega, p) \to \deg(f, \Omega, p) \) is a local degree, or degree over \( p \) (cf. [7], IV, §5). As pointed out before, for a regular value \( p \) the local degree counts the number times the set \( f^{-1}(p) \) covers \( p \) (counted with multiplicity) under the mapping \( f \), see Figure 5.1. The degree depends on \( p \) (per connected component of \( \mathbb{R}^n \setminus f(\partial \Omega) \)).
\textbf{5.1 Example.} Consider the function $f : [-3, 3] \subset \mathbb{R} \to \mathbb{R}$ given by $f(x) = x^3 - 3x$. The value $p = 1$ is regular and $\deg(f, [-3, 3], 1) = 1$. The Figure 5.1 below shows how the image $f([-3, 3])$ covers $p = 1$. If we consider $p = -20$ (see Figure 5.1) then $p$ is not covered by $\mathbb{R}$ under $f$ and the degree is zero.

![Figure 5.1.](image)

In the case $\Omega = \mathbb{R}^n$ and $f : \mathbb{R}^n \to \mathbb{R}^n$ is proper, then the mapping degree $\deg(f, \mathbb{R}^n, p)$ is independent of $p$ and is denoted by $\deg(f)$. In this case we refer to the global mapping degree (cf. [7], IV, §4). The global mapping degree counts how many times $f(\mathbb{R}^n)$ covers $\mathbb{R}^n$ counted with multiplicity. The notion of local and global degree will be discussed in more depth in Section 10.

\textbf{5.3 Example.} Consider the polynomial function $f(x) = x^4 - 2x^2 + 1$ defined on $\mathbb{R}$ is a proper mapping. Choose a weight function

$$\omega(x) = \begin{cases} 
(1 - |x|) & \text{when } x \in [-1, 1], \\
0 & \text{otherwise.}
\end{cases}$$

Via the integral definition the degree is given by

$$\deg(f) = \int_{\mathbb{R}} f^* \omega = \int_{\mathbb{R}} \left[ 1 - |x^4 - 2x^2 + 1| \right] \left[ 4x^3 - 4x \right] dx = 0,$$

which also follows from counting $f^{-1}(p)$ for any regular value $p$.

\textbf{5.4 Example.} The polynomial function $f(x) = x^3 - x/2$ defined on $\mathbb{R}$ is also a proper mapping. Let $2\omega(2x)$ be a weight function, $\omega$ as above, then

$$\deg(f) = \int_{\mathbb{R}} f^* \omega = \int_{\mathbb{R}} \left[ 1 - 2|x^3 - x/2| \right] \left[ 3x^2 - 1/2 \right] dx = 1,$$

which proves that $f(x) = p$ has at least one zero for any $p \in \mathbb{R}$. Functions $\omega$ with finite mass and which decrease monotonically to zero as $|x| \to \infty$, can be used...
to approximate weight functions. For example take \( \omega = e^{-x^2} \) and consider the mapping \( f(x) = x^3 \). Then \( \int_{\mathbb{R}} \omega(x) = \int_{\mathbb{R}} e^{-x^2} = \sqrt{\pi} \), and

\[
\text{deg}(f) = \frac{1}{\sqrt{\pi}} \int_{\mathbb{R}} f^* \omega = \frac{3}{\sqrt{\pi}} \int_{\mathbb{R}} x^2 e^{-x^2} \, dx = 1,
\]

which is of course the same answer as before.

The examples suggest that for proper mappings from \( \mathbb{R}^n \) to \( \mathbb{R}^n \) the degree is related to surjectivity.

\section{5.5 Lemma.} Let \( f : \mathbb{R}^n \to \mathbb{R}^n \) be a proper \( C^1 \)-mapping.

(i) If \( \text{deg}(f) \neq 0 \), then \( f \) is surjective.

(ii) If \( f \) is not surjective, then \( \text{deg}(f) = 0 \).

When \( f \) is surjective \( \text{deg}(f) \) counts (with multiplicity) how many times the image under \( f \) covers \( \mathbb{R}^n \).

\begin{proof}
If \( \text{deg}(f) \neq 0 \), then the equation \( f(x) = p \) has a solution for any \( p \in \mathbb{R}^n \) and thus \( f \) is surjective, proving (i).

On the other, if \( f \) is not surjective then there exists a \( p \) and \( \varepsilon > 0 \) such that \( f(\mathbb{R}^n) \cap B_\varepsilon(p) = \emptyset \), since proper mappings are closed mappings. Now choose \( \omega \), with \( \text{supp}(\omega) \subset B_\varepsilon(p) \). Then \( \text{deg}(f) = \int_{\mathbb{R}} f^* \omega = 0 \), which proves (ii) and therefore the lemma.
\end{proof}

\section{5.6 Exercise.} Give an example of an improper function \( f : \mathbb{R} \to \mathbb{R} \) such that \( f^{-1}(p) \neq \emptyset \) for all \( p \in \mathbb{R} \).

\section{5.7 Exercise.} Show that the formula \( \text{deg}(f) = \frac{1}{\pi^2} \int_{\mathbb{R}^n} e^{-\|f(x) - p\|^2} J_f(x) \, dx \), \( p \in \mathbb{R}^n \), is an alternative expression for the mapping degree for proper mappings \( f \) on \( \mathbb{R}^n \).

\section{5.8 Remark.} In terms of compactly supported De Rham cohomology (Remark 4.15) the local degree is again expressed via \( [f^* \omega] \in H^n_c(\Omega) \) with \( [\omega] \in H^n_c(D) \), \( p \in D \) a connected component of \( \mathbb{R}^n \setminus f(\partial \Omega) \). Properness is needed to ensure that \( f^* \omega \) determines a cohomology class in \( H^n_c(\Omega) \). In the case of the global degree \( [f^* \omega] = \text{deg}(f) [\omega] \). The compactly supported De Rham cohomology is homotopy invariant with respect to proper homotopies (cf. \cite{13}, \S 44, \cite{4}, I, \S 2) and \( \text{deg}(f) \) is a homotopy invariant.

\section{5b. Proper mappings on open subsets.} We can restate the degree theory developed in this chapter for smooth mappings between open subsets of \( \mathbb{R}^n \). Let \( N, M \subset \mathbb{R}^n \) be open subsets. Note that we do not assume boundedness, nor connectedness of \( N \) and \( M \). Let \( f : N \to M \) be a mapping of class \( C^1 \); \( f \in C^1(N; M) \). We start with remarking that when \( f^{-1}(p) \) is compact then \( \text{deg}(f; N, p) \) is defined.
**5.9 Exercise.** Give an example of a function $f : \mathbb{R} \to \mathbb{R}$ for which $f^{-1}(0)$ is compact and for which $f^{-1}(p)$ is unbounded for any $p \neq 0$ close to 0.

Let $\mathcal{B}_e(f^{-1}(p)) \subset M$, then the compactness of $\partial \mathcal{B}_e$ yields that $p \notin f(\partial \mathcal{B}_e)$. Define the local degree over $p$ by $\deg(f,N,p) := \deg(f,\mathcal{B}_e,p)$. This definition holds for any compact neighborhood (open interior is needed) $\mathcal{B}_e \subset N$ that contains $f^{-1}(p)$ in its interior and is independent of $\mathcal{B}_e$ by Theorem 3.2 (compare the arguments for the index of an isolated zero in Section 3b). This shows that $\deg(f,N,p)$ is well-defined. We now give a general definition of local degree over compact sets $K \subset M$ (cf. [7], IV, §5 and VIII, §4, where the local degree is defined for any continuous mapping, see also Section 10).

**5.10 Definition.** Let $K \subset M (\neq \emptyset)$ be compact, connected and $f^{-1}(K)$ is compact. Then the local degree over $K$ is defined by $\deg(f,N,K) := \deg(f,N,p)$ for any $p \in K$.

**5.11 Lemma.** The local degree $\deg(f,N,K)$ is well-defined.

**Proof:** Let $K' \subset N$ be a compact neighborhood that contains $f^{-1}(K)$ is its interior and consider the (restriction) mapping $f : K' \subset N \to M$. By the compactness of $f(\partial K')$ and $K$ it follows that $d(f(\partial K'),K) \geq \delta > 0$ and thus $K \cap f(\partial K') = \emptyset$. Since $K$ is connected it lies in a connected component $D$ of $M \setminus f(\partial K')$. From Lemma 2.18 and Definition 2.19 it follows that $\deg(f,K',D)$ only depends on $D$. Since for $p \in K \subset D$ it holds that $\deg(f,N,p) = \deg(f,K',p)$ we showed that $\deg(f,N,p)$ is the same for every $p \in K$.

**5.12 Exercise.** Show that $d(f(\partial K'),K) \geq \delta > 0$ in the proof of Lemma 5.11.

For any two connected sets $K' \subset K \subset M$ it holds that $\deg(f,N,K') = \deg(f,N,K)$. This definition is reminiscent of the degree as presented in Definition 2.19. The above considerations reveal that the local degree $\deg(f,N,K)$ can also be characterized in terms of the integral representation. Let $D \subset M \setminus f(\partial K')$ be the connected component containing $K$ and $\omega \in \Gamma_c^\omega(D)$ such that $\int_M \omega = 1$, then $\deg(f,N,K) = \int_{K'} \omega$.

A mapping $f : N \to M$ is said to be proper over $M' \subset M$ if $f^{-1}(K)$ is compact for all compact subsets $K \subset M'$. The degree $\deg(f,N,K)$ is well-defined for all compact sets $K \subset M'$. If $M'$ is open and connected then the local degree $\deg(f,N,p) = \deg(f,N,p')$ for any $p, p' \in M'$ (connect $p$ and $p'$ by a path $\gamma$ which is a compact set). In this case we have the degree $\deg(f,N,M')$. From the latter the degree for bounded domains follows as a special case.

**5.13 Example.** Let $\Omega$ be a bounded domain and let $f : \overline{\Omega} \subset \mathbb{R}^n \to \mathbb{R}^n$ be a $C^1$-mapping, then $f : \Omega \to \mathbb{R}^n$ is a smooth mapping with $N = \Omega$ and $M = \mathbb{R}^n$. Let $D \subset \mathbb{R}^n \setminus f(\partial \Omega)$ be a connected component, then for any compact subset $K \subset D$ it holds that $f^{-1}(K)$ is closed and contained in $\Omega$ which implies that $f^{-1}(K)$ is compact. Therefore $f \in C^1(\Omega, \mathbb{R}^n)$ is proper over $D$ and the local degree $\deg(f,\Omega,D)$ is well-defined.
**5.14 Definition.** Let \( f : N \rightarrow M \) be a proper \( C^1 \)-mapping and let \( M \) be connected. Then the global degree is defined by \( \deg(f) = \deg(f_N, K) \) for any compact subset \( K \subset M \) (not necessarily connected).

The global degree is well-defined since \( \deg(f_N, K) \) is independent of \( K \) (connected) and by the above arguments \( \deg(f, N, K^1 \cup K^2) = \deg(f, N, K^1) = \deg(f, N, K^2) \) (connect by a path). The degree \( \deg(f) \) counts how many times \( f(N) \) covers \( M \) counted with multiplicity, i.e. each sheet is either positively or negatively oriented.

**5.15 Example.** Let \( \Omega, \Omega' \subset \mathbb{R}^n \) be bounded domains and \( f : \overline{\Omega} \subset \mathbb{R}^n \rightarrow \overline{\Omega'} \subset \mathbb{R}^n \) be a \( C^1 \)-mapping. Suppose now that \( f : \Omega \rightarrow \Omega' \) with \( N = \Omega \) and \( M = \Omega' \), and \( f(\partial \Omega) \subset \partial \Omega' \). For any compact set \( K \subset \Omega' \) it holds that \( f^{-1}(K) \subset \Omega \) and is compact. The mapping \( f \in C^1(\Omega, \Omega') \) is proper and the degree \( \deg(f) = \deg(f, \Omega, K) \) is a global degree. If we only assume that \( f(\partial \Omega) \subset \partial \Omega' \), then \( f : \overline{\Omega} \setminus f^{-1}(\partial \Omega') \rightarrow \Omega' \). Also in this case \( f \) is proper and \( \deg(f) \) is the global mapping degree.

The homotopy of the global degree is exactly as explained before is we consider proper homotopies. For the local degree one needs to consider homotopies \( f_t \) for which \( f_t^{-1}(K) \) is compact along the homotopy. Other properties stay more or less the same and we will come back to this in a more general setting in Section 10. For mappings that a proper over an open set \( M' \subset M \) the degree can also be expressed via the integral representation. Let \( \omega \in \Gamma_c^n(M') \) with \( \int_{M'} \omega = 1 \), then \( \deg(f, N, M') = \int_{M'} f^* \omega \). For the global degree \( (M = M) \) this gives \( \deg(f) = \int_M f^* \omega \). In Example 5.15 the degree is of course given by \( \deg(f) = \int_{\Omega} f^* \omega \) where \( \omega \in \Gamma_c^n(\Omega') \) and \( \int_{\Omega} \omega = 1 \).

**5.16 Remark.** In the case that \( \Omega \) and \( \Omega' \) are bounded domains with smooth boundary and \( f(\partial \Omega') \subset \partial \Omega' \) we have the following commuting diagram. By the latter condition \( f \) is a mappings of pairs, i.e. \( f : (\overline{\Omega}, \partial \Omega) \rightarrow (\overline{\Omega'}, \partial \Omega') \) and

\[
\begin{align*}
\mathcal{H}^n_c(\Omega) & \xleftarrow{\cong} \mathcal{H}^n(\overline{\Omega}, \partial \Omega) \xleftarrow{f^*} \mathcal{H}^n(\overline{\Omega'}, \partial \Omega') \xrightarrow{\cong} \mathcal{H}^n_c(\Omega') \\
\mathcal{H}^{n-1}(\partial \Omega) & \xrightarrow{\psi^*} \mathcal{H}^{n-1}(\partial \Omega')
\end{align*}
\]

where \( \psi = f|_{\partial \Omega} \). From this diagram it follows that \( f^* \omega = \deg(f)[\omega] \) and \( \psi^* \theta = \deg(\psi)[\theta] \) and thus \( \deg(f) = \deg(\psi) \). In Section 8 we will come back to the boundary dependence of the degree. In Section 10 we will give a more detailed account of the algebraic topology.

**Notes**

The \( C^1 \)-mapping degree as considered in this first chapter was introduced by Nagumo in 1951 [15]. In his paper Nagumo diverts from the definition of the mapping the for continuous mappings by approximation via simplicial mappings, by approximation via smooth mappings and defines the mapping degree for smooth mappings as was given in Definition 2.2. The mappings degree for continuous
functions, or Brouwer degree was developed by Brouwer [5] and is treated in Chapter II, where we follow the approach of Nagumo. In a series of papers Nagumo also treats the degree in a more general settings such as the Leray-Schauder degree [14, 15, 16], see Chapter V. The generic definition of the $C^1$-mapping degree is useful often for computing the degree in specific situations. This definition of the mapping degree ties in with the homotopy argument in Lemma 2.12 and can found for example in [12] and [16]. The homotopy principle can be applied in many situations, see Chapter IV. The properties proved in Theorem 2.20 are axioms for a degree theory and can be used in a much broader context, see Chapter II. In [1] Amman & Weiss show that the properties, or axioms uniquely determine the degree. Heinz [9] gave an integral formulation of the smooth mapping degree. In Section 4 we essentially followed the treatments in [17] and [18]. The integral representation in Section 4 provides a definition of the degree for smooth functions without having to worry about regular versus non-regular values. This definition is based on the definition using compactly supported De Rham cohomology. The definition of degree extends to mappings between smooth manifolds and to mappings on unbounded domains — proper mappings, see Chapter II. In Chapter II a direct definition of the degree for continuous functions is linked to a homological definition of the degree. Further elementary accounts of the degree for smooth functions in the context of bounded domains can be found in many books on non-linear analysis. We mention in particular the books by Berger [3], Nirenberg [17] and Schwartz [16]. See also Guillemin & Pollack [8], Malchiodi & Ambrosetti [2], Brown [6], Lloyd [11], Bott & Tu [4].

Exercises

1: By identifying $\mathbb{C}$ and $\mathbb{R}^2$ the application $z \mapsto z^n$ can be identified with a smooth mapping $f$ on $\mathbb{R}^2$. Show that $t(f,0) = n$. Find a class of mappings on $\mathbb{R}^2$ for $0$ is an isolated zero and $t(f,0) = -n$.

2: Let $f \in C^1(\overline{\Omega})$, with $\overline{\Omega} \subset \mathbb{R}^n$ a bounded domain and $f$ is one-to-one. Prove that $\deg(f,\Omega,0) = \pm 1$.

3: Let $f : B_1(0) \to \mathbb{R}^n$ and $f(x) \neq \mu x$ for $\mu \geq 0$ and for all $x \in \partial B_1(0)$. Show that $f(x) = 0$ has a non-trivial solution in $B_1(0)$.

4: Let $f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$ be a polynomial with $a_n \neq 0$.
   (i) Show that for fixed coefficients $a_0, \ldots, a_n$ there exists an $r > 0$ such that $f^{-1}(0) \in (-r,r)$.
   (ii) Prove for $n$ odd that $\deg(f,(-r,r),0) = 1$.
   (iii) Prove that $n$ even that $\deg(f,(-r,r),0) = 0$.
   (Hint: use the integral representation of the degree with $\omega(x) = 1 - x^2$ on $(-1,1)$ and zero outside).

5: Prove Lemma 2.18.

6*: (Borsuk’s Theorem) Let $\Omega \subset \mathbb{R}^n$ be a bounded domain satisfying the property that $x \in \Omega$ implies that $-x \in \Omega$. Let $\varphi : \partial \Omega \subset \mathbb{R}^n \to \mathbb{R}^n \setminus \{0\}$ such that $\varphi(-x) = -\varphi(x)$. Prove that for any continuous extension $f : \Omega \subset \mathbb{R}^n \to \mathbb{R}^n$ of $\varphi$ it holds that $\deg(f,\Omega,0)$ is an odd integer.
II. The Brouwer degree and the axioms of degree theory

The $C^1$-mapping degree defined in Chapter I strongly uses the fact that $f$ is differentiable. The homotopy invariance of the $C^1$-degree can be used to extend the degree to the class of continuous functions on $\mathbb{R}^n$, which is essentially the approach due to Nagumo [15]. At the core of the definition of the $C^0$-mapping degree, or Brouwer degree is the fact that $C^1$-functions can be approximated by $C^0$-functions. We will discuss the Brouwer for bounded and unbounded domains, as well as for functions between manifolds. Another aspect is the axiomatic approach towards degree theory. This will discussed for the Brouwer degree. At the end of this chapter we will also discuss the homological definition of the degree which allows a direct definition of the Brouwer degree.

6. The Brouwer degree

Using approximation of $f$ via smooth mappings and homotopy invariance leads to the definition of the $C^0$-degree, or Brouwer degree

**6.1 Definition.** Let $f \in C^0(\overline{\Omega})$ and let $p \not\in f(\partial \Omega)$. Then, for any sequence $f^k \in C^1(\overline{\Omega})$ converging to $f$ in $C^0$, define

$$\deg(f, \Omega, p) := \lim_{k \to \infty} \deg(f^k, \Omega, p),$$

as the Brouwer degree of the pair $(f, \Omega, p)$.

The properties of the $C^1$-mapping degree imply that this definition makes sense, i.e. the limit exists and is independent of the chosen sequence $f^k$. First of all approximating sequences exist by virtue of Theorem A.3. Second, since $p \in \mathbb{R}^n \setminus f(\partial \Omega)$ it holds that $\delta = d(p, f(\partial \Omega)) > 0$ (compactness of $\overline{\Omega}$). Let $g, \tilde{g} \in C^1(\overline{\Omega})$ be approximations of $f$ such that $\|g - f\|_{C^0}, \|\tilde{g} - f\|_{C^0} < \delta/2$. Consider the homotopy $h_t(x) = (1-t)g(x) + t\tilde{g}(x), \quad t \in [0,1]$. The choices of $g$ and $\tilde{g}$ give

$$\|h_t - f\|_{C^0} \leq (1-t)\|g - f\|_{C^0} + t\|\tilde{g} - f\|_{C^0} < (1-t)\delta/2 + t\delta/2 = \delta/2,$$

and for $x \in \partial \Omega$ it holds that

$$|h_t - p| \geq |f - p| - |h_t - f| \geq \delta/2.$$

Therefore, $p \not\in h_t(\partial \Omega)$ for all $t \in [0,1]$ and the degree $\deg(h_t, \Omega, p)$ is constant in $t$ by the homotopy invariance of the degree (e.g. Lemma 4.18). We conclude that $\deg(g, \Omega, p) = \deg(\tilde{g}, \Omega, p)$. For any approximating sequence $f^k$ it holds that
\[\|f^k - f\|_{C_0} < \delta/2,\] for \(k\) large enough. Therefore, in the above definition it is assumed, without loss of generality, that \(p \notin f^k(\partial \Omega)\). These observations prove that in the above definition of the \(C^0\)-mapping degree the limit exists and is independent of the chosen sequence \(f^k\).

\[\Box \text{Remark. In approximating } C^0 \text{-functions via } C^1 \text{-functions it is not necessary to assume that } p \text{ is a regular value for the sequence } f^k. \text{ Approximations can always be chosen such that this is the case, which can be useful sometimes.} \Box\]

\[\Box \text{Exercise. Let } p \in \mathbb{R}^n \setminus f(\partial \Omega). \text{ Show that one can always approximate } f \text{ with } C^1 \text{-maps } f^k \text{ with the additional property that } p \text{ is regular value for all } f^k. \Box\]

\[\Box \text{Lemma. The Brouwer degree } d(f, \Omega, p) \text{ is continuous in } f \in C^0(\overline{\Omega}). \Box\]

**Proof:** Let \(g \in C^0(\overline{\Omega})\) be any continuous mapping such that \(\|g - f\|_{C_0} < \delta/4\). Then \(\deg(g, \Omega, p)\) well-defined, since, for \(x \in \partial \Omega\), it holds that \(\|g - p\| > \|f - p\| > \|g - f\| > \delta/4\) and thus \(p \notin g(\partial \Omega)\).

Let \(f^k \in C^1(\overline{\Omega})\) and \(g^k(\overline{\Omega}) \in C^1(\overline{\Omega})\) be sequences that converge to \(f\) and \(g\) respectively. Choose \(k\) large enough such that \(\|f^k - f\|_{C^0} < \delta/4\), and \(\|g^k - g\|_{C^0} < \delta/4\), then

\[\|g^k - f\|_{C_0} \leq \|g - f\|_{C_0} + \|g^k - g\|_{C_0} < \delta/2.\]

Then, by considering the homotopy \(h_t = (1 - t)f^k + tg^k\), \(t \in [0, 1]\), it follows that \(\|h_t - f\|_{C_0} \leq (1 - t)\|f^k - f\|_{C_0} + t\|g^k - f\|_{C_0} \leq (1 - t)\delta/4 + t\delta/2 < \delta/2\) and \(|h_t - p| \geq \delta/2\) for \(x \in \partial \Omega\). Consequently, \(\deg(f^k, \Omega, p) = \deg(g^k, \Omega, p)\) and from the definition of the Brouwer degree this then proves that \(\deg(f, \Omega, p) = \deg(g, \Omega, p)\), establishing the continuity of \(\deg\) with respect to \(f\).

Using the continuity of the degree in \(f\) the invariance under continuous homotopies can be derived.
6.6 Lemma. For any continuous path \( t \mapsto f_t \) in \( C^0(\overline{\Omega}) \), with \( f_0 = f \) and \( p \not\in f_t(\partial \Omega), \ t \in [0,1] \), it holds that \( \deg(f_t, \Omega, p) = \deg(f, \Omega, p) \) for all \( t \in [0,1] \).

Proof: By definition \( t \mapsto f_t \) is continuous in \( C^0(\overline{\Omega}) \) and therefore by Lemma 6.5, \( \deg(f_t, \Omega, p) \) depends continuously on \( t \in [0,1] \). Since the degree is integer valued it has to be constant along the homotopy \( f_t \).

6.7 Lemma. The Brouwer degree satisfies the translation property, i.e. for any \( q \in \mathbb{R}^n \) it holds that \( d(f - q, \Omega, p - q) = f(f, \Omega, p) \).

Proof: Choose a sufficiently small perturbation \( g \in C^1(\overline{\Omega}) \) of \( f \), then Axiom (A4) implies that

\[
\deg(g - q, \Omega, p - q) = \deg(g - q - (p - q), \Omega, 0) = \deg(g - p, \Omega, 0) = \deg(g, \Omega, p).
\]

By definition \( \deg(f - q, \Omega, p - q) = \deg(g - q, \Omega, p - q) \) and \( \deg(f, \Omega, p) = \deg(g, \Omega, p) \), which proves the lemma.

6.8 Remark. If \( t \mapsto p_t \) is a continuous path such that \( p_t \not\in f_t(\partial \Omega) \), then the translation property of the degree, Lemma 6.7, shows, since \( f_t - p_t \) is a homotopy, that

\[
\deg(f_t, \Omega, p_t) = \deg(f_t - p_t, \Omega, 0) = \deg(f - p, \Omega, 0) = \deg(f, \Omega, p).
\]

Therefore, the Brouwer degree is an invariant for cobordant triples \((f, \Omega, p) \sim (g, \Omega, q)\), or \((f, \Omega, D) \sim (g, \Omega, D')\). In the Section 7 the more general version will be given allowing variations in \( \Omega \).

7. Properties and axioms for the Brouwer degree

In this section a number of useful properties of the mapping degree will be given. In principle these properties can be proved using the definitions of the \( C^1 \)-mapping degree and the Brouwer degree. Another approach is to single out the most fundamental properties and show that these determine the Brouwer degree uniquely, and that all properties can be derived from the axioms. Consider triples \((f, \Omega, p)\), where \( \Omega \subset \mathbb{R}^n \) are open sets, \( f \in C(\overline{\Omega}) \), and \( \mathbb{R}^n \ni p \neq f(\partial \Omega) \). Such triple are called admissible, and the mapping \((f, \Omega, p) \mapsto \deg(f, \Omega, p)\), which satisfies the following three axioms:

(A1) if \( p \in \Omega \), then \( \deg(\operatorname{Id}, \Omega, p) = 1 \);

(A2) for \( \Omega_1, \Omega_2 \subset \Omega \), disjoint open subsets of \( \Omega \), and \( p \not\in f(\overline{\Omega \setminus (\Omega_1 \cup \Omega_2)}) \), it holds that \( \deg(f, \Omega, p) = \deg(f, \Omega_1, p) + \deg(f, \Omega_2, p) \);

(A3) for any continuous paths \( t \mapsto f_t, f_t \in C^0(\overline{\Omega}) \) and \( t \mapsto p_t \), with \( p_t \not\in f_t(\partial \Omega) \), i.e. \((f_t, \Omega, p_t)\) is admissible for all \( t \), it holds that \( \deg(f_t, \Omega, p_t) \) is independent of \( t \in [0,1] \);

is called a degree theory.

\( \blacksquare \)
\section*{7.1 Theorem} The Brouwer degree $\deg(f, \Omega, p)$ for admissible triples $(f, \Omega, p)$ satisfies the Axioms (A1)-(A3), i.e. the Brouwer degree is a degree theory.

\textbf{Proof:} In order to verify Axiom (A1) consider the equation $x = p$. Clearly, there exists a unique solution and $J_{\Omega}(x) = \text{Id}$, which proves (A1). Axiom (A3) follows from Lemma 6.6 and Remark 6.8.

Let $\Lambda = \Omega \setminus (\Omega_1 \cup \Omega_2)$, which is a closed subset in $\Omega$, and $\Omega \setminus \Lambda = \Omega_1 \cup \Omega_2$. Therefore, $f(\Omega \setminus (\Omega_1 \cup \Omega_2)) = f(\partial \Omega) \cup f(\Lambda)$, and since $p \not\in f(\Lambda)$, it follows that

$$\deg(f, \Omega \setminus \Lambda, p) = \deg(f, \Omega \setminus \Lambda, D \setminus f(\Lambda)) = \int_{\Omega \setminus \Lambda} f^* \omega.$$ 

Because, $\text{supp}(\omega) \subset D \setminus f(\Lambda)$ and $f(\Lambda) \cap (D \setminus f(\Lambda)) = \emptyset$, it holds that

$$\int_{\Omega \setminus \Lambda} f^* \omega = \int_{\Omega} f^* \omega,$$

thereby proving that $\deg(f, \Omega \setminus \Lambda, p) = \deg(f, \Omega, p)$. Now

$$\deg(f, \Omega, p) = \deg(f, \Omega_1 \cup \Omega_2, p) = \deg(f, \Omega_1, p) + \deg(f, \Omega_2, p).$$

The latter is proved as follows:

$$\deg(f, \Omega_1 \cup \Omega_2, p) = \sum_i \int_{\Omega_i} f^* \omega = \sum_i \deg(f, \Omega_i, p),$$

which proves the additivity property of the Brouwer degree.

\section*{7.2 Remark} For the Brouwer degree for maps from $\mathbb{R}^n$ to $\mathbb{R}^n$, Axiom (A3) is equivalent to the following two alternative axioms:

\begin{itemize}
  \item[(A3')] for any continuous path $t \mapsto f_t$, $f_t \in C^0(\overline{\Omega})$ and $p \not\in f_t(\partial \Omega)$, it holds that $\deg(f_t, \Omega, p)$ is independent of $t \in [0, 1]$;
  \item[(A4)] $\deg(f, \Omega, p) = \deg(f - p, \Omega, 0)$.
\end{itemize}

If the degree is considered for mappings between manifolds Axiom (A4) need not be well-defined.

\section*{7.3 Exercise} Show that the (A3') and (A4) combined are equivalent to (A3).

The above theorem shows that there exists a degree theory satisfying Axioms (A1)-(A3); the Brouwer degree. The remainder of this section is a list of properties that are derived from Axioms (A1)-(A3) and a proof that the Brouwer degree is the only degree satisfying (A1)-(A3).

\section*{7.4 Property} (Validity of the degree) If $p \not\in f(\overline{\Omega})$, then $\deg(f, \Omega, p) = 0$. Conversely, if $\deg(f, \Omega, p) \neq 0$, then there exists a $x \in \Omega$, such that $f(x) = p$.

\textbf{Proof:} By choosing $\Omega_1 = \Omega$ and $\Omega_2 = \emptyset$ it follows from Axiom (A2) that $\deg(f, \emptyset, p) = 0$. Now take $\Omega_1 = \emptyset = \Omega_2$ in Axiom (A2), then $\deg(f, \emptyset, p) = 2 \cdot \deg(f, \emptyset, p) = 0$.

Suppose that there exists no $x \in \Omega$, such that $f(x) = p$, i.e. $f^{-1}(p) = \emptyset$. Since $p \not\in f(\partial \Omega)$, it follows that $p \not\in f(\overline{\Omega})$, and thus $\deg(f, \Omega, p) = 0$, a contradiction.
7.5 Property. (Continuity of the degree) The degree \( \deg(f, \Omega, p) \) is continuous in \( f \), i.e. there exists a \( \delta = \delta(p, f) > 0 \), such that for all \( g \) satisfying \( \|f - g\|_{C^0} < \delta \), it holds that \( p \not\in g(\partial \Omega) \), and \( \deg(g, \Omega, p) = \deg(f, \Omega, p) \).

Proof: See Lemma 6.5.

7.6 Property. (Dependence on path components) The degree only depends on the path components \( D \subset \mathbb{R}^n \setminus f(\partial \Omega) \), i.e. for any two points \( p, q \in D \subset \mathbb{R}^n \setminus f(\partial \Omega) \) it holds that \( \deg(f, \Omega, p) = \deg(f, \Omega, q) \). For any path component \( D \subset \mathbb{R}^n \setminus f(\partial \Omega) \) this justifies the notation \( \deg(f, \Omega, D) \).

Proof: Let \( p \) and \( q \) be connected by a path \( t \mapsto p_t \) in \( D \), then by Axiom (A3), with \( f_t = f \), the degree \( \deg(f, \Omega, p_t) \) is constant in \( t \in [0, 1] \).

7.7 Property. (Translation invariance) The degree is invariant under translation, i.e. for any \( q \in \mathbb{R}^n \) it holds that \( \deg(f - q, \Omega, p - q) = \deg(f, \Omega, p) \).

Proof: The degree \( d(f - t q, \Omega, p - t q) \) is well-defined for all \( t \in [0, 1] \). Indeed, since \( p - t q \not\in f(\partial \Omega) - t q \), it follows from Axiom (A3) that \( \deg(f - t q, \Omega, p - t q) = \deg(f, \Omega, p) \), for all \( t \in [0, 1] \).

7.8 Property. (Excision) Let \( \Lambda \subset \Omega \) be a closed subset in \( \Omega \), and \( p \not\in f(\Lambda) \). Then, \( \deg(f, \Omega, p) = \deg(f, \Omega \setminus \Lambda, p) \).

Proof: In Axiom (A2) set \( \Omega_1 = \Omega \setminus \Lambda \) and \( \Omega_2 = \emptyset \), then \( \deg(f, \Omega, p) = \deg(f, \Omega \setminus \Lambda, p) + \deg(f, \emptyset, p) = \deg(f, \Omega \setminus \Lambda, p) \).

7.9 Property. (Additivity) Suppose that \( \Omega_i \subset \Omega \), \( i = 1, \cdots, k \), are disjoint open subsets of \( \Omega \), and \( p \not\in f(\overline{\Omega \setminus (\cup_i \Omega_i)}) \), then \( \deg(f, \Omega, p) = \sum_i \deg(f, \Omega_i, p) \).

Proof: The property holds trivially for \( k = 1 \). Now assume it holds for \( k - 1 \), then by Axiom (A2)
\[
\deg(f, \Omega, p) = \deg\left(f, \bigcup_{i=1}^{k-1} \Omega_i, p\right) + \deg(f, \Omega^k, p) = \sum_{i=1}^{k} \deg(f, \Omega^i, p),
\]
by the induction hypothesis.

7.10 Exercise. Show that the above statement holds true for countable collections of disjoint open subsets \( \Omega_i \) of \( \Omega \).

Let \( \Omega \subset \mathbb{R}^n \times [0, 1] \) be a bounded and relatively open subset of \( \mathbb{R}^n \times [0, 1] \) (Section 3a), and let \( F : \overline{\Omega} \to \mathbb{R}^n \) a continuous function on \( \overline{\Omega} \), with \( f_t = F(\cdot, t) \), such that

(i) \( f_0 = f \), and \( f_1 = g \);
(ii) \( \Omega_0 = \Omega_f \), and \( \Omega_1 = \Omega_g \);
(iii) there exists a continuous path \( t \mapsto p_t \), \( p_0 = p \) and \( p_1 = q \), such that \( (f_t, \Omega_t, p_t) \) is admissible for all \( t \in [0, 1] \);

then \( (f, \Omega_f, p) \sim (g, \Omega_g, q) \) are homotopic, or bordant (notation: \( (f, \Omega_f, p) \sim (g, \Omega_g, q) \)), and \( (f_t, \Omega_t, p_t) \) is an admissible homotopy. Compare this with Definition 3.1 for the smooth mapping degree.
\[7.11 \text{ Property.} \] (Homotopy invariance) For an admissible homotopy \((f_t, \Omega_t, p_t)\), the degree \(\deg(f_t, \Omega_t, p_t)\) is constant in \(t \in [0, 1]\).

\[7.12 \text{ Property.} \] (Orientation) Let \(A \in \text{GL}(\mathbb{R}^n)\) and \(\Omega\) any open neighborhood of \(0 \in \mathbb{R}^n\), then \(\deg(A, \Omega, 0) = \text{sign det}(A)\).

**Proof:** Following the proof Theorem 3.2 assume with loss of generality that \(p_t = p\) for all \(t\). Choose a small ball \(B_\varepsilon(p)\), then following the reasoning in the proof of Theorem 3.2, \(f_t^{-1}(B_\varepsilon(p)) \subset C_t, \ t \in (t_i - \delta_t, t_i + \delta_t)\), for finitely many sets \(C_t \subset \Omega\). By excision, Property 7.8, it follows that \(\deg(f_t, \Omega_t, p) = \deg(f_t, C_t, p)\), and since the sets \(C_t \times (t_i - \delta_t, t_i + \delta_t)\) form an open covering of \(F^{-1}(B_\varepsilon(p) \times [0, 1])\), the degree \(\deg(f_t, \Omega_t, p)\) is constant for all \(t \in [0, 1]\).

By (A4) \(\deg(f_t, \Omega_t, p_t) = \deg(f_t - p_t, \Omega_t, 0)\). Therefore, without loss of generality, assume that \(p_t = p\) is constant.

Choose \(\varepsilon > 0\) small enough such that \(B_\varepsilon(p) \subset \mathbb{R}^n \setminus \cup_{t \in [0, 1]} f((\partial \Omega)_t)\). As before, write \(\deg(f_t, \Omega_t, p) = \int_{\Omega_t} f_t^* \omega = \int_{(\Omega_t)_0} f_t^* \omega\), where \(\text{supp}(\omega) = \overline{B}_\varepsilon(p)\). By assumption, the set \(F^{-1}(B_\varepsilon(p) \times [0, 1]) \subset \Omega\) is compact. At every \(t \in [0, 1]\), the sets \(f_t^{-1}(B_\varepsilon(p)) \subset \Omega_t \times (t - \delta, t + \delta) \subset \Omega\). At each \(t\), by compactness and continuity of \(F\), \(\delta\) can be small enough such that \(f_t^{-1}(B_\varepsilon(p)) \subset \Omega_t \times (t' - \delta, t' + \delta)\) for all \(t' \in (t - \delta, t + \delta)\). For all \(t \in [0, 1]\) these sets form an open covering of \(F^{-1}(B_\varepsilon(p) \times [0, 1])\), which has a finite subcovering, \(C_t \times (t_i - \delta_t, t_i + \delta_t)\), \(i = 1, \ldots, k\). Therefore, for \(t' \in (t_i - \delta_t, t_i + \delta_t)\), \(f_t^* \omega = f_t^{-1}(B_\varepsilon(p)) f_t^* \omega = \int_{C_t \times (t - \delta, t + \delta)} f_t^* \omega\), which is continuous in \(t\) by Lemma 4.16 and thus constant in \(t\). Since the sets \(C_t \times (t_i - \delta_t, t_i + \delta_t)\), \(i = 1, \ldots, k\), form an open covering, the degree is the same for all \(t \in [0, 1]\).

**Proof:** The group \(\text{GL}(\mathbb{R}^n)\) consists of two path components \(\text{GL}^+\) and \(\text{GL}^-\). If \(A \in \text{GL}^+\) choose a path \(t \mapsto A_t\), connecting \(\text{Id}\) and \(A\). Clearly, \((A_t, \Omega_t, 0)\) is admissible for all \(t\), and therefore by Axioms (A1) and (A3) it follows that \(\deg(A, \Omega_t, 0) = \deg(A_t, \Omega_t, 0) = \deg(\text{Id}, \Omega_t, 0) = 1\), which proves the statement for \(A \in \text{GL}^+\).

For \(A \in \text{GL}^-\) choose a path \(t \mapsto A_t\), connecting \(R = \text{diag}(-1, 1, \ldots, 1)\) and \(A\). As before, \((A_t, \Omega_t, 0)\) is admissible for all \(t\), and thus by Axiom (A3) it follows that \(\deg(A, \Omega_t, 0) = \deg(A_t, \Omega_t, 0) = \deg(R, \Omega_t, 0)\). It remains therefore to determine \(\deg(R, \Omega_t, 0)\). Consider the homotopy

\[f_t(x) = \left(\frac{|x_1|}{2} + t, x_2, \ldots, x_n\right) : (-1, 1) \times \Omega' \to \mathbb{R}^n,\]

where \(\Omega' \subset \mathbb{R}^{n-1}\) is an open neighborhood of \(0 \in \mathbb{R}^{n-1}\). It is clear that \((f_t, (-1, 1) \times \Omega', 0)\) is admissible for all \(t \in [0, 1]\), and thus by Axiom (A3)

\[\deg(f_t, (-1, 1) \times \Omega', 0) = \deg(f_1, (-1, 1) \times \Omega', 0) = 0,\]

since the equation \(f_t(x) = 0\) has no solutions (Property 7.4). At \(t = 0\), the value \(0\) is a regular value and the equation \(f_0(x) = 0\) has exactly two non-degenerate solutions \(x^- = (-\frac{1}{2}, 0, \ldots, 0)\) and \(x^+ = (\frac{1}{2}, 0, \ldots, 0)\). Choosing two sufficiently small open
neighborhoods $\Omega^-$ and $\Omega^+$ of $x^-$ and $x^+$ respectively, Axiom (A2) yields that
\[
\deg(f_0, (-1, 1) \times \Omega', 0) = \deg(f_0, \Omega^-, 0) + \deg(f_0, \Omega^+, 0) = 0.
\]

For $f_0$ it holds that $f_0(x) = (-x_1 - \frac{1}{2}, x_2, \cdots, x_n)$ on $\Omega^-$ and $f_0(x) = (x_1 - \frac{1}{2}, x_2, \cdots, x_n)$ on $\Omega^+$. Set $p = (\frac{1}{2}, 0, \cdots, 0)$, then by Property 7.7
\[
\deg(f_0, \Omega^-, 0) = \deg(\text{Id} - p, \Omega^+, 0) = \deg(\text{Id}, \Omega^+, p) = 1.
\]
The latter follows using Property 7.11, with $\Omega = \{(x_1 - t/2, x_2, \cdots, x_n) \mid x \in \Omega^+\}$, and $p_t = (\frac{1-t}{2}, 0, \cdots, 0)$, i.e. $\deg(\text{Id}, \Omega^+, p) = \deg(\text{Id}, \Omega_1, 0) = 1$ by Axiom (A1). Similarly,
\[
\deg(f_0, \Omega^-, 0) = \deg(R - p, \Omega^-, 0) = \deg(R, \Omega^-, p) = -\deg(f_0, \Omega^+, 0) = -1.
\]
Using the homotopy property (Property 7.7) as above it follows that $\deg(R, \Omega, 0) = -1$.

\textbf{7.13 Theorem.} If $(f, \Omega, p)$ is an admissible triple, with $f \in C^1(\overline{\Omega})$ and $p$ regular, then
\[
\deg(f, \Omega, p) = \sum_{x \in f^{-1}(p)} \text{sign}(J_f(x)).
\]

For an admissible triple in general there exists an admissible triple $(g, \Omega, q)$ with $g \in C^1(\overline{\Omega})$ and $q$ regular, which homotopy to $(f, \Omega, p)$. Moreover, $\deg(f, \Omega, p) = \deg(g, \Omega, q)$.

\textit{Proof:} For a regular value $p$ the inverse image $f^{-1}(p) = \{x^j\}$ is a finite set in $\Omega$ (see Lemma 4.1). For $\varepsilon > 0$ sufficiently small, $f^{-1}(B_\varepsilon(p))$ consists of disjoint homeomorphic balls $N_\varepsilon(x^j) \subset \Omega$. By the additivity (Property 7.9)
\[
\deg(f, \Omega, p) = \sum_j \deg(f, N_\varepsilon(x^j), p).
\]

If $\varepsilon > 0$ is chosen small enough then
\[
\deg(f, N_\varepsilon(x^j), p) = \deg(f'(x^j), B_{\varepsilon}(0), 0) = \text{sign}(J_f(x^j)),
\]
which proves the first statement. The latter identity can be proved as follows. By assumption $f'(x^j)$ is invertible for all $j$. Define the homotopy $f_t = (1-t)f + tp + tf'(x^j)(x-x^j)$, then for $x \in N_\varepsilon(x^j)$ it holds that $f_t - p = f'(x^j)(x-x^j) + (1-t)R(x, x^j)$, and $\|R\| = o(\|x-x^j\|)$, for $\|x-x^j\|$ sufficiently small. This gives the estimate
\[
\|f_t(x) - p\| \geq \|f'(x^j)(x-x^j)\| - (1-t)\|R(x, x^j)\| \geq C\|x-x^j\| - o(\|x-x^j\|),
\]
and thus $\|f_t - p\| > 0$, for all $t \in [0, 1]$, provided $\|x-x^j\| = \varepsilon$ is small enough. Using Axiom (A3) it follows that $\deg(f, N_\varepsilon(x^j), p) = \deg(f'(x^j)(x-x^j), B_{\varepsilon}(x^j), p)$.

Now use the Properties 7.11 and 7.12, as in the previous proof, to show that $\deg(f'(x^j)(x-x^j), B_{\varepsilon}(x^j), p) = \deg(f'(x^j), B_{\varepsilon}(0), 0) = \text{sign}(J_f(x^j)).$

In Section 6 it was proved that for each admissible triple $(f, \Omega, p)$ there exists an admissible triple $(g, \Omega, q)$, with $g \in C^1(\overline{\Omega})$ and $q$ regular, so that $(f, \Omega, p) \sim (g, \Omega, q)$. Then, by Axiom (A3), $\deg(f, \Omega, p) = \deg(g, \Omega, q)$.\hfill \blacksquare
Theorem 7.13 shows that the Brouwer degree is the unique degree theory satisfying Axioms (A1)-(A3).

\begin{itemize}
  \item \textbf{7.14 Property. (Composition)} Let \( f \in C^0(\overline{\Omega}) \), \( g \in C^0(\overline{\Lambda}) \), with \( f(\Omega) \subset \Lambda \) and \( \Omega \) both bounded and open. Let \( D_i \) be the path components of \( \Lambda \setminus f(\partial \Omega) \). Assume that \( p \notin g(\partial \Lambda) \cup g(f(\partial \Omega)) \), then
  \[
  \deg(g \circ f, \Omega, p) = \sum_i \deg(g, D_i, p) \cdot \deg(f, \Omega, D_i),
  \]
  which is finite sum.
\end{itemize}

Identify \( \mathbb{R}^n \) with \( \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \), and let \( \Omega_1 \subset \mathbb{R}^{n_1} \), and \( \Omega_2 \subset \mathbb{R}^{n_2} \), be open and bounded subsets.

\begin{itemize}
  \item \textbf{7.15 Property. (Cartesian product)} Let \( (f, \Omega_1, p) \) and \( (g, \Omega_2, q) \), with \( f \in C^0(\overline{\Omega}_1) \), \( \Omega_1 \subset \mathbb{R}^{n_1} \), and \( g \in C^0(\overline{\Omega}_2) \), be admissible triples. Then \( (f \times g, \Omega_1 \times \Omega_2, p \times q) \) is admissible, and
  \[
  \deg(f \times g, \Omega_1 \times \Omega_2, p \times q) = \deg(f, \Omega_1, p) \cdot \deg(g, \Omega_2, q).
  \]
  \textit{Proof:} By Theorem 7.13 it suffices to prove this statement for \( C^1 \)-functions \( f \) and \( g \), and regular values \( p \) and \( q \) respectively. The product \( f \times g \) is also \( C^1 \), \( p \times q \) a regular value, and
  \[
  (f \times g)^{-1}(p \times q) = f^{-1}(p) \times g^{-1}(q) = \{\xi^I, \xi^J\}_{i, j}.
  \]
  For the degree this yields
  \[
  \deg((f \times g, \Omega_1 \times \Omega_2, p \times q) = \sum_{i, j} \text{sign} \left( J_{f \times g}(\xi^I, \xi^J) \right)
  \]
  \[
  = \left[ \sum_i \text{sign} \left( J_f(\xi^i) \right) \right] \cdot \left[ \sum_j \text{sign} \left( J_g(\xi^j) \right) \right],
  \]
  since for \( f \times g \) it holds that
  \[
  J_{f \times g}(\xi^I, \xi^J) = J_f(\xi^i) \cdot J_g(\xi^j),
  \]
  which completes the proof of Property 7.15. \hfill \( \blacksquare \)

8. **Boundary dependence of the degree**

The homotopy invariance of the degree can be used to prove that the Brouwer degree depends only on the restriction of \( f \) to the boundary \( \partial \Omega \).

\begin{itemize}
  \item \textbf{8.1 Lemma.} Let \( \varphi : \partial \Omega \to \mathbb{R}^n \setminus \partial \Omega \) be a continuous mapping. Then, for any two continuous extensions\(^3\) \( f, g \in C^0(\overline{\Omega}) \), such that
    \[
    f|_{\partial \Omega} = g|_{\partial \Omega} = \varphi,
    \]
  it holds that \( \deg(f, \Omega, p) = \deg(g, \Omega, p) \) and therefore the Brouwer degree only depends on the restriction \( f|_{\partial \Omega} \) to \( \partial \Omega \).
  \end{itemize}

\textit{Proof:} Consider the homotopy \( h_t = (1-t)f + tg, t \in [0, 1] \). Then, since \( f = g = \varphi \) on \( \partial \Omega \), it holds that \( h_t = \varphi \) on \( \partial \Omega \) for all \( t \in [0, 1] \) and therefore \( p \notin h_t(\partial \Omega) = \varphi(\partial \Omega) \) for all \( t \in [0, 1] \). Consequently, \((h_t, \Omega, p)\) is an admissible homotopy and by the homotopy invariance of the degree \( \deg(h_t, \Omega, p) \) is independent of \( t \in [0, 1] \). \( \blacksquare \)

Lemma 8.1 makes it possible to define a degree theory for continuous mappings on compact sets that occur as boundaries of bounded open sets in \( \mathbb{R}^n \). Continuous\(^3\) Continuous extensions exist by virtue of Tietze’s Extension Theorem, see Appendix 1b.
mappings from \( \partial \Omega \) to \( \mathbb{R}^n \) cannot be surjective. Therefore assume, without loss of generality, that maps \( \varphi \) act from \( \partial \Omega \) to \( \mathbb{R}^n \setminus p \) for some \( p \notin \varphi(\partial \Omega) \).

**8.2 Definition.** Let \( \varphi : \partial \Omega \to \mathbb{R}^n \setminus p \) be a continuous mapping. The mapping degree is defined by

\[
W_{\partial \Omega}(\varphi, p) := \deg(f, \Omega, p),
\]

for any continuous extension \( f : \Omega \subset \mathbb{R}^n \to \mathbb{R}^n \), with \( f|_{\partial \Omega} = \varphi \). The mapping degree for mappings \( \varphi : \partial \Omega \to \mathbb{R}^n \setminus p \) can be regarded as a generalized notion of the winding number.

**8a. Generalized winding numbers.** From the translation property of the degree (see Property 7.7) it follows that \( \deg(f, \Omega, p) = \deg(f - p, \Omega, 0) \), which implies that \( W_{\partial \Omega}(\varphi, p) = W_{\partial \Omega}(\varphi - p, 0) \). Define the normalized mapping

\[
\psi := \frac{\varphi - p}{|\varphi - p|} : \partial \Omega \to S^{n-1},
\]

where \( S^{n-1} \subset \mathbb{R}^n \) denotes the standard unit sphere in \( \mathbb{R}^n \). Consider the homotopy \( \zeta_t = (1-t)[\varphi - p] + t\psi \), then by Tietze’s Extension Theorem there exists a continuous homotopy \( h_t \) on \( \overline{\Omega} \times [0,1] \) with \( h_t|_{\partial \Omega} = \zeta_t \). From the homotopy property of the degree it then follows that \( W_{\partial \Omega}(\varphi, p) = \deg(h_0, \Omega, 0) = \deg(h_1, \Omega, 0) =: \deg(\psi, \partial \Omega, S^{n-1}) \), and thus

\[
\deg(\psi) := \deg(\psi, \partial \Omega, S^{n-1}) = W_{\partial \Omega}(\varphi - p, 0),
\]

which defines the degree for \( \psi \). Homotopy defines an equivalence relation on mappings \( \varphi \), called the homotopy type, and the degree only depends on the homotopy type of the map.

In order to derive an integral formula for the degree \( \deg(\psi) \) assume now that \( f \) is a \( C^1 \)-mapping on \( \overline{\Omega} \) and \( \partial \Omega \) is a piecewise \( C^1 \)-boundary.

**8.3 Theorem.** Let \( \mu \in \Gamma^{n-1}(S^{n-1}) \), with \( \int_{S^{n-1}} \mu = 1 \), and let \( \psi \) be as defined above. Then

\[
\deg(\psi) = \int_{\partial \Omega} \psi^* \mu,
\]

where \( \psi^* \mu \in \Gamma^{n-1}(\partial \Omega) \).

*Proof:* Let \( f \) be as before, and since \( \deg(f, \Omega, p) \) does not depend on \( p \), choose \( p \) to be a regular value, so that \( f^{-1}(p) = \{x'\} \) is a finite set. Let \( B_\varepsilon(p) \) be a sufficiently small ball in \( \mathbb{R}^n \setminus f(\partial \Omega) \) such that \( N \subset \Omega \), where \( N = f^{-1}(B_\varepsilon(p)) \). Since \( p \) is regular, \( N = \bigcup_j N^j \) (finite), where the sets \( N^j \) are all mutually disjoint and diffeomorphic to \( B_\varepsilon(p) \). Consider the disjoint open sets \( \Lambda = \Omega \setminus \overline{N} \subset \Omega \) and \( N \subset \Omega \). Then \( p \notin f(\overline{\Omega \setminus (\Lambda \cup N)}) \) and by Property 7.9 \( \deg(f, \Omega, p) = \deg(f, \Lambda, p) + \deg(f, N, p) \). Since \( p \notin f(\overline{\Lambda}) \) it follows from Property 7.4 that \( \deg(f, \Lambda, p) = 0 \) and therefore \( \deg(f, \Omega, p) = \deg(f, N, p) \). According to the Definition in 2.2 and Property 7.9 the degree on \( N \) is given by \( \deg(f, N, p) = \sum_j \deg(f, N^j, p) \) and \( \deg(f, N^j, p) = \text{sign}(f_j(x')) \).
The mapping \( \Psi \) has a \( C^1 \)-extension to \( \Lambda \) denoted by \( \Psi \) and given by the formula in (8.1), i.e. \( \Psi = (f - p)/(f - p) \). The restrictions to \( \partial \Lambda \) is again denoted by \( \Psi \) and the restriction to \( \partial N \) by \( \Psi^* \). Choose an \((n - 1)\)-form \( \mu \in \Gamma^{n-1}(S^{n-1}) \), with \( \int_{S^{n-1}} \mu = 1 \), which can be regarded as the restriction of \((n - 1)\)-form \( \mu \) on an open neighborhood of \( S^{n-1} \) in \( \mathbb{R}^n \). The orientation on \( \Lambda \) induces the Stokes orientation on \( \partial \Lambda = \partial \Omega - \partial N \). By Stokes’ Theorem
\[
\int_{\partial \Omega} \Psi^* \mu = \int_{\partial \Omega} \Psi^* \mu - \int_{\partial N} \Psi^* \mu = \int_{\Lambda} d(\Psi^* \mu) = \int_{\Lambda} \Psi^* d\mu = 0.
\]
The latter follows from the fact that \( \Psi^* d\mu = 0 \). Indeed, \( (\Psi^* d\mu)_x(\xi^1, \cdots, \xi^n) = d\mu_{f(x)}(\Psi_x \xi^1, \cdots, \Psi_x \xi^n) = 0 \), since the set of tangent vectors \( \{(\Psi_x \xi^1, \cdots, \Psi_x \xi^n)\} \) are linearly dependent. Consequently,
\[
\int_{\partial \Omega} \Psi^* \mu = \int_{\partial N} \Psi^* \mu = \sum_j \int_{\partial N_j} (\Psi^j)^* \mu.
\]
Since \( f|_{N_j} \) is a \( C^1 \)-change of variables that is either orientation preserving or reversing, the same holds for the renormalized restrictions \( \Psi^j \) via the Stokes orientation of \( \partial N_j \). This yields the following identity
\[
\int_{\partial N_j} (\Psi^j)^* \mu = \pm \int_{S^{n-1}} \mu = \pm 1 = \text{sign}(J_f(x^j)) = \deg(f, N^j, p).
\]
Combining these identities finally gives
\[
\int_{\partial \Omega} \Psi^* \mu = \sum_j \int_{\partial N_j} (\Psi^j)^* \mu = \sum_j \deg(f, N^j, p) = \deg(f, N, p)
\]
which proves the theorem.

\begin{rmk}
The integral representation can also be used to compute the degree of \( \varphi \) as defined in Definition 8.2. Let \( \mu \in \Gamma^{n-1}(\mathbb{R}^p \setminus p) \) with \( \int_{\partial \Omega} \mu = 1 \). Then
\[
W_{\partial \Omega}(\varphi, p) = \int_{\partial \Omega} \varphi \Psi^* \mu.
\]
\end{rmk}

8b. Winding numbers in the plane. Let \( \Omega = B_1(0) \subset \mathbb{R}^2 \), \( f : B_1(0) \subset \mathbb{R}^2 \to \mathbb{R}^2 \) a continuous mapping then if \( 0 \neq f(\partial B_1) \), then the degree \( \deg(f, B_1, 0) \) is well-defined. Denote the restriction of \( f \) to \( \partial B_1 = S^1 \) by \( \varphi \), and the renormalization by \( \Psi = \varphi^{\#} : S^1 \to S^1 \), then \( \deg(\Psi) = \deg(f, B_1, 0) \). The degree of \( \Psi \) can be expressed as follows \( \deg(\Psi) = \int_{S^1} \Psi^* \mu \), where \( \mu \) is a 1-form on \( S^1 \). The 1-forms on \( S^1 \) can be expressed as \( \mu = (c + h(\theta))d\theta \), where \( h \) is 2\( \pi \)-periodic. Via polar coordinates \( x_1 = r \cos(\theta), x_2 = r \sin(\theta) \), \( \mu \) extends to \( \mathbb{R}^2 \setminus \{(0,0)\} \) and is given by
\[
\mu = c \frac{-x_2 dx_1 + x_1 dx_2}{x_1^2 + x_2^2} + dh(x_1, x_2).
\]

\footnote{The Stokes orientation is the induced orientation on the boundary using the outward pointing normal. The orientation of \( \partial N \) induced by \( \Lambda \) is opposite the orientation induced by \( N \). This explains the notation \( \partial \Omega - \partial N \) instead of \( \partial \Omega \cup \partial N \).}
By taking $c = 1/2\pi$ it follows that $\int_{S^1} \mu = 1$, and set $\theta = \frac{1}{2\pi} \frac{x_2 dx_1 - x_1 dx_2}{x_1^2 + x_2^2}$, as the standard ‘volume’ form on $S^1$. A direct calculation shows that $\int_{S^1} \psi^* \theta = \int_{S^1} \varphi^* \theta$ and therefore

$$W(\varphi, 0) := \frac{1}{2\pi} \int_{S^1} \psi^* \theta = \deg(\varphi) = \deg(\psi),$$

which is called the winding number $\varphi$ about 0. Conversely, starting with a mapping $\varphi : S^1 \to S^1$, Tietze’s extension theorem yields that for any extension $f$ to $B_1(0)$ the degree $\deg(f, B_1(0))$ is given by the winding number defined in (8.2).

9. Mappings between smooth manifolds and the mapping degree

So far the mapping degree is discussed for continuous mappings between compact subsets of Euclidean spaces. Most of the ideas and proofs translate to the setting of continuous mappings between compact subsets of orientable differentiable manifolds. In the previous section we already discussed the degree for maps from $S^{n-1} \subset \mathbb{R}^n$ into $\mathbb{R}^n$. Before discussing the degree for maps between manifolds some preliminary notation and definitions needs to be introduced.

9a. Topological and smooth manifolds. A topological space $M$ is called an $m$-dimensional (topological) manifold if the following conditions hold:

(i) $M$ is a Hausdorff space,
(ii) for any $p \in M$ there exists a neighborhood $U$ of $p$ which is homeomorphic to an open subset $V \subset \mathbb{R}^m$, and
(iii) $M$ has a countable basis of open sets.

A manifold is covered by countably map open sets: $M = \cup_i U_i$. A pair $(U, \varphi)$, where $\varphi : U \to \mathbb{R}^m$ is a homeomorphism, is called a chart and for any two charts $(U_i, \varphi_i)$ and $(U_j, \varphi_j)$ the transition maps

$$\varphi_{ij} = \varphi_j \circ \varphi_i^{-1} : \varphi_i(U_i \cap U_j) \to \varphi_j(U_i \cap U_j)$$

are homeomorphisms. If we model the above definition of manifold over $\mathbb{H}^m = \{x = (x_1, \cdots, x_m) \mid x_m \geq 0\}$ instead of $\mathbb{R}^m$ we obtain a manifold with boundary $(M, \partial M)$. We use the terminology manifold with boundary, or $\partial$-manifold. A manifold is special case of a $\partial$-manifold with empty boundary. By defining the model of manifold over $\mathbb{H}^m_+ = \{x = (x_1, \cdots, x_m) \mid x_i \geq 0\}$ we obtain $\partial$-manifold with piecewise smooth boundaries.

A topological manifold $M$ for which all the transition maps $\varphi_{ij} = \varphi_j \circ \varphi_i^{-1}$ are diffeomorphisms is called a differentiable, or smooth manifold. The transition maps are mappings between open subsets of $\mathbb{R}^m$. Diffeomorphisms between open subsets of $\mathbb{R}^m$ are $C^\infty$-maps, whose inverses are also $C^\infty$-maps. Points in $M$ are denoted by $p, q$, etc. and the associated points in $\mathbb{R}^n$ by $x, y$, etc.

A $C^\infty$-atlas is a set $\mathcal{A} = \{(U_i, \varphi_i)\}_{i \in I}$ such that

\(^5\)\text{Usually an open neighborhood $U$ of a point $p \in M$ is an open set containing $p$. A neighborhood of $p$ is a set containing an open neighborhood containing $p$. Here we will always assume that a neighborhood is an open set.}
(i) \( M = \bigcup_{i \in I} U_i \),
(ii) the transition maps \( \phi_{ij} \) are diffeomorphisms between \( \phi_i(U_i \cap U_j) \) and \( \phi_j(U_i \cap U_j) \), for all \( i \neq j \).

Let \( N, M \) be smooth manifolds of dimensions \( n \) and \( m \) respectively and let \( \Omega \subset N \) be an open subset with \( \overline{\Omega} \) compact. The space of continuous maps on \( \overline{\Omega} \) is denoted by \( C^0(\overline{\Omega}; M) \). A mapping \( f : \Omega \subset N \to M \) is said to be \( k \)-times continuously differentiable if for every \( x \in N \) there exist charts \((U, \varphi)\) of \( x \) and \((V, \psi)\) of \( p = f(x) \), with \( f(U) \subset V \), such that \( \tilde{f} = \psi \circ f \circ \varphi^{-1} : \varphi(U) \to \psi(V) \) is a \( C^k \)-mapping (from \( \mathbb{R}^n \) to \( \mathbb{R}^m \)). The space of \( k \)-times continuously differentiable mappings is denoted by \( C^k(\Omega; M) \).

At every point \( p \in M \) there exists a tangent space \( T_p M \) which is defined as the space of all equivalence classes \([\gamma]\) of curves \( \gamma \) through \( x \). A tangent vector \( X_\gamma \), as the equivalence class of curves, is given by

\[
X_\gamma := [\gamma] = \{ \gamma : \gamma(0) = \gamma(0), (\varphi \circ \gamma)'(0) = (\varphi \circ \gamma)'(0) \}.
\]

Choose charts \((U, \varphi)\) for \( x \in N \), and \((V, \psi)\) for \( p \in M \). Define the derivative or pushforward of \( f \) at a point \( p \) as follows. For a given tangent vector \( X_\gamma \) at \( N \) or \( M \),

\[
f'(x) : T_x N \to T_p M, \quad f'(x)([\gamma]) = [f \circ \gamma].
\]

The following commutative diagram shows that \( f'(x) \) is a linear map and its definition does not depend on the charts chosen at \( x \in N \), or \( p \in M \).

\[
\begin{array}{ccc}
T_x N & \overset{f'(p)}{\longrightarrow} & T_p M \\
\tau_\varphi \downarrow & & \downarrow \tau_\psi \\
\mathbb{R}^n & \overset{(\varphi \circ f \circ \varphi^{-1})'((\xi))}{\longrightarrow} & \mathbb{R}^m
\end{array}
\]

where \( \tau_\varphi([\gamma]) = (\varphi \circ \gamma)'(0) \) and similarly \( \tau_\psi \).

\[\text{degree of degree for mappings between manifolds.} \]

Let \( f : \overline{\Omega} \subset N \to M \) be continuous and \( C^1 \) on \( \Omega \) and \( N \) and \( m \) smooth orientable manifolds with \( \dim N = \dim M = n \). The definition of the degree works in exactly the same way as for subsets of \( \mathbb{R}^n \). One starts with the regular case. A value \( p \in M \) is regular if \( f'(x) \) has maximal rank for all \( x \in f^{-1}(p) \), i.e. \( f'(x) \) is invertible. If \( \overline{\Omega} \) is compact then \( f^{-1}(p) \) is a discrete set by the Inverse Function Theorem and

\[
\deg(f, \Omega, p) = \sum_{x \in f^{-1}(p)} \text{sign} \left( J_f(x) \right),
\]

where \( J_f(x) = \det(f'(x)) \) as before. If \( f \) is proper, or if \( N \) is compact then \( \deg(f) := \deg(f, N, p) \) is well-defined for any regular value \( p \in M \). The construction of the mapping degree via integration can be repeated verbatim for compactly supported \( n \)-forms \( \omega \) on \( M \) and their pull-back \( f^* \omega \) under \( f \). We conclude, for any \( p \in D \subset M \setminus f(\partial \Omega) \) and \( \omega \in \Gamma^n_c(D) \) with \( \int_D \omega = 1 \), that \( \deg(f, \Omega, p) = \int_M f^* \omega \). The latter definition of degree is homotopy invariant with respect to \( p \) and \( f \) and retrieves the sign-definition given above. This extends the \( C^1 \)-degree to arbitrary values \( p \in M \).

In the case that \( M \) is compact and \( f \) is proper, or when also \( N \) is compact, then any
n-form on $M$ is compactly supported and $\deg(f) := \deg(f, N, p) = \int_N f^* \omega$ (here $\int_M \omega = 1$). This generalizes the degree for mappings from $\partial \Omega \to S^{n-1}$ as described in the previous section.

\textbf{9.1 Lemma.} Let $f : N \to M$ with $N$ and $M$ compact, orientable manifolds of dimension $\dim N = \dim M = n$. Then

(i) If $\deg(f) \neq 0$, then $f$ is surjective.

(ii) If $f$ is not surjective, then $\deg(f) = 0$.

The degree gives the number of times the image $f(N)$ covers $M$ counted with orientation.

\textit{Proof:} See proof of Lemma 5.5.

\textbf{9c. Local degree and proper mappings.} The local mapping degree for mappings between open subsets of $\mathbb{R}^n$ as explained in Section 5b extends without modification to the case of continuous mappings between orientable $n$-dimensional manifolds. We start with the case of manifolds (without boundary). The case of manifolds with boundary will be discussed thereafter.

Let $f : N \to M$ be a $C^1$-mapping, i.e. $f \in C^1(N; M)$ and $K \subset M$ a non-empty compact and connected subset such that $f^{-1}(K) \subset N$ is compact. Then the local degree $\deg(f, N, K)$ is defined as in Definition 5.10. The local degree can be expressed via the integral representation as described in Sections 5b and 9b. For a proper mapping $f \in C^1(N; M)$, $M$ connected, the global degree $\deg(f) = \deg(f, N, K)$ for any compact subset $K \subset M$. In the case $N$ and $M$ are compact $\partial$-manifolds and $f(\partial N) \subset \partial M$, then the mapping $f : N \setminus f^{-1}(\partial M) \to M^6$ is proper and the global degree is given by $\deg(f) := \deg(f, N \setminus f^{-1}(\partial M), \hat{M})$. In Section 10 we will come back to these notions in context of continuous mappings and direct definition using homology.

\textbf{10. The homological definition of the Brouwer degree}

Give an account of a direct definition of the mapping degree for continuous maps using homology theory. Describe the treatment of the degree as was done in Dold. p. 266 and p. 66.

\textbf{Notes}
To be written

\textbf{Exercises}

1: Let $S^1 = \mathbb{R}/\mathbb{Z}$ be the set of equivalence classes $[x]$ of $x \sim y$ if $x - y \in \mathbb{Z}$, and let $f : S^1 \to S^1$ be a smooth mapping. A lift $\tilde{f}$ of $f$ is a mapping $\tilde{f} : \mathbb{R} \to \mathbb{R}$ such that $f([x]) = [\tilde{f}(x)]$.

(a) Show that $\tilde{f}$ is a smooth mapping which is uniquely determined up to an additive constant, and $\tilde{f}(x + 1) = \tilde{f}(x) + k$ for some $k \in \mathbb{Z}$.

(b) Prove that $\deg(f) = \tilde{f}(x + 1) - \tilde{f}(x)$ for any $x \in \mathbb{R}$.

\footnote{The interior $N \setminus \partial N$ is denoted by $\hat{N}$ and $N \setminus f^{-1}(\partial M) \subset \hat{N}$ since $\partial N \subset f^{-1}(\partial M)$.}
2: Give a proof of Property 7.15 via the integral representation of the degree (Hint: set $\deg(f, \Omega, p) = \int_{\Omega} f^* \omega$ and $\deg(g, \Omega, q) = \int_{\Omega} g^* \lambda'$ and let $\mu = \omega \otimes \lambda = \omega(x) \cdot \lambda(y) dx \wedge dy$ be a $(n+m)$-form on $\mathbb{R}^{n+m}$ and compute $\int_{\Omega \times \Lambda} (f \times g)^* \mu$).
III. Applications of finite dimensional degree theory

11. The Brouwer fixed point theorem

A classical application of the Brouwer degree is the Brouwer fixed point theorem for continuous maps of the $n$-disc. A fixed point for a mapping $f : \mathbb{R}^n \to \mathbb{R}^n$ is a point $x \in \mathbb{R}^n$ which satisfies the equation

$$f(x) = x.$$ 

As a matter of fact the Brouwer fixed point theorem can be stated for sets homeomorphic to the $n$-disc, or closed unit ball $B_1(0)$.

\begin{itemize}
\item \textbf{11.1 Theorem.} Let $\Omega \subset \mathbb{R}^n$ be an open subset such that $\overline{\Omega}$ is homeomorphic to $B_1(0)$, and let $f : \overline{\Omega} \to \mathbb{R}^n$ be any continuous map. If $f(\overline{\Omega}) \subset \overline{\Omega}$, then $f$ has a fixed point in $\overline{\Omega}$.
\end{itemize}

\begin{proof}
Let $\varphi : \overline{\Omega} \to B_1(0)$ be a homeomorphism. Then the mapping $g := \varphi \circ f \circ \varphi^{-1} : B_1(0) \to B_1(0)$ is continuous. The maps $f$ and $g$ are conjugate and thus $f$ has a fixed point if and only $g$ has a fixed point.
\end{proof}

\begin{itemize}
\item \textbf{11.2 Exercise.} Show the above claim for conjugate mappings.
\end{itemize}

The Brouwer fixed point theorem can be proved by showing the theorem holds for $g$. Suppose that $g$ has no fixed points in $B_1(0)$, then $g(x) \neq x$, for all $x \in B_1(0)$. Consider the line $y = x + \lambda(g(x) - x)$, which intersects $\partial B_1(0)$ in exactly two points. If $x \in \partial B_1(0)$, then $\lambda = 0$ and if $g(x) \in \partial B_1(0)$, then $\lambda = 1$. Since $g(x) \neq x$ for all $x$ it follows that there are two solutions $-\lambda(x) \leq 0$ and $\lambda_-(x) \geq 1$ to the quadratic equation $\|x + \lambda(g(x) - x)\|_2 = 1$. Choose the intersection corresponding to $\lambda_- \leq 0$. The function $x \mapsto \lambda_-(x)$ is continuous in $x \in \overline{B_1(0)}$. Indeed, by an application of the Implicit Function Theorem to the quadratic equation $F(\lambda; a, b, c) = a\lambda^2 + b\lambda + c = 0$, continuity of $\lambda(a, b, c)$ holds provided

$$2a\lambda + b = 2[(g(x) - x)\lambda + x, g(x) - x] = 2[\lambda(x), g(x) - x] \neq 0.$$ 

This holds since $g(x) \neq x$, and therefore the vector $\lambda$ cannot be orthogonal to $g(x) - x$. Upon substitution this yields the continuity of the mapping $h : B_1(0) \to \partial B_1(0)$, defined by $h(x) = x + \lambda_-(x)(g(x) - x)$. Moreover, $h(x) = x$, for $x \in \partial B_1(0)$, i.e. $h|_{\partial B_1} = \text{Id}$. From Lemma 8.1 if follows that

$$\deg(h, B_1(0), 0) = \deg(\text{Id}) = 1,$$

which implies that the equation $h(x) = 0$ has a solution. This is clearly a contradiction, since $h(B_1(0)) = \partial B_1(0)$.

\end{itemize}
The proof of the Brouwer fixed point theorem is based on the observation that continuous mappings from $B_1(0)$ to $\partial B_1(0)$, which are the identity on $\partial B_1(0)$ do not exit. This uses the boundary dependence property of the Brouwer degree discussed in Section 8, and holds in a much more general setting of bounded and open subset $\Omega \subset \mathbb{R}^n$.

**11.3 Theorem.** There are no continuous maps $f : \Omega \to \partial \Omega$, with $f|_{\partial \Omega} = \text{Id}$. ▶

*Proof:* By Lemma 8.1 $\deg(f, \Omega, p) = \deg(\text{Id}) = 1$, for any point $p \in \Omega$, which implies that the equation $f(x) = p$ has a solution, a contradiction. ■

Another theorem worth mentioning in this context is the Hairy Ball Theorem, which, in dimension two, asserts that a 2-sphere ‘covered with hair’ cannot be combed in a continuous manner. Here the theorem is formulated for the embedded sphere $S^{n-1} = \partial B_1(0)$. Consider a function $X : S^{n-1} \to \mathbb{R}^n$, with the property that $(X(x), x) = 0$, for all $x \in S^{n-1}$. Such a function is called a tangent vector field on $S^{n-1}$.

**11.4 Theorem (Hairy Ball Theorem).** The $(n-1)$-sphere $S^{n-1}$ allows a non-vanishing tangent vector field $X(x) \neq 0$ if and only if $n - 1$ is odd. ▶

*Proof:* If $n - 1$ is odd a non-vanishing vector field is easily given:

$$X(x) = (-x_2, x_1, -x_4, x_3, \ldots, -x_n, x_{n-1}),$$

which is clearly tangent to $S^{n-1}$ and non-vanishing.

As for the converse argue as follows. Suppose there exists a non-vanishing tangent vector field $X(x)$ on $S^{n-1}$, then normalization defines a unit tangent vector field $Y = X/\|X\|$. Consider

$$h_t = \cos(\pi t)x + \sin(\pi t)Y(x).$$

It is clear, since $\langle x, Y \rangle = 0$, that $\|h_t\| = 1$ and $h_t : S^{n-1} \to S^{n-1}$ for all $t \in [0, 1]$. Moreover, $h_0 = \text{Id}$ and $h_1 = -\text{Id}$ and are thus homotopic mappings. From Property 7.12 and Lemma 8.1 it follows that $\deg(h_1) = \deg(-\text{Id}) = (-1)^n$. By the homotopy invariance of the degree $1 = \deg(\text{Id}) = \deg(h_0) = \deg(h_1) = (-1)^n$ and thus $n - 1$ is odd. ■

12. The mapping degree for holomorphic functions

The Brouwer degree can also be used in complex function theory. A complex function $f : \mathbb{C} \to \mathbb{C}$ can be regarded as a mapping from $\mathbb{R}^2$ to $\mathbb{R}^2$ via the following correspondence. Set $z = x_1 + ix_2$ and $f(z) = u(x_1, x_2) + iv(x_1, x_2)$, then $f : \mathbb{R}^2 \to \mathbb{R}^2$ is defined by

$$(x_1, x_2) \mapsto f(x_1, x_2) = (u(x_1, x_2), v(x_1, x_2)).$$

A complex mapping $f$ is holomorphic on a bounded open set $\Omega \subset \mathbb{C}$, if the Cauchy-Riemann equations are satisfied, i.e. $\bar{\partial}f = 0$, which is equivalent to

$$\frac{\partial u}{\partial x_1} = \frac{\partial v}{\partial x_2}, \quad \frac{\partial v}{\partial x_1} = -\frac{\partial u}{\partial x_2},$$
for all $z = x_1 + ix_2 \in \Omega$. The Brouwer degree for the triple $(f, \Omega, z)$, with $z \in \mathbb{C} \setminus \partial \Omega$, is defined as the degree of the mapping $f = (u, v)$ on $\mathbb{R}^2$.

From complex function theory it follows that zeroes of holomorphic functions are isolated, or the function is identically equal to zero. This leads to a following result about the mapping degree for holomorphic functions.

**12.1 Lemma.** Let $f : \overline{\Omega} \subset \mathbb{C} \rightarrow \mathbb{C}$ be a holomorphic function, not identically equal to zero, and $f(z^0) = 0$, for some $z^0 \in \Omega$. Then there exists an $\varepsilon > 0$, and a ball $B_{\varepsilon}(z^0) \subset \Omega$ such that $f(z) \neq 0$, for all $z \in B_{\varepsilon}(z^0) \setminus \{z^0\}$, and

$$\deg(f, B_{\varepsilon}(z^0), 0) = m \geq 1,$$

where $m$ is the order of $z^0$, i.e. $f(z) = (z - z^0)^m g(z)$, $z \in B_{\varepsilon}(z^0)$, and $|g(z)| \geq a > 0$, for all $z \in B_{\varepsilon}(z^0)$.

**Proof:** Since $f$ is not identically equal to zero, $z^0$ is an isolated zero of $f$, and there exists a ball $B_{\varepsilon}(z^0) \subset \Omega$ on which $f$ is non-zero, except at $z^0$. Also, by analyticity, it follows that $z^0$ is a finite order zero; $f(z) = (z - z^0)^m g(z)$, $m \geq 1$, and $|g(z)| \geq a > 0$ in $B_{\varepsilon}(z^0)$. These considerations make that the degree $\deg(f, B_{\varepsilon}(z^0), 0)$ is well-defined, since $|f(z)| = \varepsilon^m a > 0$, for $z \in \partial B_{\varepsilon}(z^0)$. In the case $m = 1$ the degree can be easily computed from the definition. In general, for a holomorphic function, $J_f(z) = \frac{1}{2\pi} \|\nabla f\|^2$. Since 0 is a regular value, $J_f(z^0)$ can be computed as follows: $f(z) = (z - z^0)g(z)$, and thus $f'(z) = g(z) + (z - z^0)g'(z)$. Therefore

$$J_f(z^0) = |g(z^0)|^2 = a^2 > 0,$$

and $\deg(f, B_{\varepsilon}(z^0), 0) = 1$.

Consider the holomorphic function $p(z) = (z - z^0)^m g(z^0)$, and the homotopy $f_\lambda(z) = \lambda f(z) + (1 - \lambda) p(z)$, $\lambda \in [0, 1]$, which is a homotopy of holomorphic functions. Choose $\varepsilon > 0$ small enough such that $|g(z) - g(z^0)| < \frac{1}{2} |g(z^0)|$, for all $z \in B_{\varepsilon}(z^0)$. In order to use the homotopy property of the degree it needs to be verified that $0 \notin \partial f(B_{\varepsilon}(z^0))$, for all $\lambda \in [0, 1]$. Let $|z - z^0| = \varepsilon$, then

$$|f_\lambda(z)| = |\lambda(z - z^0)^m g(z) + (1 - \lambda)(z - z^0)^m g(z^0)|$$

$$= \varepsilon^m |\lambda g(z) + (1 - \lambda) g(z^0)|$$

$$= \varepsilon^m |g(z^0) + \lambda g(z) - g(z)|$$

$$\geq \varepsilon^m |g(z^0)| - \lambda |g(z) - g(z^0)|$$

$$\geq \frac{1}{2} \varepsilon^m |g(z^0)|.$$  

If we choose $\delta = \frac{1}{4} \varepsilon^m |g(z^0)|$, then $f_\lambda(z) = \delta$ has no solutions on $\partial B_{\varepsilon}(z^0)$, for all $\lambda \in [0, 1]$. Consequently,

$$\deg(f, B_{\varepsilon}(z^0), \delta) = \deg(p, B_{\varepsilon}(z^0), \delta).$$

It remains to evaluate $\deg(p, B_{\varepsilon}(z^0), \delta)$. The associated equation is

$$p(z) = (z - z^0)^m g(z^0) = \delta = \frac{1}{4} \varepsilon^m |g(z^0)|.$$
This implies that zeroes lie on $|z - z^0| = e^{4^{-\frac{1}{n}}}$. For the arguments it holds that

$$m \arg (z - z^0) + \arg (g(z^0)) = 2\pi n, \quad n \in \mathbb{Z}.$$ 

It follows immediately that the above equation has exactly $m$ non-degenerate solutions, and therefore, $\deg(p, B_\varepsilon(z^0), \delta) = m$.

At the boundary $\partial B_\varepsilon(z^0), |(z - z^0)| = \varepsilon$ one has $|g(z)| = \varepsilon^m a$. Consider the path $\tilde{\xi}_\lambda = \frac{1}{\varepsilon} \lambda \varepsilon^m a$, then $f(z) \neq \tilde{\xi}_\lambda$ on $\partial B_\varepsilon(z^0)$, for all $\lambda \in [0, 1]$. Consequently, $d(f, B_\varepsilon(z^0), \tilde{\xi}_\lambda)$ is constant all $\lambda \in [0, 1]$, and

$$\deg(f, B_\varepsilon(z^0), 0) = \deg(f, B_\varepsilon(z^0), \delta) = \deg(p, B_\varepsilon(z^0), \delta) = m,$$

which completes the proof.

A direct consequence of the above lemma is a result on mapping degree holomorphic functions in general.

\textbf{12.2 Corollary.} Let $f : \overline{\Omega} \subset \mathbb{C} \to \mathbb{C}$ be a holomorphic function. Assume that $0 \not\in f(\partial \Omega)$. Then

$$d(f, \Omega, 0) \geq 0.$$ 

\textbf{Proof:} By analyticity $f$ has only isolated zeroes $z^i \in \Omega$. Let $B_\varepsilon(z^i) \subset \Omega$ be sufficiently small neighborhoods containing exactly one zero each. The excision and summation properties of the degree then give

$$d(f, \Omega, 0) = d(f, \cup_i B_\varepsilon(z^i), 0) = \sum_i d(f, B_\varepsilon(z^i), 0) = \sum_i m_i,$$

where the numbers $m_i \geq 1$ are the orders of the zeroes $z^i$. Since $\sum_i m_i \geq 0$ this yields the desired result.

Another consequence of Lemma 12.1 is the Fundamental Theorem of Algebra.

\textbf{12.3 Corollary.} Any polynomial $p(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_0$, with real coefficients $a_i$, has exactly $n$ complex roots counted with multiplicity.

\textbf{Proof:} Write $p(z) = z^n + r(z)$, then $|p(z) + r(z)| \geq ||p(z)| - |r(z)||$. On the circle $|z| = R > 0$, for $R$ sufficiently large, we have $|r(z)| \leq CR^n - 1$, and thus

$$|p(z) + r(z)| \geq ||p(z)| - |r(z)|| \geq R^n - CR^n - 1 > 0,$$

which proves that all zeroes are contained in the ball $B_R(0)$, and $\deg(f, B_R(0), 0)$ is well-defined. The same holds for the homotopy $p_\lambda(z) = z^n + \lambda r(z), \lambda \in [0, 1]$. This gives

$$\deg(p, B_R(0), 0) = \deg(z^n, B_R(0), 0) = n > 0,$$

implying that $p(z) = 0$ has at least one solution $z^1$ in $B_R(0)$. Now repeat the argument for the polynomial $p^1(z) = \frac{p(z)}{z^{n_1}}$. This again produces a zero $z^2$. This process terminates after $n$ steps, proving the desired result.
13. **Linking numbers**

The usual example of linking are two tangled closed loops in $\mathbb{R}^3$, but also the winding of a closed loop around a point in the plane is an example of linking in $\mathbb{R}^2$. Similarly, a compact orientable surface in $\mathbb{R}^3$ separating the inside from the outside is an example of lining in $\mathbb{R}^3$. The concept of linking can be formulated in terms of degree degree for objects of higher dimension as well.

Let $K, L \subset \mathbb{R}^n$ be smooth embedded manifolds of dimension $k$ and $\ell$ respectively. Assume that both $K$ and $L$ are compact and orientable. Moreover $K \cap L = \emptyset$ and $k + \ell = n - 1$.

Define the mapping

$$
\Psi : K \times L \subset \mathbb{R}^{2n} \to S^{n-1} \subset \mathbb{R}^n, \quad (x, y) \mapsto \frac{y-x}{|y-x|},
$$

which is a continuous mapping between orientable manifolds. The orientation on $K \times L$ is the product orientation and the orientation on $S^{n-1}$ the orientation induced by $\mathbb{R}^n$.

**\[13.1\] Definition.** For two disjoint, smoothly embedded compact and orientable submanifolds $K$ and $L$ in $\mathbb{R}^n$, the linking number is defined by

$$
\text{link}(K, L) := \deg(\Psi),
$$

provided that $k + \ell = n - 1$.

For the traditional linking of embedded circles in $\mathbb{R}^3$ we can compute some simple examples.

**\[13.2\] Example.** Consider embedded circles $K$ and $L$ in $\mathbb{R}^3$. In order to compute the linking number we need to compute the degree of the map $\Psi : K \times L \cong \mathbb{R}^2 \to S^2$. We start with a volume form on $S^2$. Define $\omega = i_\alpha dx$, where $dx = dx_1 \wedge dx_2 \wedge dx_3$ and $n = x_1 \frac{\partial}{\partial x_1} + x_2 \frac{\partial}{\partial x_2} + x_3 \frac{\partial}{\partial x_3}$ the unit normal vector field to $S^2 \subset \mathbb{R}^3$, then

$$
\omega = x_1 dx_2 \wedge dx_3 - x_2 dx_1 \wedge dx_3 + x_3 dx_1 \wedge dx_2.
$$

The integral $\int_{S^2} \omega = 4\pi$ gives the area (volume) of $S^2$. The map $\Psi$ is a composition of the $\Phi(x, y) = y - x : K \times L \to \mathbb{R}^3 \setminus \{0\}$ and the retraction $\rho(x) = \frac{x}{|x|} : \mathbb{R}^3 \setminus \{0\} \to S^2$.

Now

$$
\rho^* \omega(\tilde{x}, \eta) = \omega(\rho_*(\tilde{x}), \rho_*(\eta)) = \frac{x_1}{|x|^3} dx_2 \wedge dx_3(\tilde{x}, \eta) - \frac{x_2}{|x|^3} dx_1 \wedge dx_3(\tilde{x}, \eta) + \frac{x_3}{|x|^3} dx_1 \wedge dx_2(\tilde{x}, \eta) = \det(x, \tilde{x}, \eta),
$$

where we used the fact that for $\tilde{x}, \eta \in T_x S^2$ it holds that $\rho_*(\tilde{x}) = \frac{1}{|x|} \tilde{x}$ and $\rho_*(\eta) = \frac{1}{|x|} \eta$. Under the map $\Phi$ we obtain

$$
\Phi^* \omega(\tilde{x}, \eta) = \omega(-\tilde{x}, \eta) = -\omega(\tilde{x}, \eta) = -\det(y - x, \tilde{x}, \eta) = \det(x - y, \tilde{x}, \eta).
$$
For the map $\Psi$ this implies that
\[
\Psi^*\omega(\xi, \eta) = \frac{\det(x - y, \xi, \eta)}{|x - y|^3}.
\]

Parametrize $K$ and $L$ and denote the parametrizations by $\kappa$ and $\lambda$ respectively. Then,
\[
\int_{K \times L} \Psi^*\omega = \int_0^{2\pi} \int_0^{2\pi} \frac{\det(\kappa(t) - \lambda(s), \kappa'(t), \lambda'(s))}{|\kappa(t) - \lambda(s)|^3} dt ds.
\]
The linking number is given by
\[
(13.1) \quad \text{link}(K, L) = \frac{1}{4\pi} \int_0^{2\pi} \int_0^{2\pi} \frac{\det(\kappa(t) - \lambda(s), \kappa'(t), \lambda'(s))}{|\kappa(t) - \lambda(s)|^3} dt ds.
\]

This integral may be hard to compute. Consider an example of two circles in the $x_1, x_2$-plane, then
\[
t \mapsto (\cos(t), \sin(t), 0), \quad s \mapsto (2 \cos(s), 2 \sin(s), 0),
\]
and $\det(\kappa(t) - \lambda(s), \kappa'(t), \lambda'(s)) = 0$, which shows that $\text{link}(K, L) = 0$.

Before doing some more elaborate examples let us derive some properties of the linking number.

\textbf{\textsection 13.3 Theorem.} The linking number satisfies the following properties:

(i) $\text{link}(L, K) = (-1)^{(k+1)(\ell+1)} \text{link}(K, L)$;
(ii) if $K$ and $L$ are separated by a hyperplane in $\mathbb{R}^n$, then $\text{link}(K, L) = 0$;
(iii) let $K_t$ and $L_t$ be 1-parameter families of embedded circles such that $K_t \cap L_t = \emptyset$ for all $t \in [0, 1]$, then $\text{link}(K_0, L_0) = \text{link}(K_1, L_1)$.

As a matter of fact the linking number is an isotopy invariant.

\textbf{Proof:} For the pair $L, K$ we have the map $\Psi(x, y) = \frac{x - y}{|x - y|}$. Define the maps
\[
r(x, y) = (y, x) \quad \text{and} \quad a(x, y) = (-x, -y).
\]
Then $\deg(r) = (-1)^{k\ell}$ and $\deg(a) = (-1)^{k+\ell+1}$. For the map $\Psi$ it holds that $\Psi = r^{-1} \circ \Psi \circ a$ and and by the composition property of the degree we derive the desired statement in (i).

As (ii) if a hyperplane exists then $\Psi$ is not surjective onto $\mathbb{S}^{n-1}$ and therefore $\deg(\Psi) = 0$, which proves (ii).

Property (iii) is a direct consequence of he homotopy principle for the degree. □

\textbf{\textsection 13.4 Example.} Consider the circles $K = \{x \in \mathbb{R}^3 \mid x_1^2 + x_2^2 = 1, \ x_3 = 0\}$ and $L = \{x \in \mathbb{R}^3 \mid (x_2 - 1)^2 + x_3^2 = 1, \ x_1 = 0\}$. On $K$ consider the orientation form $\theta_K = -x_2 dx_1 + x_1 dx_2$ and on $L$ the orientation form $\theta_L = -x_3 dx_2 + (x_2 - 1)dx_3$. Choose the following parametrizations
\[
t \mapsto (-\sin(t), \cos(t), 0), \quad s \mapsto (0, 1 + \cos(s), \sin(s)),
\]
again denoted by $\kappa$ and $\lambda$ respectively. Upon substitution in Equation (13.1) yields the following expression
\[
\text{link}(K, L) = \frac{1}{4\pi} \int_0^{2\pi} \int_0^{2\pi} \frac{\cos(s) - \cos(t) \cos(s) - \cos(t)}{(3 + 2\cos(s) - 2\cos(t) \cos(s) - 2\cos(t))^{3/2}} dt ds.
\]
Under the mapping $\Psi$ the inverse image of a value $p \in S^2$ is characterized by the following relation

$$\Psi^{-1}(p) = \{(x, y) \in K \times L \mid y - x = \mu p, \mu > 0\}.$$ 

Such a value is regular if $\Psi^* \omega$ is nondegenerate at points in $\Psi^{-1}(p)$. By our previous calculations this means when $\det(x - y, \xi, \eta) \neq 0$, where $\xi \in T_xK$ and $\eta \in T_yL$. If we choose $p$ to be a regular value, then the degree can be computed by adding the signs of the determinants at points in $\Psi^{-1}(p)$. Let us carry out this calculation for the above situation. Choose $p = (0, 1, 0)$, then $\Psi^{-1}(p)$ consists of the point pairs $(0, 1, 0) \in K$, $(0, 2, 0) \in L$, $(0, -1, 0) \in K$, $(0, 2, 0) \in L$ and $(0, -1, 0) \in K$, $(0, 0, 0) \in L$. The determinants are $-1$, $+2$ and $-1$ respectively, and therefore $\text{link}(K, L) = -1$.

**13.5 Remark.** In order to compute $\text{link}(K, L)$ in Example 13.4 one can also try to evaluate the integral with brute force. We the help of Maple we obtain that

$$\text{link}(K, L) = \frac{1}{4\pi} \int_0^{2\pi} \int_0^{2\pi} \frac{\cos(s) - \cos(t) \cos(s) - \cos(t)}{(3 + 2\cos(s) - 2\cos(t) \cos(s) - 2\cos(t))^3/2} ds dt$$

$$= \frac{1}{4\pi} \int_0^{2\pi} 2 \cdot \text{EllipticK} \left( 2 \sqrt{-1 + \cos(t)} \right)$$

$$- 6 \cdot \text{EllipticE} \left( 2 \sqrt{\cos(t) - 1} \right) \frac{1}{5 - 4\cos(t)} dt = -1,$$

which follows by numerically integrating the elliptic integrals.
IV. Extensions of the degree and elementary homotopy theory

For continuous mappings $f$ from a compact domain $\overline{\Omega} \subset \mathbb{R}^n$ to $\mathbb{R}^n$, the question of solvability of the $f(x) = p$ is determined only by the mapping degree, when formulated in the following setting. It was proved in Section 8 that the degree $\text{deg}(f, \Omega, p)$ is determined only by the degree of the boundary map $\varphi = f|_{\partial \Omega} : \partial \Omega \to \mathbb{R}^n \setminus \{p\}$. Non-triviality of $\text{deg}(\varphi)$ implies that any continuous extension $f$ of $\varphi$ to $\overline{\Omega}$ has a solution to the equation $f(x) = p$. In this chapter it is proved that the converse also holds, i.e. if every continuous extension $f$ of $\varphi$ to $\overline{\Omega}$ has a solution to $f(x) = p$, then $\text{deg}(\varphi) \neq 0$. This already indicates that the question of solvability is strongly related to the problem of extending a mapping $\varphi$ to all of $\overline{\Omega}$. To be more precise, if $\varphi : \partial \Omega \subset \mathbb{R}^n \to \mathbb{R}^n \setminus \{p\}$ has a continuous extension $f : \overline{\Omega} \to \mathbb{R}^n \setminus \{p\}$, then the boundary map $\varphi$ does not force solvability of $f(x) = p$ for all continuous extensions $f$, with $f|_{\partial \Omega} = \varphi$. In this case $\varphi$ is said to be inessential with respect to $\Omega$. When $\varphi$ is not inessential with respect to $\Omega$ it is said to be essential with respect to $\Omega$, which implies there are no continuous extension $f : \overline{\Omega} \to \mathbb{R}^n \setminus \{p\}$, and thus for continuous extension $f$ takes values in $\mathbb{R}^n$ in general and the equation $f(x) = p$ has non-trivial solutions in $\Omega$. A fundamental theorem by Hopf is used to prove that essential versus inessential is completely determined by the mapping degree of $\varphi$.

The goal of this chapter is to broaden the above question to cases where the degree cannot decide between essential versus inessential, or when the mapping degree is not defined. An important case is for maps

$$f : \overline{\Omega} \subset \mathbb{R}^n \to \mathbb{R}^k,$$

where $n$ is not necessarily equal to $k$. In this case the degree as introduced in Chapter ?? is not defined. The question is whether $\varphi = f|_{\partial \Omega} : \partial \Omega \to \mathbb{R}^k \setminus \{p\}$ still determines the solvability if $f(x) = p$, for any continuous extension $f$ of $\varphi$.

14. Homotopy types and Hopf’s Theorem

Consider mappings $\psi : \partial \Omega \subset \mathbb{R}^n \to S^{n-1}$, where $S^{n-1} \subset \mathbb{R}^n$ is the standard unit sphere. In this equal dimension situation an important version of the extension problem holds which can be regarded as a version of Hopf’s Theorem.
14.1 Theorem. Let $\Omega \subset \mathbb{R}^n$ be a connected, bounded domain. A continuous mapping $\psi : \partial \Omega \subset \mathbb{R}^n \to S^{n-1}$ extends to a continuous mapping $f : \overline{\Omega} \subset \mathbb{R}^n \to S^{n-1}$, with $f|_{\partial \Omega} = \psi$ if and only if $\deg(\psi) = 0$.

Theorem 14.1 is also referred to the extension problem for mappings $\psi : \partial \Omega \to S^{n-1} \subset \mathbb{R}^n$ and connected boundaries of bounded open sets $\Omega \subset \mathbb{R}^n$. In the forthcoming sections this problem will be put in a more general context. As explained above the extension problem is directly linked to the solvability problem, see Corollary 14.13.

14.2 Remark. The connectivity condition Theorem 14.1 can be omitted by replacing the condition on the degree. Let $\psi' = \psi|_{\partial \Omega'}$, where $\Omega'$ are the connected components of $\Omega$, then the condition on the degree becomes $\deg(\psi') = 0$ for all connected components $\Omega'$ of $\Omega$. The proof is obvious by applying Theorem 14.1 to each component.

14.3 Definition. A family of mappings $\psi_t = \Psi(\cdot, t)$, with $\Psi : \partial \Omega \times [0,1] \to S^{n-1}$ is continuous, is called a homotopy between $\psi_0, \psi_1 : \partial \Omega \to S^{n-1}$. The mappings $\psi_0$ and $\psi_1$ are called homotopic.

Homotopy is an equivalence relation on $C^0(\partial \Omega; S^{n-1})$ and its equivalence classes are called homotopy types or homotopy classes. The homotopy type of a map $\psi$ in $C^0(\partial \Omega; S^{n-1})$ is denoted by $[\psi]$ and the collection of all homotopy types or equivalence classes is denoted by $[\partial \Omega; S^{n-1}] = \{ [\psi] \mid \psi \in C^0(\partial \Omega; S^{n-1}) \}$.

14.4 Exercise. Show that homotopy type introduced above defines an equivalence relation on $C^0(\partial \Omega; S^{n-1})$.

Theorem 14.1 can be proved by using the following fundamental property of the generalized winding number (see Section 8), which is a special case of Hopf’s Theorem.

14.5 Lemma. A continuous mapping $\psi : S^{n-1} \to S^{n-1}$, where $S^{n-1} \subset \mathbb{R}^n$ is the standard unit sphere, has trivial homotopy type if and only if $\deg(\psi) = 0$.


14.6 Exercise. Give an elementary proof of Lemma 14.5 (Hint: use an induction argument in $n$).

Proof of Theorem 14.1. If there exists a continuous extension $f : \overline{\Omega} \subset \mathbb{R}^n \to S^{n-1} \subset \mathbb{R}^n \setminus \{0\}$, then $\deg(f, \Omega, 0) = 0$, and thus by definition $\deg(\psi) = 0$.

Now suppose $\deg(\psi) = 0$. Let $g : \overline{\Omega} \subset \mathbb{R}^n \to \mathbb{R}^n$ be a continuous extension (use Tietze’s Extension Theorem), with $g|_{\partial \Omega} = \psi$. By construction $g^{-1}(0) \subset U \subset \Omega$, where $U$ is compact. Moreover, $g$ can be chosen to be $C^1$ on $U$, and such that $0$ is a regular value (see Chapter ??). In this case $g^{-1}(0)$ is a finite set of points in $U \subset \Omega$. Now connect the points $x_i \in g^{-1}(0)$ via a path $\gamma_i$ such that $\gamma_i$ has no self-intersections. Since $\gamma \subset U$ is a compact set it can be covered by finitely many small open ball $B_i$, which yields a compact set $\overline{V} \subset U$ which contains $\gamma$, and $g$ has a piecewise smooth boundary $\partial V$. Moreover, $\overline{V}$ is homeomorphic to the unit
The zeroes of $g$ are contained in $U \subset \Omega$ and are connected by a non-intersecting path $\gamma$ [left]. The path $\gamma \subset U$ can be covered by a union of open balls $V \subset U$ [right].

The following theorem due to E. Hopf shows that the homotopy types in $C^0(\partial \Omega; S^{n-1})$ are characterized by the mapping degree, which is therefore the only homotopy invariant on $C^0(\partial \Omega; S^{n-1})$, which generalizes Lemma 14.5 and is a direct consequence of Theorem 14.1. Theorem 14.8 below is referred to as the classification problem and generalizations will be discussed in forthcoming sections.

**14.8 Theorem.** Let $\partial \Omega \subset \mathbb{R}^n$ be compact, connected, smooth hypersurface.\(^7\) Then, two continuous mappings $\psi_0, \psi_1 : \partial \Omega \rightarrow S^{n-1}$ are homotopic if and only if $\deg(\psi_0) = \deg(\psi_1)$.

---

\(^7\)A smooth hypersurface is the level set $H^{-1}(0)$ of a smooth function $H : \mathbb{R}^n \rightarrow \mathbb{R}$, where 0 a regular value. Such a hypersurface is an embedded codimension-1 submanifold of $\mathbb{R}^n$, see also Chapter ???. Moreover, $\partial \Omega$ is orientable.
Proof: Two mappings $\psi_0, \psi_1 : \partial \Omega \to S^{n-1}$ are homotopic if and only if there exists a homotopy $\psi_t = \Psi(\cdot, t)$ between $\psi_0$ and $\psi_1$, where $\Psi : \partial \Omega \times [0, 1] \subset \mathbb{R}^{n+1} \to S^{n-1}$ is a continuous mapping. Let $F : \Omega \times [0, 1] \to \mathbb{R}^n$ be an extension of $\Psi$ (use Tietze’s Extension Theorem), then

$$\deg(f, \Omega, 0) = \deg(f_0, \Omega, 0) = \deg(f_1, \Omega, 0),$$

and therefore $\deg(\psi_0) = \deg(\psi_1)$.

Now suppose $\deg(\psi_0) = \deg(\psi_1)$. By assumption $\partial \Omega = H^{-1}(0)$ for some smooth function $H : \mathbb{R}^n \to \mathbb{R}$, with $0$ a regular value. Therefore the interval $[-\varepsilon, \varepsilon]$, $\varepsilon > 0$ sufficiently small, consists of regular values. The function is assumed to be negative on $\Omega$, $H < 0$, and thus bounded from below. Define the domain

$$\Lambda := \{ x \in \mathbb{R}^n \mid -\varepsilon < H(x) < 0 \},$$

is connected with $\partial \Lambda = \partial \Omega - H^{-1}(-\varepsilon)$. The deformation lemma in Section 28 can be used now to show that there exists an isotopy\(^8\) from $\partial \Omega = H^{-1}(0)$ to $H^{-1}(-\varepsilon)$. Indeed, consider the normalized gradient flow

$$\frac{dx}{dt} = -\frac{\nabla H(x)}{||\nabla H(x)||^2},$$

The solution of the initial value problem for $x \in \partial \Omega$ is given by $\xi(x, t)$, with

$$\xi(x, 0) = x, \quad H(\xi(x, t)) = H(x) - t.$$ 

For details see Section 28. fig:fig-deform1 The mapping $\eta(x, t) = \xi(x, t(H(x) + \varepsilon))$ defines an isotopy from $\partial \Omega$ to $H^{-1}(-\varepsilon)$;

$$\eta : \partial \Omega \times [0, 1] \to \mathbb{R}^n,$$

where each $\eta_t(\cdot) = \eta(\cdot, t)$ is diffeomorphism from $\partial \Omega$ to $H^{-1}(-\varepsilon t)$. Let $\psi$ be a mapping from $\partial \Lambda$ to $S^{n-1}$ defined as $\psi_0$ on $\partial \Omega$ and $\psi_1 \circ \eta_{-1}$ on $H^{-1}(-\varepsilon)$. By Theorem 8.3

$$\deg(\psi) = \int_{\partial \Lambda} \psi^* \mu = \int_{\partial \Omega} \psi_0^* \mu - \int_{H^{-1}(-\varepsilon)} \left( \psi_1 \circ \eta_{-1} \right)^* \mu.$$

\(^8\)An isotopy is a homotopy $h_t$ for which $h_t$ is a diffeomorphism for each $t \in [0, 1]$. 

\[\text{Figure 14.2. Deformation of } \partial \Omega \text{ via the normalized gradient flow on } H.\]
Since $\eta_1$ is a diffeomorphism, it holds that $\int_{H^{-1}(-\varepsilon)}(\psi_1 \circ \eta_1^{-1})^* \mu = \int_{\partial \Omega} \psi_1^* \mu$, and therefore $\deg(\psi) = 0$. By Theorem 14.1 there exists a continuous mapping $f : \overline{X} \subset \mathbb{R}^n \to S^{n-1} \subset \mathbb{R}^n$. Now define

\[ \Psi(x,t) = f(\eta(x,t)) : \partial \Omega \times [0,1] \to S^{n-1}, \]

which is a homotopy between $\psi_0$ and $\psi_1$, and therefore proves the theorem.

\[\blacklozenge\]

**14.10 Corollary.** There exists a mapping $\psi : \partial \Omega \to S^{n-1}$ of any degree $m \in \mathbb{Z}$.

In particular $[\partial \Omega, S^{n-1}] \cong \mathbb{Z}$.

**Proof:** Under construction.

Theorem 14.8 is derived from the extension problem in Theorem 14.1. On the other hand Theorem 14.1 can be derived from the classification problem in Theorem 14.8 in the special case when $\partial \Omega = S^{n-1}$.

**14.11 Remark.** Hopf’s Theorem (Theorem 14.8) still holds in the case that $\partial \Omega$ is a triangulable set. Recall that a set is triangulable if it is homeomorphic to $n$-dimensional simplicial complex $\Delta_n$. The result also holds for maps $\psi : X \to S^{n-1}$, where $X$ is a triangulable topological space, with $\dim(X) = n - 1$, see [10]. In particular (abstract) smooth manifolds $M$ are triangulable, and therefore Hopf’s Theorem extends to maps $\psi : M \to S^{n-1}$. In Chapter ?? the notion of degree for maps between smooth manifolds will be introduced.

**14.12 Example.** Consider the annulus $\Omega = \{(x,y) \in \mathbb{R}^2 \mid 1 \leq x^2 + y^2 \leq 2\}$, and the mapping

\[ f(x,y) = \left( \begin{array}{c} y/\sqrt{x^2 + y^2} \\ x/\sqrt{x^2 + y^2} \end{array} \right), \]

acting from $\overline{\Omega}$ to $\mathbb{R}^2 \setminus \{0\}$. The boundary $\partial \Omega$ of the annulus is disconnected and $\deg(\psi) = 0$, where $\psi = f|_{\partial \Omega} : \partial \Omega \to S^1$. Clearly, $[\psi] \neq 0$, which shows that the connectivity condition in Hopf’s Theorem cannot be removed.

Connectivity of $\partial \Omega$ is not required for Theorem 14.1. The degree gives the proper invariant and a straightforward calculation shows that $\deg(\psi) = 0$, which is in compliance with the extension $f$.

The above theorem states that $\psi$ is inessential with respect to $\Omega$ if and only if $\deg(\psi) = 0$. The extension problem in Theorem 14.1 can be rephrased into a solvability property for the equation $f(x) = p$, with $f : \overline{\Omega} \subset \mathbb{R}^n \to \mathbb{R}^n$, and $\varphi = f|_{\partial \Omega}$. This problem will be referred to as the solvability problem. In the latter case the boundary mapping is denoted by $\varphi : \partial \Omega \subset \mathbb{R}^n \to \mathbb{R}^k \setminus \{p\}$, and the extension problem becomes; given $\varphi$, does there exist a continuous extension $f : \overline{\Omega} \subset \mathbb{R}^n \to \mathbb{R}^n$, with $\varphi = f|_{\partial \Omega}$. This version of the extension problem is equivalent to the version in 14.1. Indeed, a normalized mapping $\psi = \frac{\varphi - p}{\|\varphi - p\|} : \partial \Omega \subset \mathbb{R}^n \to S^{k-1}$ is inessential with respect to $\Omega$ — there exists a continuous extension $g : \overline{\Omega} \subset \mathbb{R}^n \to S^{k-1}$ — if and only if $\varphi : \partial \Omega \subset \mathbb{R}^n \to \mathbb{R}^k \setminus \{p\}$ is inessential — there exists a continuous extension $f : \overline{\Omega} \subset \mathbb{R}^n \to \mathbb{R}^k \setminus \{p\}$. Indeed, if $\varphi$ is inessential then $g = \frac{\varphi - p}{\|\varphi - p\|}$ gives the desired extension for $\psi$, and conversely, if $\psi$ is inessential, then $f = r \cdot g + p$ is
the desired extension for $\varphi$, where $r : \overline{\Omega} \subset \mathbb{R}^n \to \mathbb{R}^+$ is a continuous extension of $\rho = |\varphi - p| : \partial \Omega \subset \mathbb{R}^n \to \mathbb{R}^+$ via Tietze’s Extension Theorem.

**14.13 Corollary.** Let $\Omega \subset \mathbb{R}^n$ be a connected domain. A continuous mapping $\varphi : \partial \Omega \subset \mathbb{R}^n \to \mathbb{R}^n \setminus \{p\}$ is essential with respect to $\Omega$ if and only if $\deg(\varphi) \neq 0$. ▶

*Proof:* By the discussion above and Theorem 14.1 $\varphi$ is inessential with respect to $\Omega$ if and only if $\deg(\varphi) = 0$. Therefore, $\varphi$ is essential with respect to $\Omega$ if and only if $\deg(\varphi) \neq 0$. □

**14.14 Remark.** If $\Omega$ is not necessarily connected, then the condition on the degree has to be replaced with $\deg(\varphi') \neq 0$, for some $i$, where $\varphi' = f|_{\partial \Omega^i}$, and $\Omega^i$ are the connected components of $\Omega$. ▶

## 15. The extension problem for mappings on a ball

The main task in this section is to investigate the extension and classification problems for $n$ is not necessarily equal to $k$, in the special case when $\overline{\Omega}$ is homeomorphic to a closed ball. As it turns out the degree cannot be used as a invariant, but homotopy type plays a crucial role in this special case, as well the appropriate homotopy invariants that will be discussed in Section 17. The generalization of the extension problem for mappings $\psi : \partial \Omega \cong S^{n-1} \to S^{k-1}$ can be formulated as follows.

**15.1 Theorem.** Let $\overline{\Omega}$ be homeomorphic to $\overline{B_1(0)}$. Then a continuous mapping $\psi : \partial \Omega \cong S^{n-1} \to S^{k-1}$ is inessential with respect to $\Omega$ — in other words there exists a continuous extension $f : \overline{\Omega} \to S^{k-1}$ with $f|_{\partial \Omega} = \psi$ — if and only if $\psi$ has trivial homotopy type. ▶

In the previous section the degree was the appropriate invariant for extension problem. Just the homotopy type does not fully describe the problem as Example 14.12 shows. Her, in the case that $\partial \Omega \cong S^{n-1}$ homotopy type and degree contain the same information by Hopf’s Theorem. Homotopy type makes sense when $n \neq k$, which explains the formulation of the above theorem using homotopy type. When $\partial \Omega$ is not homeomorphic to a sphere the situation becomes more complicated.

*Proof:* Suppose $\psi$ is inessential with respect to $\Omega$, i.e. there exists a continuous extension $f : \overline{\Omega} \subset \mathbb{R}^n \to S^{k-1}$. Let $g : \overline{\Omega} \to \overline{B_1(0)}$ be a homeomorphism, then $\tilde{f} = f \circ g^{-1} : B_1(0) \to S^{k-1}$. Define the homotopy

$$h(x,t) = \tilde{f}(tx),$$

which, when restricted to $S^{n-1}$, becomes a homotopy between $\varphi \circ g^{-1}$ and the constant map $x \mapsto \tilde{f}(0)$. Via the homeomorphism $g$, the homotopy $k = h \circ g$, which provides a homotopy between $\psi$ and the constant map. Therefore, $\psi$ has trivial homotopy type.
Assume $|\psi| = 0$, then there exists a homotopy $h : \partial \Omega \times [0, 1] \to \mathbb{R}^k \setminus \{0\}$ between $\psi$ and a constant map. The map $k = h \circ g^{-1} : S^{n-1} \times [0, 1] \to S^{k-1}$ is then a homotopy between $\varphi \circ g^{-1}$ and a constant map. Now define the continuous extension

$$\tilde{f} : \overline{B}_1(0) \to S^{k-1}, \quad \tilde{f}(tx) = k(x, t), \ x \in S^{n-1}.$$ 

The map $f = \tilde{f} \circ g$ now is continuous extension of $\psi$ to $\overline{\Omega}$.

As in the previous section above corollary is a characterization of the solvability problem for domains homeomorphic to a ball.

\begin{itemize}
  \item \textbf{15.2 Corollary.} Let $\overline{\Omega}$ be homeomorphic to $\overline{B}_1(0)$. Then a continuous mapping $\varphi : \partial \Omega \cong S^{n-1} = \partial B_1(0) \subset \mathbb{R}^n \to \mathbb{R}^k \setminus \{0\}$ is essential with respect to $\Omega$ — in other words any continuous extension $f : \overline{\Omega} \to \mathbb{R}^k$ of $\varphi$ has a nontrivial solution to the equation $f(x) = 0$ — if and only if the normalized boundary mapping $\psi = \varphi/|\varphi| : S^{n-1} \to S^{k-1}$ has nontrivial homotopy type.
  
  In this section $\partial \Omega$ is homeomorphic to the standard unit sphere $S^{n-1} \subset \mathbb{R}^n$, the homotopy classes $[\partial \Omega, S^{k-1}]$ can be linked to the standard homotopy groups $\pi_{n-1}(S^{k-1})$.

  \begin{itemize}
    \item \textbf{15.3 Exercise.} Show that there exists a canonical isomorphism $[\partial \Omega, S^{k-1}] \cong \pi_{n-1}(S^{k-1})$, and give the group structure on $[\partial \Omega, S^{k-1}]$.
    \item \textbf{15.4 Exercise.} Prove that $\pi_n(S^n) \cong \mathbb{Z}$ (Hint: construct the isomorphism).
    \item \textbf{15.5 Exercise.} Show that $\pi_{n-1}(S^{k-1}) \cong 0$ for all $n < k$.
  \end{itemize}

In Section 14 the homotopy types $[\partial \Omega, S^{n-1}]$ were characterized by the mapping degree. In this section the boundary $\partial \Omega$ is restricted to the special case of a sphere, but is more general in the sense that $k$ is not necessarily equal to $n$. The degree cannot be used to classify the homotopy types $[S^{n-1}, S^{k-1}]$. The framed cobordism theory of Pontryagin gives an satisfactory answer to this question. Framed cobordisms will be considered in Section 17 and will actually apply to the general case of smooth boundaries $\partial \Omega$.

For proper mappings $f : \mathbb{R}^n \to \mathbb{R}^k$ the above theory can be used the give necessary and sufficient condition for the solvability problem.

\section{The general extension problem}

Let $f : \overline{\Omega} \subset \mathbb{R}^n \to \mathbb{R}^k$, be a continuous mapping with $\varphi = f|_{\partial \Omega} \neq 0$, and consider the equation

$$f(x) = 0.$$ 

In the case $n = k$ the existence of a solution for any continuous extension $f$, with $\varphi = f|_{\partial \Omega} \neq 0$, is equivalent to nontriviality of $|\psi|$ (assuming $\partial \Omega$ is connected). This a consequence of Hopf’s Theorem and a more general result for any $n$ and $k$. Before formulating the general result and proving the above statement it is should be pointed out that solvability and homotopy type are not necessarily equivalent when $n \neq k$, as the following example shows.
Consider the mapping $f : \mathbb{R}^3 \setminus \{0\} \to \mathbb{R}^2$ and the solid torus $\Omega \cong B^2 \times S^1$ both given by

$$f(x,y,z) = \left( \frac{-y}{\sqrt{x^2 + y^2}}, \frac{x}{\sqrt{x^2 + y^2}} \right), \quad (\alpha, \beta, r) \mapsto \begin{pmatrix} (2 + r\cos(\beta)) \cos(\alpha) \\ (2 + r\cos(\beta)) \sin(\alpha) \end{pmatrix},$$

where $(\alpha, \beta, r) \in (\mathbb{R}/2\pi \mathbb{Z})^2 \times [0, 1)$. Figure 16.1 clearly shows that $f$ is a continuous extension to $\overline{\Omega}$ which is nowhere vanishing. This explains that non-trivial homotopy type of a map $\varphi$ on a boundary $\partial \Omega$ which is not homeomorphic to sphere, and $n \neq k$, does not necessarily imply that every continuous extension has a zero.

In order to give criteria for the general solvability problem, the problem will be reformulated in terms of the general extension problem. As before consider mappings $\psi : \partial \Omega \to S^{k-1}$. An if and only if criterion for extendibility to all of $\overline{\Omega}$ also provides an if and only if criterion for solvability of $f(x) = 0$ in terms of $\varphi$ restricted to $\partial \Omega$. Therefore, a criterion for the extension problem is formulated first.

**16.3 Lemma.** The extension problem for $\psi$ only depends the homotopy type of $\varphi$. To be more precise, given an extension $f : \overline{\Omega} \to S^{k-1}$, then for every if for every (partial) homotopy $h_t = h(\cdot, t) : \partial \Omega \times [0, 1] \to S^{k-1}$, for which $h_0 = \psi$, there exists an homotopy extension $f_t = f(\cdot, t) : \overline{\Omega}[0, 1] \to S^{k-1}$. In case the pair $(\overline{\Omega}, \partial \Omega)$ is said to satisfy the homotopy extension property (HEP).

**Proof:** Under construction.

**16.4 Remark.** The homotopy extension property holds for an arbitrary pair $(X, A)$, with $X$ a metric space, and $A$ a closed subspace, with respect to $S^{k-1}$. The latter can also be relaxed to be a finitely triangulable space $Y$. If $(X, A)$ is a finitely triangulable pair, then the homotopy extension property is satisfied for any topological space $Y$, in which case $(X, A)$ said to satisfy the absolute homotopy extension property (AHEP).

![Figure 16.1.](image-url) The map $\varphi = f|_{\partial \Omega}$ maps to the unit circle in $\mathbb{R}^2 \setminus \{0\}$, and can be viewed as constant vector field along the torus. Clearly the homotopy type of $\varphi$ is non-trivial.
The the sets of homotopy classes \([\Omega;S^k-1]\) and \([\partial\Omega;S^k-1]\) do not a priori have an algebraic structure. For example if \(\partial\Omega = S^{n-1}\), then \([S^{n-1};S^k-1]\) can be identified with the homotopy groups \(\pi_{n-1}(S^k-1)\). Some functorial properties of \([\cdot;\cdot]\) can be easily derived. Consider topological spaces \(X\) and \(Y\) and continuous mappings \(g : X \to Y\), then by considering the set of homotopy classes \([X;S^k-1]\) and \([Y;S^k-1]\), there is an induced morphism, or mapping
\[
g^* : [Y;S^k-1] \to [X;S^k-1],
\]
defined as follows. A class \(\theta \in [Y;S^k-1]\) is represented by a mapping \(h : Y \to S^k-1\), and \(h \circ g : X \to S^k-1\) represents a the homotopy class
\[
g^*(\theta) := [h \circ g] \in [X;S^k-1].
\]

\[\textbf{xer:funcl}\]

\[\textbf{Exercise.}\] Show that \([\cdot, S^k-1]\) defines a contravariant functor from the category of topological spaces to the category of sets.

Let \(i : A \subset X\) be the inclusion map, and \(\psi : A \to S^k-1\), which has an extension \(f : X \to S^k-1\), such that \(f|_A = f \circ i = \psi\). Clearly, \(\psi^* : [S^k-1;S^k-1] \to [A;S^k-1]\), and \([S^k-1,S^k-1] \cong \pi_{k-1}(S^k-1) \cong \mathbb{Z}\). Let \(1 \in \pi_{k-1}(S^k-1) \cong \mathbb{Z}\) correspond to the identity \(\text{Id}\) in \([S^k-1,S^k-1]\), then \(\psi^*(1) = [\psi]\). The assumption that there exists an extension \(f\) then yields
\[
i^*(f^*(1)) = \psi^*(1), \quad f^*(1) \in [X;S^k-1],
\]
which shows that \(\psi^*(1) \in i^*(|[X;S^k-1]|)\). The converse also holds, which gives the following result.

\[\textbf{16.6 Theorem.}\] Let \((X,A)\) be a topological pair and \(\psi : A \to S^k-1\) a continuous mapping. There exists a continuous extension \(f : X \to S^k-1\) if and only if
\[
[\psi] = \psi^*(1) \in i^*(|[X;S^k-1]|),
\]
where \(1 = [\text{Id}]\), and \(\text{Id} : S^k-1 \to S^k-1\).

\[\textbf{Proof:}\] The necessity of (16.1) was shown above. As for sufficiency the following holds. Since \(\psi^*(1) \in i^*(|[X;S^k-1]|)\), there exists a class \(\alpha \in [X;S^k-1]\), generated by \(g : X \to S^k-1\), such that \(\psi^*(1) = i^*(\alpha)\). Now
\[
[g \circ i] = i^*(\alpha) = \psi^*(1) = [\psi],
\]
and thus \(g \circ i \cong \psi\). By Lemma 16.3 and Remark 16.4, the homotopy extension property, there exists an extension \(f : X \to S^k-1\) such that \(f \circ i = \psi\).

This theorem can be applied to the pair \((\Omega, \partial\Omega)\) in particular. Recall that solvability of \(f(x) = 0\) is equivalent to \(\phi\) being essential with respect to \(\Omega\). Indeed, given a mapping \(\phi : \partial\Omega \to \mathbb{R}^k \setminus \{0\}\), then \(\phi\) can be extended to a mapping \(f : \Omega \to \mathbb{R}^k \setminus \{0\}\) if and only if \(\psi^*(1) \in i^*(|[X;S^k-1]|)\), where \(\psi = \phi/|\phi|\). This criterion can also be formulated as \([|\phi|] = \phi^*(1) \in i^*(|[X;\mathbb{R}^k \setminus \{0\}]|)\).
16.7 Theorem. A continuous mapping \( \varphi : \partial \Omega \subset \mathbb{R}^n \to \mathbb{R}^n \setminus \{0\} \) is essential with respect to \( \Omega \) if and only if

\[
\psi = \frac{\varphi}{|\varphi|} \notin i^* \left( [X; S^{k-1}] \right),
\]
or in other words if and only if \( \varphi \notin i^* \left( [X; \mathbb{R}^k \setminus \{0\}] \right) \).

Proof: Theorem 16.6 shows that \( \varphi \) is inessential with respect to \( \Omega \) if and only if

\[
\psi = \frac{\varphi}{|\varphi|} \in i^* \left( [X; S^{k-1}] \right).
\]
Clearly, \( \varphi \) is the essential with respect to \( \Omega \) if and only if \( \varphi \notin i^* \left( [X; \mathbb{R}^k \setminus \{0\}] \right) \).

As pointed out Theorem 16.7 generalizes the results for \( \Omega \cong B^n \), and \( k = n \). Without any algebraic structure on the homotopy classes the criterion in Theorem 16.7 may be hard to verify. In the next section we will discuss various algebraic structures and applications to a number of spacial cases. For example when \( n = k \) it follows from Hopf’s theorem that the isomorphism

\[
\deg \left( \partial \Omega; S^{n-1} \right) \to \mathbb{Z},
\]
can be used to further simplify (16.1). From Theorem 11.3 it follows that 

\[
\deg(i^* \left( [X; \mathbb{R}^k \setminus \{0\}] \right)) = 0,
\]
where \( i : \partial \Omega \to \Omega \). Theorem 16.6 then gives the criterion \( \deg(\psi) \neq 0 \), which yields Corollary 14.13. In the next section algebraic structures needed to understand (16.1) are discussed and a part of algebraic topology dealing with extension problems, called obstruction theory, is introduced.

17. Framed cobordisms

This section gives an introduction to Pontryagin’s theory of framed cobordisms. The treatment of framed cobordisms here is reminiscent of the homotopy principle in Subsection 2b. In Chapter ?? the theory of framed cobordisms will be considered in a more general setting.

In order to give a sufficient introduction to the theory of cobordisms the notion of smooth (embedded) \( n \)-dimensional manifold in \( M \subset \mathbb{R}^\ell \) is required. In the appendix a detailed account of elementary facts about manifolds can be found. From this point on \( M \) will be referred to as an \( n \)-dimensional manifold.

Let \( M \) be \( n \)-dimensional manifold and \( N \subset M \) a smooth, closed submanifold of codimension \( k \). For brevity \( N \) will be referred to as a codimension \( k \) submanifold of \( M \). It is important to point out that \( N \) is not necessarily connected. This plays an important role for the group structure of cobordisms. When \( n = k \), then \( N \) is a finite set of points.

17.1 Definition. Two codimension \( k \) submanifolds \( N, N' \subset M \), are said to be cobordant, if there exists a smooth, compact manifold \( P \subset M \times [0, 1] \) such that

\[
\partial P = (N \times \{0\}) \cup (N' \times \{1\}).
\]

Notation: \( N \approx N' \). The manifold \( M \) is called cobordism between \( N \) and \( N' \), and the equivalence classes are called cobordism classes.

---

A closed submanifold is compact and its relative boundary is the emptyset.
\section*{17.2 Exercise.} Prove that cobordism defines an equivalence relation on smooth, closed, codimension $k$ submanifolds of $\Omega$ (Hint: Show that by ‘gluing’ two cobordisms it is possible to construct a smooth cobordism).

If $T_xN$ denotes the tangent at a point $x \in N$, then $T_xN \subset T_xM \cong \mathbb{R}^n$. The latter allows a decomposition

$$T_xM = T_xN \oplus (T_xN)^\perp,$$

where the orthogonal complement is taken with respect to the standard inner product in $\mathbb{R}^n$. Clearly $T_xN \cong \mathbb{R}^{n-k}$ and $(T_xN)^\perp \cong \mathbb{R}^k$.

\section*{17.3 Definition.} A framing of a codimension $k$ submanifold $N$ is a smooth function $\nu$ on $N$, defined by

$$x \mapsto \nu(x) = (v^1(x), \ldots, v^k(x)) \in (T_xN)^\perp \times \cdots \times (T_xN)^\perp,$$

such that $\nu(x)$ is a basis for $(T_xN)^\perp$ for all $x \in N$. This function is called a framing of $N$. Together with the framing $\nu$, $(N, \nu)$ is called a framed submanifold.

A submanifold does not necessarily allow a framing. For example, the Möbius strip is not orientable and does not allow any framing!

Of special importance are submanifolds given as level sets of smooth functions $f : M \to M'$, where $M'$ is a $k$-dimensional manifold. By Sard’s Theorem most value are regular, and by the Implicit Function Theorem

$$N = f^{-1}(p),$$

is a codimension $k$ submanifold (closed and smooth) contained in $M$, $k \leq n$.

\section*{17.4 Example.} Consider $f : \mathbb{R}^2 \to \mathbb{R}$, defined by $f(x) = x_1^2 + x_2^2 - 1$. Then $0$ is a regular value and $N = S^1 = f^{-1}(0)$ is the unit circle in $\mathbb{R}^2$. The tangent space at a point $x$ is given by $T_xS^1 = \{ (\xi_1, \xi_2) \in \mathbb{R}^2 \mid x_1 \xi_1 + x_2 \xi_2 = 0 \}$. This gives a ‘bundle’ of straight lines tangent to $S^1$. The gradient $\nabla f$ gives a vector field in $(T_xS^1)^\perp$ at each $x$. This is an example of a framing of $S^1$.

For submanifolds given as regular level sets the tangent spaces are defined as follows. For $x \in N = f^{-1}(p)$ set

$$T_xN := \{ \xi \in \mathbb{R}^k \mid df(x)\xi = 0 \}.$$

Example 17.4 gives a framing via $\nabla f$. The differential

$$df(x) : T_xM \cong \mathbb{R}^n \to T_{f(x)}M' \cong \mathbb{R}^k,$$

is a mapping whose null space at $x$ is the tangent space $T_xN$, and maps $(T_xN)^\perp$ isomorphically onto $\mathbb{R}^k$. Therefore, $v^j(x) = (df(x))^{-1}y^j$, with $y = \{ y^j \} \subset T_{f^j}M'$ a basis for $\mathbb{R}^k$, provides a framing of $N = f^{-1}(p)$. Notation

$$f^\ast y = (df(x))^{-1}y,$$

which gives a framing of $N$. A basis $y$ can also be regarded as a element in $\text{Gl}(\mathbb{R}^k)$. Therefore, for a given framing $\nu(x)$, the function $x \mapsto df(x)\nu(x) = y(x)$ can be interpreted as smooth path in either $\text{Gl}^+(\mathbb{R}^k)$, or $\text{Gl}^-(\mathbb{R}^k)$. The notation $y$ may
indicate a fixed choice of a basis of $\mathbb{R}^k$, or a path in $\text{Gl}^\pm(\mathbb{R}^k)$. The framed submanifold $(f^{-1}(p), f^*y)$ is called a Pontryagin framed manifold associated with $f$, or Pontryagin manifold for short.

Framing can also be incorporated within the definition of cobordism.

- **17.5 Definition.** Two framed submanifolds $(N, v)$ and $(N', v')$ of $M$ are said to be framed cobordant if there exists a cobordism $P$, and a framing $w = w(x, t)$ of $P$, such that
  
  $$\pi_x w(x, 0) = v(x), \quad \pi_x w(x, 1) = v'(x),$$
  
  where $\pi_x$ is the projection onto the first $n$ components.

  Cobordism is the smooth regular analogue of homotopy, which allows for additional structures to be carried across, such as framing.

- **17.6 Remark.** In some definitions of cobordism $P$ has the property that it contains $(N \times [0, 1)) \cup (N' \times (1 - \varepsilon, 1])$. Such a cobordism can be obtained by adding a collar at $\partial P$, and will be referred to as a cobordism with collar. See appendix for existence of collars. One can easily prove that both definitions are equivalent.

- **17.7 Exercise.** Show that $N, N'$ are cobordant if and only if there exists a cobordism $P$ with collar.

- **17.8 Lemma.** Framed cobordism defines an equivalence relation on the set of smooth, oriented, codimension $k$ framed submanifolds of $M$. The equivalence classes are called framed cobordism classes, and are denoted by $\Pi^k(M)$.

  **Proof:** Clearly $N$ is cobordant to itself by the trivial cobordism, and $N \approx N'$ implies $N' \approx N$ by reflecting the $t$-coordinate; $t \mapsto 1 - t$. Transitivity can be obtained as follows. Say $N \approx N'$ with cobordism $P$, and $N' \approx N''$ cobordism $P'$. Both $P$ and $P'$ can be assumed to be cobordisms with collar. Define $P \# P'$ as the cobordism between $N$ and $N'$ by gluing the interval $[0, 1]$ and $[1, 2]$ and rescaling $t$. Clearly, $P \# P'$ is a cobordism with collar between $N$ and $N''$. Since two collars are glued at $N'$ the two framings $v$ and $v'$ automatically glue to a framing $w = v \# v'$.

  In the case of a manifold $M$ with boundary $\partial M$ the framed cobordism classes are denoted by $\Pi^k(M, \partial M)$.

18. **Pontryagin manifolds**

As explained in the previous section sublevel sets of smooth functions are particular cases of framed submanifolds, or Pontryagin manifolds. As pointed out before cobordism is a smooth analogue of homotopy. When dealing with Pontryagin manifolds it becomes apparent that cobordisms are strongly related to smooth homotopies. Pontryagin manifolds are a different perspective on the solvability problem $f(x) = p$. For this reason in this section $M$ is either a bounded domain $\Omega$ or its boundary $\partial \Omega$, and mappings that map $M$ into $\mathbb{R}^k$. 


18a. **Pontryagin manifolds of bounded domains.** Let \( M = \Omega \subset \mathbb{R}^n \) a bounded domain in \( \mathbb{R}^n \), with boundary \( \partial \Omega \). As open set, \( \Omega \) is a smooth \( n \)-dimensional manifold. Consider smooth mappings
\[
f : \overline{\Omega} \subset \mathbb{R}^n \rightarrow \mathbb{R}^k, \quad k \leq n.
\]
Consider Pontryagin manifolds of the following form. Let \( p \in \mathbb{R}^k \setminus f(\partial \Omega) \) be a regular then
\[
(N, v) = (f^{-1}(p), f^*y), \quad y \in \text{Gl}(\mathbb{R}^k),
\]
is a framed submanifold of \( \Omega \). Such Pontryagin manifolds lie in certain cobordism classes in \( \Pi^k(\Omega, \partial \Omega) \), and may be able to give more insight into the structure of \( \Pi^k(\Omega, \partial \Omega) \) and its relation to mappings \( f \) as described above.

The lemmas about framed cobordism classes below follow exact along the lines as the proof of the homotopy principle of Lemma 2.12.

- **18.1 Lemma.** For any two bases \( y, y' \) of \( T_p \mathbb{R}^k \) of the same orientation (positively, or negatively oriented\(^{10}\)), the framed submanifolds \( (N, f^*y) \) and \( (N, f^*y') \) are framed cobordant, i.e. \( [(N, f^*y)] = [(N, f^*y')] \in \Pi^k(\Omega, \partial \Omega) \).

**Proof:** By assumption \( y \) and \( y' \) are of the orientation and therefore as matrix, \( \det(y) = \det(y') \), and \( y, y' \) lie in the same connected component of \( \text{Gl}(\mathbb{R}^k) \). Let \( y_t \) be a smooth path in \( \text{Gl}^+(\mathbb{R}^k) \).\(^{11}\) Define \( F(x, t) = f(x) \), and \( w(x, t) = f^*y_t \), then \( P = N \times [0, 1] \) is a framed cobordism between \( (N, f^*y) \) and \( (N, f^*y') \), with framing \( w \).

- **18.2 Lemma.** Let \( f, f' : \overline{\Omega} \subset \mathbb{R}^n \rightarrow \mathbb{R}^k \) be smooth mappings such that \( \|f - f'\|_C < \varepsilon \), and \( p \) is a regular value for both \( f \) and \( f' \). Let \( N, N' \) be smooth submanifolds of \( \Omega \) defined by \( N = f^{-1}(p) \) and \( N' = (f')^{-1}(p) \). If \( \varepsilon > 0 \) is small enough, then \( (N, f^*y) \) and \( (N', (f')^*y) \) are framed cobordant, i.e. \( [(N, f^*y)] = [(N', (f')^*y)] \in \Pi^k(\Omega, \partial \Omega) \).

**Proof:** As in Section 6 define a smooth homotopy \( f_t(x) = F(x, t) = (1 - t) f(x) + t f'(x) \). If \( \varepsilon > 0 \) is chosen sufficiently small then \( p \in \mathbb{R}^k \setminus f_t(\partial \Omega) \), and \( p \) is a regular value for \( f_t \), for all \( t \in [0, 1] \). By definition \( P = F^{-1}(p) \) is a cobordism between \( N \) and \( N' \). Indeed, \( dF(x,t)(\xi, \tau) = df(x)\xi + t(df - df')(x)\xi - (f - f')(x)\tau \), and since \( \varepsilon > 0 \) is small, \( dF \) is onto \( \mathbb{R}^k \), and therefore \( p \) is a regular value of \( F \). Since \( p \) is regular \( F^*y \) gives a framing for \( P \), which proves the lemma.

As in Chapter ?? denote the connected components of \( \mathbb{R}^k \setminus f(\partial \Omega) \) by \( D \). The next step is to prove that the cobordism class of \( N \) is independent of the chosen regular value \( p \in D \).

\(^{10}\)This condition is equivalent to \( \det(y) = \det(y') \).

\(^{11}\)A smooth path can be found because \( \text{Gl}(\mathbb{R}^k) \) is a smooth manifold.
18.3 Lemma. Let \( p, p' \in D \) be regular values. Then, the framed submanifolds \((N, f^* y)\) and \((N', f^* y)\), given by \( N = f^{-1}(p) \), and \( N' = f^{-1}(p') \), are framed cobordant, i.e. \([ (N, f^* y) ] = [(N', f^* y)] \) \( \in \Pi^k(\partial \Omega, \partial M) \).

Proof: Since \( D \subset \mathbb{R}^k \setminus f(\partial \Omega) \) is open there exists a smooth path \( p_t \in D \) connecting \( p \) and \( p' \). Define the smooth homotopy \( F(x, t) = f(x) - p_t \). By Sard’s Theorem there exists a regular value \( 0' \) arbitrarily close to \( 0 \). The level set \( M = F^{-1}(0') \) is a framed cobordism between \( \tilde{N} = f^{-1}(p + 0') \) and \( \tilde{N}' = f^{-1}(p' + 0') \), and therefore

\[
[(\tilde{N}, f^* y)] = [(\tilde{N}', f^* y)].
\]

By Lemma 18.3 it holds that when \( |0 - 0'| < \varepsilon \), sufficiently small, then

\[
[(N, f^* y)] = [(\tilde{N}, f^* y)] \quad \text{and} \quad [(N', f^* y)] = [(\tilde{N}', f^* y)].
\]

Conclusion

\[
[(N, f^* y)] = [(\tilde{N}, f^* y)] = [(\tilde{N}', f^* y)] = [(N', f^* y)],
\]

which proves the lemma.

Combining the above lemmas yields the following theorem on framed cobordisms.

18.4 Theorem. Let \( f : \overline{\Omega} \subset \mathbb{R}^n \to \mathbb{R}^k \) be a smooth mapping, and \( p \in D \) regular value. Then, for any regular value \( p' \in D \), and any oriented basis \( y' \) with the same orientation as \( y \), it holds that \((N', f^* y')\) is framed cobordant to \((N, f^* y)\), where \( N = f^{-1}(p) \) and \( N' = f^{-1}(p') \).

Proof: Combine Lemmas 18.1, 18.2, and 18.3.

Homotopies of mappings \( f \) discussed here need to have the property that a distinguished point \( p \) is not contained in \( f(\partial \Omega) \). The proper way of discussing homotopies in this setting and their homotopy types is via homotopy of pairs. A topological pair \((X, A)\) consists of a topological space \( X \) and a subspace \( A \subset X \). In general, a continuous mapping \( f \) between topological pairs \((X, A)\) and \((Y, B)\),

\[
f : (X, A) \to (Y, B),
\]

is a continuous mapping \( f : X \to Y \), with the additional property that \( f(A) \subset B \).

Here, consider topological pairs \((\overline{\Omega}, \partial \Omega)\) and \((\mathbb{R}^k, \mathbb{R}^k \setminus \{p\})\), and continuous mappings \( f : (\overline{\Omega}, \partial \Omega) \to (\mathbb{R}^k, \mathbb{R}^k \setminus \{p\}) \). By definition this means that mappings \( f : \overline{\Omega} \to \mathbb{R}^k \), have the additional property that \( f(\partial \Omega) \subset \mathbb{R}^k \setminus \{p\} \), which is equivalent to saying that \( p \notin f(\partial \Omega) \), or \( p \in \mathbb{R}^k \setminus f(\partial \Omega) \).

18.5 Definition. A family of mappings \( f_t = F(\cdot, t) \), with \( F : (\overline{\Omega}, \partial \Omega) \times [0, 1] \to (\mathbb{R}^k, \mathbb{R}^k \setminus \{p\}) \) continuous, is called a homotopy between \( f, g : (\overline{\Omega}, \partial \Omega) \to (\mathbb{R}^k, \mathbb{R}^k \setminus \{p\}) \). The mappings \( f \) and \( g \) are called homotopic as mappings of topological pairs.

Homotopy is an equivalence relation and its equivalence classes are called homotopy types or homotopy classes. The homotopy type of a map \( f \) is denoted by \([f]\) and the collection of all homotopy types is denoted by

\[
[(\overline{\Omega}, \partial \Omega) ; (\mathbb{R}^k, \mathbb{R}^k \setminus \{p\})] = \left\{[f] \mid f : (\overline{\Omega}, \partial \Omega) \to (\mathbb{R}^k, \mathbb{R}^k \setminus \{p\})\right\}.
\]
\begin{enumerate}
\item \textbf{18.6 Exercise.} Show that the homotopy type introduced above defines an equivalence relation.
\item \textbf{18.7 Theorem.} Let \( f, f' : (\overline{\Omega}, \partial \Omega) \to (\mathbb{R}^k, \mathbb{R}^k \setminus \{p\}) \) be a smooth mappings, with \( p \in \mathbb{R}^k \) a regular value, which are smoothly homotopic with respect to a homotopy \( F : (\overline{\Omega}, \partial \Omega) \times [0, 1] \to (\mathbb{R}^k, \mathbb{R}^k \setminus \{p\}) \). Then, the submanifolds \( (N, f^*y) \) and \( (N', (f')^*y) \) are framed cobordant, where \( N = f^{-1}(p) \) and \( N' = (f')^{-1}(p) \).
\end{enumerate}

\textbf{Proof:} Let \( p' \in \mathbb{R}^k \) be a regular value for \( f, f' \) and \( F \) (use Sard’s Theorem), such that \( F : (\overline{\Omega}, \partial \Omega) \times [0, 1] \to (\mathbb{R}^k, \mathbb{R}^k \setminus \{p'\}) \) is again a smooth homotopy. By definition \( P = F^{-1}(p') \) is a framed cobordism between the Pontryagin manifolds \( f^{-1}(p') \) and \( (f')^{-1}(p') \). The theorem now follows from Lemma 18.3 (or Theorem 18.4).

The Theorems 18.4 and 18.7 reveal that the cobordism class
\[ [(f^{-1}(p), f^*y)] \in \Pi^k(\Omega, \partial \Omega), \]
is invariant under homotopy in \( f, p \) and \( y \), and thus a homotopy invariant of Pontryagin manifolds \( (f^{-1}(p), f^*y) \). The classes of Pontryagin manifolds are denoted by \( \text{Pon}^k(\Omega, \partial \Omega) \) form a subset of cobordism classes in \( \Pi^k(\Omega, \partial \Omega; \mathbb{R}^k, \mathbb{R}^k \setminus \{p\}) \).

\textbf{18b. Pontryagin manifolds of smooth boundaries.} Let \( M = \partial \Omega \subset \mathbb{R}^n \) be the smooth boundary of a compact domain \( \Omega \subset \mathbb{R}^n \). The boundary \( \partial \Omega \) is a smooth hypersurface in \( \mathbb{R}^n \) and therefore a smooth, compact \((n - 1)\)-dimensional manifold. As such \( \partial \Omega \) is a 1-framed submanifold of \( \mathbb{R}^n \), which framing corresponding with the inward, or outward pointing normal. Consider smooth mappings
\[ \varphi : \partial \Omega \subset \mathbb{R}^n \to \mathbb{R}^k \setminus \{0\}, \quad k \leq n, \]
which are restrictions to \( \partial \Omega \) of smooth mappings \( f : U \subset \mathbb{R}^n \to \mathbb{R}^k \setminus \{0\} \), where \( U \) is a tubular neighborhood with coordinates \((x, u) \in \partial \Omega \times (-\varepsilon, \varepsilon) \). The mapping \( f \) is called a tubular extension.

\textbf{18.8 Remark.} By definition \( \varphi \) is the restriction to \( \partial \Omega \) of smooth mapping \( f : U \to \mathbb{R}^k \setminus \{0\} \). By Teitze’s Extension Theorem, \( g \) extends to a continuous mapping \( \tilde{f} : \mathbb{R}^n \to \mathbb{R}^k \). Let \( V \) be an open neighborhood of \( \mathbb{R}^n \setminus U \), such that \( \partial \Omega \subset V \). Now smoothen \( \tilde{f} \) on \( V \) (use Lemma ??), leaving \( \tilde{f} = f \) unchanged on a neighborhood of \( \partial \Omega \). This map is now the desired extension and is again denoted by \( f \).

\textbf{18.9 Exercise.} Carry out the smoothing procedure in the above remark (Hint: Use a covering of open balls for \( V \)).

A value \( p \in \mathbb{R}^{k-1} \setminus \{0\} \) is called regular if it is a regular value for some tubular extension \( f \). Consider Pontryagin manifolds of the following form. Let \( p \in \mathbb{R}^{k-1} \setminus \{0\} \) be a regular, then
\[ (N, v) = (\varphi^{-1}(p), \varphi^*y), \quad y \in \text{Gl}(\mathbb{R}^k), \]
is a \( k \)-framed, or codimension \( k \) framed submanifold of \( \partial \Omega \). Such Pontryagin manifolds lie in the cobordism classes in \( \Pi^{k-1}(\partial \Omega) \).
The Lemmas 18.1, 18.2, and 18.3 are still true in this case and the proof are almost identical, and will therefore be omitted. The main results can be phrased as follows.

\section*{18.10 Theorem.} Let \( \varphi : \partial \Omega \subset \mathbb{R}^n \to \mathbb{R}^k \setminus \{0\} \) be a smooth mapping. Then, for any pair of regular values \( p, p' \in \mathbb{R}^k \setminus \{0\} \), and any oriented bases \( y, y' \) with the same orientation, it holds that \( (\varphi^{-1}(p), \varphi^*y) \) is framed cobordant to \( (\varphi^{-1}(p'), \varphi^*y') \).

Homotopies of mappings \( \varphi \) were discussed in Section 14. The homotopy type of a map \( \varphi \) is denoted by \( [\varphi] \) and the collection of all homotopy types is denoted by

\[ \partial \Omega; \mathbb{R}^k \setminus \{0\} = \left\{ [\varphi] : \partial \Omega \to \mathbb{R}^k \setminus \{0\} \right\}. \]

\section*{18.11 Theorem.} Let \( \varphi, \varphi' : \partial \Omega \to \mathbb{R}^k \setminus \{0\} \) be a smooth mappings, which are smoothly homotopic with respect to a homotopy \( \Phi : \partial \Omega \times [0, 1] \to \mathbb{R}^k \setminus \{0\} \). Then, the Pontryagin manifolds \( (\varphi^{-1}(p), \varphi^*y) \) and \( ((\varphi')^{-1}(p'), (\varphi')^*y') \) are framed cobordant, for any regular value \( p \in \mathbb{R}^k \setminus \{0\} \).

The Theorems 18.10 and 18.11 reveal that the cobordism class

\[ \langle (\varphi^{-1}(p), \varphi^*y) \rangle \in \Pi^{k-1}(\partial \Omega; \mathbb{R}^k \setminus \{0\}), \]

is invariant under homotopy in \( \varphi, p \) and \( y \), and thus a homotopy invariant of Pontryagin manifolds \( (\varphi^{-1}(p), \varphi^*y) \). The classes of Pontryagin manifolds are denoted by \( \text{Pont}^{k-1}(\partial \Omega; \mathbb{R}^k \setminus \{0\}) \) and they form a subset of \( \Pi^{k-1}(\partial \Omega) \).

\section*{18c. Homotopy types.} The results of the previous subsections imply that if two mappings \( \varphi, \varphi' \) lie in the same homotopy class in \( \partial \Omega; \mathbb{R}^k \setminus \{0\} \), then the associated Pontryagin manifolds are framed cobordant. This defines a homomorphism

\[ \alpha_{k-1} : \partial \Omega; \mathbb{R}^k \setminus \{0\} \to \text{Pont}^{k-1}(\partial \Omega). \]

It actually holds that \( \alpha_{k-1} \) is onto and one-to-one, and thus an isomorphism. This is the subject in the next section. A first step toward this result is:

\section*{18.12 Lemma.} The homotopy classes of mappings \( \varphi : \partial \Omega \to \mathbb{R}^k \setminus \{0\} \) and \( \psi : \partial \Omega \to S^{k-1} \) are isomorphic:

\[ \partial \Omega; \mathbb{R}^k \setminus \{0\} \cong \partial \Omega; S^{k-1}. \]

The isomorphism is given by \( [\varphi] \mapsto \left[ \varphi / \left| \varphi \right| \right] \).

\section*{Proof:} The inclusion map \( \iota : S^{k-1} \hookrightarrow \mathbb{R}^k \setminus \{0\} \) shows that each mapping \( \psi \) yields a mapping \( \varphi = \iota \circ \psi \), and thus the homotopy classes \( \partial \Omega; S^{k-1} \) are contained in \( \partial \Omega; \mathbb{R}^k \setminus \{0\} \).

Let \( \Phi \) be a homotopy between mapping \( \varphi \) and \( \varphi' \), then \( \Psi = \Phi / |\Phi| \) defines a homotopy between the mappings \( \varphi / |\varphi|, \varphi' / |\varphi'| : \partial \Omega \to S^{k-1} \), which, by the previous, shows that \( \varphi \mapsto \varphi / |\varphi| \) defines a isomorphism bewteen \( \partial \Omega; \mathbb{R}^k \setminus \{0\} \) and \( \partial \Omega; S^{k-1} \).
A similar correspondence holds for mappings \( f : \overline{\Omega} \to \mathbb{R}^k \) and the homotopy types of pairs. The considerations in Subsection 18a yield a homomorphism

\[
\beta_k : \left[ (\overline{\Omega}, \partial \Omega); (\mathbb{R}^k, \mathbb{R}^k \setminus \{p\}) \right] \to \text{Pont}^k (\overline{\Omega}, \partial \Omega; \mathbb{R}^k, \mathbb{R}^k \setminus \{p\}).
\]

The homotopy type can be described as homotopy types of the restrictions to the boundary.

\[\blacktriangleleft\]

**18.13 Lemma.** The homotopy classes of mappings \( f : (\overline{\Omega}, \partial \Omega) \to (\mathbb{R}^k, \mathbb{R}^k \setminus \{p\}) \) and \( \psi : \partial \Omega \to S^{k-1} \) are isomorphic:

\[
\left[ (\overline{\Omega}, \partial \Omega); (\mathbb{R}^k, \mathbb{R}^k \setminus \{p\}) \right] \cong \left[ \partial \Omega; S^{k-1} \right],
\]

where the isomorphism is given by \([\phi] \mapsto [\phi/|\phi|], \) with \( \phi = f|_{\partial \Omega}. \)

\[\blacktriangleleft\]

**Proof:** Clearly, if two mappings \( f, f' : (\overline{\Omega}, \partial \Omega) \to (\mathbb{R}^k, \mathbb{R}^k \setminus \{p\}) \) are homotopic via \( F, \) then \( \Psi = \Phi/|\Phi|, \) with \( \Phi = F|_{\partial \Omega}, \) defines a homotopy between the boundary restrictions.

On the other if two mappings \( \phi, \phi' : \partial \Omega \to \mathbb{R}^k \setminus \{p\} \) are homotopic via \( \Phi, \) then by Tietze’s Extension Theorem there exists an extension \( F \) to \( \overline{\Omega}, \) which defines a homotopy between mappings of pairs. Combining this with Lemma 18.12 then proves the lemma.

\[\blacktriangleleft\]

19. Framed cobordism classes and homotopy types

The objective of this section is to characterize homotopy types by framed cobordism classes, which turn can be equipped with an algebraic structure (Subsection 19d). The homotopy classes will thus be classified by algebraic invariant. In the case \( k = n \) this recovers the Brouwer degree. The first result concerns the relation between framed cobordism classes and Pontryagin manifolds. After that the Pontryagin manifolds will be linked to homotopy classes.

19a. Framed cobordism classes as Pontryagin manifolds. Let \( M = \Omega \subset \mathbb{R}^n \) a bounded domain in \( \mathbb{R}^n, \) with boundary \( \partial \Omega. \) As open set, \( \Omega \) is a smooth \( n \)-dimensional manifold. Consider smooth mappings

\[
\phi : \partial \Omega \to \mathbb{R}^k \setminus \{0\},
\]

as defined in the previous subsection. The following lemma shows that a framed submanifold in \( \partial \Omega \) is always a Pontryagin manifold for some mapping \( \phi. \)

\[\blacktriangleleft\]

**19.1 Theorem.** Any framed submanifold \( (N, v) \) in \( \partial \Omega \) is a Pontryagin manifold \( (\phi^{-1}(p), \phi' y) \) for some smooth mapping \( \phi : \partial \Omega \to \mathbb{R}^k \setminus \{0\} \) and regular value \( p \in \mathbb{R}^k \setminus \{0\}. \) Consequently,

\[
\text{Pont}^{k-1}(\partial \Omega; \mathbb{R}^k \setminus \{0\}) \cong \Pi^{k-1}(\partial \Omega).
\]

**Proof:** Consider the mapping \( h : N \times \mathbb{R}^k \to \mathbb{R}^n, \) defined by

\[
h(x, y) := x + y_1 v^1(x) + \cdots + y_k v^k(x), \quad x \in N, \ y = (y_1, \cdots, y_k) \in \mathbb{R}^k.
\]

\[\blacktriangleleft\]
The set $N \times \{0\}$ plays a special role, and the derivative is given by
\[
dh(x,0)(\xi,\eta) = \tau^1(x)\xi_1 + \cdots + \tau^{n-k} \eta_{n-k} + v^1(x)\eta_1 + \cdots + v^k(x)\eta_k,
\]
where the vectors $\tau^1(x), \ldots, \tau^{n-k}(x)$ span $T_xN$. This shows that $dh(x,0)$ is invertible for all $x \in N$, and therefore, $h$ maps $U_\eps \times B_k(0)$, $U_\eps \ni x$ an open neighborhood of $x$, diffeomorphically onto an open set $V_\eps \subset \mathbb{R}^n$. By the compact of $N$, $\eps > 0$ can be chosen uniformly for all $x \in N$. Denote the image of $N \times B_\eps(0)$ by $V = \cup \Omega V_\eps$.

It remains to show that $h(x,y) = h(x',y')$ if and only if $(x,y) = (x',y')$. Assume $(x,y) \neq (x',y')$, then since $y,y' \in B_\eps(0)$, it holds that $|x-x'| < C$, uniformly for $x \in N$. If $\eps > 0$ is sufficiently small local behavior of $h$ yields a contradiction.

Compose $h$ with the mapping $y_i \mapsto \mathbb{E}^2 y_i/(\mathbb{E}^2 - |y|^2)$, and denote it by $\tilde{h}$. Then $\tilde{h}$ maps $N \times \mathbb{R}^k$ diffeomorphically onto $V$. When $\eps > 0$ is sufficiently small, then $V \subset \Omega \subset \mathbb{R}^n$. The relation $\pi(h(x,y)) = y$ defines a smooth mapping $\pi : V \subset \Omega \to \mathbb{R}^k$. It holds that $0$ is a regular value and $\pi^{-1}(0) = N$.

Let $s_q : \mathbb{R}^k \to S^k \backslash \{q\}$ be the inverse stereographic projection, then $\sigma(y) = s_q(x/\sigma(|x|))$, with supp($\sigma$) $\subset B_1(0)$, maps smoothly from $\mathbb{R}^k$ to $S^k$. Now define $\varphi : \overline{\Omega} \to S^k$ as $\sigma \circ \pi$ for points in $V$ and $\varphi = q$ for points in $\overline{\Omega} \backslash V$.

Consider smooth mappings
\[
g : (\Omega, \partial \Omega) \to (S^k, q), \quad q \in S^k,
\]
where $S^k \subset \mathbb{R}^{k+1}$ is the unit sphere in $\mathbb{R}^{k+1}$. Then the framed cobordisms in $\Pi^k(\Omega, \partial \Omega)$ are related to Pontryagin manifolds in $\text{Pont}^k(\overline{\Omega}, \partial \Omega; S^k, q)$.

\section{19.2 Theorem.} Any framed submanifold $(N,v)$ in $\Omega$ is a Pontryagin manifold $(g^{-1}(p,q), g^* y)$ for some smooth mapping $g : (\Omega, \partial \Omega) \to (S^k, q), \quad q \in S^k$ and regular value $p \in \mathbb{R}^k \backslash \{0\}$. Consequently,
\[
\text{Pont}^k(\overline{\Omega}, \partial \Omega; S^k, q) \cong \Pi^k(\overline{\Omega}, \partial \Omega).
\]

\textbf{Proof:} The proof follows along the same lines as the proof of Theorem 19.1.

\section{19.3 Theorem.} Two mappings $\varphi, \varphi' : \partial \Omega \to \mathbb{R}^k \backslash \{0\}$ are smoothly homotopic if and only if their associated Pontryagin manifolds are framed cobordant. In other words
\[
\text{Pont}^{k-1}(\partial \Omega; \mathbb{R}^k \backslash \{0\}) \cong [\partial \Omega; \mathbb{R}^k \backslash \{0\}].
\]

\section{19.4 Corollary.} For smooth mappings $\varphi : \partial \Omega \to \mathbb{R}^k \backslash \{0\}$ it holds that
\[
[\partial \Omega; \mathbb{R}^k \backslash \{0\}] \cong \Pi^{k-1}(\partial \Omega).
\]
\[ 19.5 \textbf{Corollary.} \] For mappings \( f : (\Omega, \partial \Omega) \to (\mathbb{R}^k, \mathbb{R}^k \setminus \{p\}) \) it holds that
\[ \left( (\Omega, \partial \Omega) ; (\mathbb{R}^k, \mathbb{R}^k \setminus \{p\}) \right) \cong \Pi^{k-1}(\partial \Omega). \]

One can even prove that \( \text{Pont}^k(\Omega, \partial \Omega; \mathbb{R}^k, \mathbb{R}^k \setminus \{p\}) \) is in fact isomorphic to \( \text{Pont}^{k-1}(\partial \Omega; \mathbb{R}^k \setminus \{0\}) \) and thus to \( \left( (\Omega, \partial \Omega) ; (\mathbb{R}^k, \mathbb{R}^k \setminus \{p\}) \right) \).

\[ 19.6 \textbf{Example.} \] Consider the homotopy \( F(x,t) = x^2 + t^2 - \frac{1}{4} \) between the maps \( f(x) = x^2 - \frac{1}{4} \) and \( f'(x) = x^2 + \frac{1}{4} \). The submanifold \( N \) consists of the points \( x = \pm \frac{1}{2} \), and \( df(x) : \mathbb{R}^n \to \mathbb{R}^n \) is an isomorphism for all \( x \in N = f^{-1}(p) \). Consequently, the determinant \( \det(f^*) \) is either positive or negative. For two framed cobordant submanifolds \( (N, f^*y) \) and \( (N', f'^*y) \) only points of the same sign in \( N \) and \( N' \) respectively, can be connected by a component of a cobordism \( M \), or points in \( N \) (or in \( N' \)) with opposite signs.

\[ 19.8 \textbf{Lemma.} \] Let \( y = \{e_1, \cdots, e^n \} \) be the standard basis for \( \mathbb{R}^n \), then \( f^*y = df(x) \), and \( \det(f^*y) = J_f(x) \).

\[ \textbf{Proof:} \] By definition \( df(x)f^*y = \text{Id} \), and since \( df(x) \) is invertible, it follows that \( f^*y = (df(x))^{-1}y \). Clearly, \( \det(f^*y) = J_f(x) \).
19.9 Theorem. Let \( n \geq 2 \), then the set of \( n \)-framed cobordism classes \( \Pi^n(\Omega) \) is isomorphic to \( \mathbb{Z} \). In particular, \( \text{deg}(N, f^*y) = \sum_N \text{sign} \det(f^*y) \) can be regarded as an isomorphism

\[
\text{deg} : \Pi^n(\Omega) \rightarrow \mathbb{Z},
\]

in the sense that \( \text{deg}(N, f^*y) = \text{deg}(f, \Omega, p) \), where \( N = f^{-1}(p) \).

Proof: By Lemma 19.8 \( \text{deg}(N, f^*y) = \text{deg}(f, \Omega, p) \), and the map \( \text{deg} \) is onto by virtue of Corollary 14.10 and the fact that \( \text{deg}(f, \Omega, p) \) is given by the degree of the restriction of \( f \) to \( \partial \Omega \). This proves that the map \( \text{deg} \) is onto \( \mathbb{Z} \).

Injectivity can be proved as follows. Given a mapping \( f : \overline{\Omega} \rightarrow \mathbb{R}^n \) of degree \( m \geq 0 \) can be chosen to have exactly \( m \) zeroes of positive orientation. Let \( g : \overline{\Omega} \rightarrow \mathbb{R}^n \) any other admissible function with \( \text{deg}(g, \Omega, p) = m \). By the definition of degree \( g \) has \( m + m' \) positively oriented zeroes and \( m' \) negatively oriented zeroes, see Figure 19.2. Now connects the \( m \) positively oriented zeroes of \( f \) with \( m \) positively oriented zeroes of \( f' \) via \( m \) smooth, non-intersection curves. The remaining \( m' \) positively and negatively oriented zeroes of \( g \) are pairwise connected by \( m' \) smooth, non-intersecting curves, which also avoid the first \( m \) curves.
These $m + m'$ non-intersecting, smooth curve form $M = F^{-1}(p)$, for some smooth homotopy $F$, and is a framed cobordism between $N = f^{-1}(p)$ and $N' = g^{-1}(p)$. The case $m \leq 0$ follows along the same lines.

For $n = 1$, there are three framed cobordism classes, characterized by the functions $f(x) = x$, $f(x) = -x$, and $f(x) = 1$ respectively, and $\deg$ is an isomorphism from $\Pi^1(\Omega)$ to the set $\{-1, 0, 1\}$.

\textbf{Exercise 19.11} Prove the above statement for $n = 1$.

Using the characterization of the mapping degree in terms of the boundary restriction, Theorem 19.9 shows that if two mappings $\psi, \psi' : \partial \Omega \to S^{n-1}$ have the same degree, then their extension to $\Omega$ yield the same framed cobordism class, and therefore the same homotopy type. This proves Lemma 14.5.

19d. The group structure of framed cobordism classes and cohomotopy groups. Define the group operation on $\Pi^{k-1}$ and explain the link to cohomotopy groups. In the case of $k = n$ the degree is rediscovered. If computable these algebraic invariants are useful for studying the extension problem. The Pontryagin manifolds and framed cobordisms are a differentiable tool for studying cohomotopy. Compare the proof in the previous subsection and the cobordism proof of the homotopy invariance of degree.
V. The Leray-Schauder degree

A natural question to ask is if there exists a degree theory for mappings on infinite dimensional spaces? The answer to this question is not so straightforward as the following example will show. Consider the space of sequences defined by infinite dimensional spaces: The answer to this question is not so straightforward

\[ \ell^2 := \{ x = (x_1, x_2, \cdots) \mid \sum i x_i^2 < \infty \} \]

The space \( \ell^2 \cong \mathbb{R}^\infty \) has a natural norm \( \| x \|_{\ell^2}^2 := \sum i x_i^2 \) and inner product \( \langle x, y \rangle := \sum i x_i y_i \) and is a complete normed space, called a Hilbert space. Let \( B^m = \{ x \in \ell^2 \mid \| x \|_{\ell^2} \leq 1 \} \) and define a mapping \( f \) as follows:

\[ f(x) = \left( \sqrt{1 - \| x \|_{\ell^2}^2}, x_1, x_2, \cdots \right), \]

which is a continuous mapping from \( B^m \) to \( \partial B^m =: S^m \). It follows that \( f \) has no fixed points in \( B^m \). Indeed, for \( x \in S^m \) it holds that \( f(x) = (0, x_1, x_2, \cdots) \neq x \). On the other hand \( f(B^m) \subset B^m \) and the mapping \( f \) satisfies the requirements of the Brouwer fixed point theorem, which therefore does not holds for continuous mappings of on \( \ell^2 \). If a degree theory for continuous maps on \( \ell^2 \) exists so does the Brouwer fixed point theorem. This is already an indication that a degree for continuous mappings on infinite dimensional spaces does not exist. More precisely, following the proof of the Brouwer fixed point theorem, define \( r(x) = x + \lambda_-(x)(f(x) - x) \), where \( \lambda_-(x) \leq 0 \). Since \( \deg(S, \text{Id}) = 0 \) the homotopy property of the degree yields that

\[ \deg(r, B^m, 0) = \deg(\text{Id}, B^m, 0) = 1, \]

which implies that \( r(x) = 0 \) has a non-trivial solution. On the other hand, since \( f \) has no fixed points it holds that \( r(B^m) = S^m \), which is a contradiction with the existence of a degree theory.

Another consequence of the above construction is that any two mappings \( g_1, g_2 \) from \( S^m \) to \( S^m \) are homotopic. Indeed, \( h(x,t) = r((1-t)g_1(x) + tg_2(x))^{12} \) gives a homotopy between \( g_1 \) and \( g_2 \). If \( g_1 = \text{Id} \) then \( \deg(g_1) = 1 \) and if \( g_2 = f \) then \( \deg(g_2) = 0 \), which implies degree cannot characterize homotopy classes. In particular \( S^m \) is contractible. This is far from the situation in finite dimensions.

The goal of this chapter is to find an adequate degree theory for the infinite dimensional setting and to extend the theory of homotopy classes of maps from \( \mathbb{R}^n \) to \( \mathbb{R}^n \) to homotopy classes of maps on infinite dimensional spaces.

20. Notation

The infinite dimensional spaces under consideration in this chapter are normed linear vector spaces are their subsets. Extensions will be discussed in the next chapter. Let \( X \) be a (real) linear vector space. On \( X \) we define a norm \( \| \cdot \|_{X} \), or \( \| \cdot \| \) for short which satisfies the following hypotheses:

\[ \| (1-t)g_1 + tg_2 \|_{X}^2 \leq 1 \]

provided \( \| g_i \|_{X}^2 = 1, i = 1, 2. \)

\[ \footnote{One easily verifies that \( \| (1-t)g_1 + tg_2 \|_{X}^2 \leq 1 \) provided \( \| g_i \|_{X}^2 = 1, i = 1, 2. \)} \]
(i) \( \|x + y\| \leq \|x\| + \|y\| \), for all \( x, y \in X \),
(ii) \( \|\lambda x\| = |\lambda|\|x\| \), for all \( x \in X \), and for all \( \lambda \in \mathbb{R} \),
(iii) \( \|x\| = 0 \) if and only if \( x = 0 \).

If there is no ambiguity about the space involved we simply write \( \| \cdot \| \).

The combination \( (X, \| \cdot \|) \) is called a normed linear vector space. If in addition \( X \) is complete it is called a Banach space. A normed linear space is complete if every Cauchy sequence has a limit in \( X \); \( \{x^n\} \subset X \), with \( \|x^n - x^m\| \to 0 \), as \( n,m \to \infty \), implies that there exists a \( x \in X \) such that \( \|x^n - x\| \to 0 \), as \( n \to \infty \).

Normed vector spaces and Banach spaces are examples of metric and complete metric spaces respectively, where the metric is given by

\[
d(x, y) := \|x - y\|.
\]

For the remainder of this chapter \( X \) is assumed to be complete, i.e. a Banach space.

As in the previous chapter \( \Omega \subset X \) denotes an open and bounded subset of \( X \).

The closure in \( X \) or the remainder of this chapter \( X \) respectively if the metric is given by \( \partial \Omega = \overline{\Omega} \setminus \Omega \).

**20a. Continuity.** Throughout this section \( X \), and \( Y \) are (real) Banach spaces with norms \( \| \cdot \|_X \) and \( \| \cdot \|_Y \) respectively. We omit the subscripts is there is no ambiguity about the notation.

\[20.1 \textbf{Definition.} \quad \text{A mapping } f : X \to Y \text{ is continuous if } x^n \to x \text{ (in } X \text{) implies that } f(x^n) \to f(x) \text{ (in } Y \text{). A map is uniformly continuous on } X, \text{ if for for any } \varepsilon > 0 \text{ there exists a } \delta_\varepsilon > 0 \text{ such that } \|x - y\| < \delta \implies \|f(x) - f(y)\| < \varepsilon. \]

The latter can also be defined with respect to a closed subset \( A \subset X \).

A continuous function \( f : X \to Y \) is bounded if \( f(\Omega') \subset X \) is bounded for any bounded subset \( \Omega' \subset X \). Continuous mappings on \( \mathbb{R}^n \) are necessarily bounded, i.e. bounded sets in \( \mathbb{R}^n \) are mapped to bounded set under \( f \). This is however not the case in general Banach spaces.

\[20.2 \textbf{Exercise.} \quad \text{Give an example of continuous map between Banach spaces that is not bounded.} \]

\[20.3 \textbf{Lemma.} \quad \text{A uniformly continuous map is bounded.} \]

\[
\text{Proof: } \text{We need to show that for any bounded set } A \subset X \text{ the image } f(A) \subset Y \text{ is also bounded. Choose } R > 0 \text{ such that } A \subset B_R(0), \text{ and let } n > \frac{2R}{\delta}. \text{ Then for any two points } x, y \in A \text{ it holds that } \|x - y\| \leq 2R, \text{ and one can define the line-segment } x^t = x + t(y - x), t \in [0, 1], \text{ in } B_R(0). \text{ For } t_i = \frac{i}{n} \text{ we obtain point } x^{t_i} \subset B_R(0), \text{ with } \|x^{t_i} - x^{t_{i+1}}\| < \delta, \text{ by the choice of } n. \text{ Since } f \text{ is uniformly continuous it follows that } \|f(x^{t_i}) - f(x^{t_{i+1}})\| < \varepsilon, \text{ for all } i. \text{ From the triangle inequality we then get } \|f(x) - f(y)\| \leq \sum_i \|f(x^{t_i}) - f(x^{t_{i+1}})\| < n\varepsilon,
\]

which proves the boundedness of \( f \).
of bounded continuous functions on $\Omega$ is denoted by $C^0_b(\Omega; Y)$, or $C^0_b(\overline{\Omega})$. On $C^0_b$ the following norm is defined

$$\|f\|_{C^0_b} := \sup_{x \in \Omega} \|f(x)\|_Y,$$

which makes $C^0_b(\Omega)$ a normed linear space.

20b. Differentiability. \begin{definition} A mapping $f \in C(X, Y)$ is called Fréchet differentiable at a point $x_0 \in X$, if there exists a bounded linear map $A : X \to Y$ such that

$$\|f(x) - f(x_0) - A(x - x_0)\| = o(\|x - x_0\|),$$

in a neighborhood $N$ of $x_0$. \end{definition}

We use the notation $A = f'(x_0)$ for the Fréchet derivative. If the map $x \mapsto f'(x)$ is continuous as a map from $X$ to $B(X, Y)$, then $f$ is of class $C^1$; notation $f \in C^1(X, Y)$.

\begin{definition} A mapping $f \in C(X, Y)$ is called Gateaux differentiable in the direction $h \in X$, at a point $x_0$, if there exists a $y \in Y$ such that

$$\lim_{t \to 0} \|f(x_0 + th) - f(x_0) - ty\| = 0,$$

with $x_0 + th$ defined in a neighborhood $N$ of $x_0$. \end{definition}

The Gateaux derivative at at point is usually denoted by $df(x_0, h)$ and is commonly referred to as the directional derivative in the direction $h$. In $\mathbb{R}^n$ it is known as partial derivative and is a weaker notion of differentiability.

\begin{exercise} Give an example of a function that is Gateaux differentiable in a point, but is not Fréchet differentiable. \end{exercise}

For functions on $\mathbb{R}^n$ there is an important relation between the two notions of differentiability, i.e. the partial derivatives exist and are continuous, then the function is differentiable. In the Banach space setting the same result holds.

\begin{theorem} If $f \in C(X, Y)$ is Fréchet differentiable at a point $x_0$, then $f$ is Gateaux differentiable at $x_0$. Conversely, if a function $f \in C(X, Y)$ is Gateaux differentiable at $x_0$ for all directions $h \in X$, and the mapping $x \mapsto df(x, \cdot) \in B(X, Y)$ is continuous at $x_0$, then $f$ is Fréchet differentiable at $x_0$. \end{theorem}

In the latter case we write, by the linearity of $df$ in $h$, that $df(x, h) = df(x_0)h = f'(x)h$.

\begin{proof} The first claim of the theorem simply follows from the definition of the Fréchet derivative. For the converse we argue as follows. By assumption the map $f(x_0 + th)$ is differentiable in $t$ (sufficiently small), and $df(x_0 + th, h) = df(x_0 + th)h$ is continuous in $t$. Therefore,

$$f(x_0 + h) - f(x) = \int_0^1 df(x_0 + th)h dt.$$ \end{proof}
Using this identity we find the following estimate:

\[ \|f(x_0 + h) - f(x_0) - df(x_0)h\| = \int_0^1 \| (df(x_0 + th) - df(x_0), h) \| dt \]

\[ \leq \int_0^1 \| df(x_0 + th) - df(x_0) \| dt \]

\[ \leq \int_0^1 \| df(x_0 + th) - df(x_0) \|_{B(X,Y)} \| h \| X dt \]

\[ = o(\|h\|), \]

by the continuity of \( df(x_0 + th) \).

The notions differentiability can be further extended to higher derivatives and we leave this to the reader. Furthermore, one can easily prove various basic properties of derivatives:

**20.8 Exercise.** Prove the chain rule: Let \( f : X \rightarrow Y, g : Y \rightarrow Z \), and \( f \) and \( g \) are differentiable at \( x_0 \) and \( y_0 = f(x_0) \) respectively. Then \( \left( g(f(x_0)) \right)' = g'(f(x_0)) \cdot f'(x_0) \).

**20.9 Exercise.** Prove the product rule: Let \( f : X \rightarrow \mathbb{R} \), and \( g : X \rightarrow Y \) be differentiable at \( x_0 \), then \( f \cdot g \) is differentiable at \( x_0 \), and \( (f \cdot g)'(x_0)h = f'(x_0)h \cdot g(x_0) + f(x_0) \cdot g'(x_0)h \).

A value \( p \in Y \) is called a regular value if \( f'(x) \in B(X,Y) \) is surjective for all \( x \in f^{-1}(p) \) and \( p \) is singular, or critical if it is not a regular value. A point \( x \in X \) is called regular if \( f'(x) \) is surjective and otherwise a point is called singular, or critical point.

20c. **Fredholm mappings and proper mappings.** Let \( f \) be a \( C^2 \) proper Fredholm mapping of index 0.

Important! Mention the Kuiper result

21. **Compact and finite rank maps**

An important subspace of continuous mappings are the compact mappings from \( f : X \rightarrow X \). A mapping \( f : X \rightarrow X \) is compact if \( \overline{f(\Omega')} \) is compact for any bounded subset \( \Omega' \subset X \). Compact mappings are bounded since \( \overline{f(\Omega')} \) is bounded for any bounded set \( \omega' \subset X \). The space of compact mappings on \( X \) is denoted by \( K(X) \). This definition of compact mappings also holds for continuous mappings on subsets of \( X \).

**21.1 Definition.** A continuous map \( k : \overline{\Omega} \subset X \rightarrow X \) is called compact if \( k(\overline{\Omega}) \) is compact.

In particular for any \( \Omega' \subset \overline{\Omega} \) it holds that \( \overline{f(\Omega')} \) is compact since \( f(\Omega') \subset f(\overline{\Omega}) \). The space of compact mappings on \( \overline{\Omega} \) is denoted by \( K(\overline{\Omega}) \subset C^0(\overline{\Omega}) \). Compact maps are examples of mappings which are close to mappings in finite dimensional Euclidean space. The following lemma explains how compact maps can be approximated by maps of finite rank. To be more precise, a finite rank map is a mapping whose range is contained in a finite dimensional subspace of \( X \). The subspace of finite ranks mappings is denoted by \( F(\overline{\Omega};X) \subset C^0(\overline{\Omega};X) \).
Lemma 21.2. Let \( k \in K(\overline{\Omega}) \), then for any \( \varepsilon > 0 \), there exists a finite rank map \( k^\varepsilon \in F(\overline{\Omega}) \) such that \( \| k - k^\varepsilon \|_{C^0_b} < \varepsilon \). i.e. \( k^\varepsilon \in F(\overline{\Omega}) \cap C^0_b(\overline{\Omega}) \).

**Proof:** Since \( k(\overline{\Omega}) \) is compact it can be covered by finitely many balls \( B_\varepsilon(x^j) \), with \( x^j \in k(\overline{\Omega}) \). Define
\[
\mu^j(x) = \frac{\lambda^j(x)}{\sum_j \lambda^j(x)},
\]
where \( \lambda^j(x) = \max(0, \varepsilon - \| k(x) - x^j \|) \). This maximum is zero whenever \( k(x) \notin B_\varepsilon(x^j) \) and therefore \( \mu^j(x) = 0 \), unless \( \| k(x) - x^j \| < \varepsilon \). Set
\[
k^\varepsilon(x) = \sum_i \mu^i(x) \cdot x^i.
\]
Now \( k^\varepsilon(\Omega) \subset \text{span}(x^i) \). As for the approximation we obtain
\[
\| k - k^\varepsilon \|_{C^0_b} = \left\| k - \sum_i \mu^i \cdot x^i \right\|_{C^0_b} = \left\| \sum_i \mu^i \cdot (k - x^i) \right\|_{C^0_b},
\]
using the fact that \( \sum_i \mu^i = 1 \). By construction \( \| \mu^i \cdot (k(x) - x^i) \| < \mu^i \varepsilon \) and thus
\[
\| k - k^\varepsilon \|_{C^0_b} = \sup_{x \in \Omega} \left\| \sum_i \mu^i(x) (k(x) - x^i) \right\| 
\leq \sup_{x \in \Omega} \left\| \sum_i \mu^i(x) (k(x) - x^i) \right\| < \sum_i \mu^i(x) \varepsilon = \varepsilon,
\]
which completes the proof.

The converse of this lemma can be formulated as follows:

Lemma 21.3. For any sequence \( \{ k^\varepsilon \} \subset F(\overline{\Omega}) \cap C^0_b(\overline{\Omega}) \), with \( k^\varepsilon \to k \) in \( C^0_b(\overline{\Omega}) \), as \( \varepsilon \to 0 \), it holds that \( k \in K(\overline{\Omega}) \).

Finally, using the above characterization of compact mappings, it is worth mentioning a version of Tietze’s extension theorem for compact mappings.

Lemma 21.4. Any compact mapping \( f \in K(\overline{\Omega}) \) extends to a compact mapping \( \tilde{f} \in K(X) \).

Lemma 21.5. Let \( k \in K(\overline{\Omega}) \cap C^1(\Omega) \), then for any \( x \in \Omega \) the linear operator \( k'(x) : X \to X \) is compact.

**Proof:** Under construction.

22. **Definition of the Leray-Schauder degree**

The problem with degree theory in infinite dimensional spaces is that homotopy invariance, a basic property of the degree, prevents the existence of a non-trivial degree theory (compare the axioms for degree theory, Section 7). We can alter the notion of homotopy invariance in order to build a degree theory, or limit the types
of maps for which a degree is well-defined. The Leray-Schauder degree does both by considering specific types of mappings, namely mappings of the form\[
f = \text{Id} - k,\]
where \(\text{Id}\) is the identity map on \(X\) and \(k \in K(\Omega)\). Homotopies are considered in the same class. Denote the function class by \(C^0_{\text{Id}}(\Omega) = \{f = \text{Id} - k \mid k \in K(\Omega)\}\) and by \(C^0_f(X)\) for mapping defined on \(X\). These classes are affine subspaces of \(C^0_{\text{Id}}(\Omega)\) and \(C^0_f(X)\) respectively.

The set \(\partial \Omega\) is a closed and bounded set in \(X\). Due to the specific form of \(f\) the set \(f(\partial \Omega)\) is also closed and bounded. Indeed, let \(x^* \in \partial \Omega\) such that \(f(x^*) \to x^*\). Since \(k\) is compact we have that \(k(x^*)\) has a convergent subsequence and \(k(x^{n_k}) \to x^{*\ast}\). Therefore, \(x^{n_k} = f(x^{n_k}) + k(x^{n_k}) \to x^* + x^{*\ast} = x\), which, by continuity, implies that \(x^* = x - k(x) = f(x)\), proving the closedness of \(f(\partial \Omega)\). For the boundedness we argue as follows: \(k(\partial \Omega)\) is pre-compact and thus \(f(\partial \Omega)\) is bounded. Combining these facts we conclude that \(p \notin f(\partial \Omega)\) implies that
\[
\inf_{x \in \partial \Omega} \| p - f(x) \| = \inf_{y \in f(\partial \Omega)} \| p - y \| = \delta > 0.
\]

Indeed, if not, there exists a minimizing sequence \(x^* \in \partial \Omega\) such that \(f(x^*) \to p\). By the closedness of \(f(\partial \Omega)\) then \(p \in f(\partial \Omega)\), a contradiction.

\[\textbf{22.1 Definition.}\] Let \(f\) be a continuous map of the form \(f \in C^0_{\text{Id}}(\Omega)\), and let \(p \notin f(\partial \Omega)\). Let \(k^e\) be a finite rank perturbation with \(\|k - k^e\|_{C^0_{\text{Id}}} < \varepsilon\) and \(\varepsilon < \delta/2\) (\(\delta\) as given above) and with \(k^e(\Omega) \subset Y^e \subset X\) (subspace). Then for any finite dimensional subspace \(X^e\) containing both \(Y^e\) and \(p\), define the Leray-Schauder degree as
\[\deg_{\text{LS}}(f, \Omega, p) := \deg(f^e, \Omega \cap X^e, p),\]
where \(f^e = \text{id} - k^e\). If there is no ambiguity about the context we mostly omit the subscript for the notation.

The remainder of this section is devoted to showing that the Leray-Schauder degree is well-defined. By the choice of domain \(\Omega \cap X^e\) it follows that \(f : \Omega \cap X^e \to X^e\) and \(p \in X^e\). Moreover, \(p \notin f(\partial \Omega \cap X^e)\), which follows from the following inequality:
\[
\inf_{x \in \partial \Omega \cap X^e} \| p - f^e(x) \| = \inf_{x \in \partial \Omega \cap X^e} \| p - x + k^e(x) \| \geq \inf_{x \in \partial \Omega} \| p - x + k^e(x) \|
\geq \inf_{x \in \partial \Omega} \| p - x + k(x) \| - \delta/2 > \delta/2 > 0.
\]

The degree \(d(f^e, \Omega \cap X^e, p)\) is well-defined. We show now that the definition is independent of the chosen subspace \(X^e\) and approximation \(k^e\).
\textbf{22.2 Lemma.} Let $\widetilde{X}^e \subset X$ be any finite dimensional subspace such that $Y^e \subset \widetilde{X}^e$ and $p \in X^e$. Then $\deg(f^e, \Omega \cap \widetilde{X}^e, p) = \deg(f^e, \Omega \cap X^e, p)$. \hfill $\blacktriangleright$

\textbf{Proof:} Step (i): Consider mappings of the form $g = \Id - \delta : D \subset \mathbb{R}^n \oplus \mathbb{R}^m \to \mathbb{R}^n \oplus \mathbb{R}^m$, with $h(D) \subset \mathbb{R}^n$. Suppose $p \in \mathbb{R}^n$ and $p \notin g(D)$. Then, $\deg(g, D, p) = \deg(g_n, D \cap \mathbb{R}^n, p)$, where $g_n = g|_{D \cap \mathbb{R}^n}$. We prove the above statement in the case that $h$ is $C^1$, since the degree is defined via $C^1$ approximations and with $p = 0$ by virtue of the Property (v) in Section 7. Let $\omega_1$ and $\omega_2$ be top forms on $\mathbb{R}^n$ and $\mathbb{R}^m$ respectively with $\int_{\mathbb{R}^n} \omega_1 = \int_{\mathbb{R}^m} \omega_2 = 1$ and their supports contained in a sufficiently small neighborhood of the origin. In terms of coordinates we write $x = x_1 + x_2, x_1 \in \mathbb{R}^n$ and $x_2 \in \mathbb{R}^m$. For the degree this yields

$$
\deg(g, D, p) = \int_D g^* (\omega_1 \otimes \omega_2)
= \int_D \omega_1 (x_1 - h(x_1 + x_2)) \omega_2 (x_2) J_g(x_1 + x_2) dx_1 dx_2.
$$

By the specific form of $g$ we have that $J_g(x_1 + x_2) = \det \left( \Id - \frac{\partial h}{\partial x_1}(x) \right)$. Since the expression for the degree is independent of $\omega_1$ and $\omega_2$ we can choose $\omega_2$ to approximate a density function that peaks at 0 and has integral equal to 1 — approximating a delta distribution. Due to the independence on $\omega_2$ this gives

$$
\deg(g, D, p) = \int_{D \cap \mathbb{R}^n} \omega_1 (x_1 - h(x_1)) \det \left( \Id - \frac{\partial h}{\partial x_1}(x_1) \right) dx_1,
= \deg(g_n, D \cap \mathbb{R}^n, p).
$$

Step (ii): Since $Y^e \subset X^e \cap \widetilde{X}^e$ and $p \in X^e \cap \widetilde{X}^e$ we may assume without loss of generality that $X^e \subset \widetilde{X}^e$. By construction it holds that $f^e : \overline{\Omega} \cap X^e \to X^e$ and $f^e : \overline{\Omega} \cap \widetilde{X}^e \to \widetilde{X}^e$. Consider the linear change of variables $y = q(x)$ such that $q(X^e) = \mathbb{R}^n \oplus 0$ and $q(\widetilde{X}^e) = \mathbb{R}^n \oplus \mathbb{R}^m$. Then $g = q \circ f^e \circ q^{-1} = \Id - h$ and $h(D) \subset \mathbb{R}^n \oplus 0$, where $D = q(\overline{\Omega} \cap \widetilde{X}^e) \subset \mathbb{R}^n \oplus \mathbb{R}^m$. From Step (i) it follows that $\deg(g, D, p) = \deg(g_n, D \cap \mathbb{R}^n, p)$. It remains to prove that the degree is invariant under the change of coordinates. Using differential calculus and the integral characterization of the degree we obtain:

$$
\int_D g^* \omega = \int_D (q \circ f^e \circ q^{-1})^* \omega = \int_D (q^{-1})^* ((f^e)^*(q^* \omega))
= \sign(J_{q^{-1}}(x)) \int_{\overline{\Omega} \cap \widetilde{X}^e} (f^e)^*(q^* \omega)
= \sign(J_{q^{-1}}(x)) \deg(f^e, \overline{\Omega} \cap \widetilde{X}^e, p) \int_{\mathbb{R}^{n+m}} q^* \omega
= \sign(J_{q^{-1}}(x)) \deg(f^e, \overline{\Omega} \cap \widetilde{X}^e, p) \sign(J_q(y)) \int_{\mathbb{R}^{n+m}} \omega
= \deg(f^e, \overline{\Omega} \cap \widetilde{X}^e, p) \int_{\mathbb{R}^{n+m}} \omega.
$$

\footnote{For a linear subspace $X^e \subset X$ it holds that $\partial(\overline{\Omega} \cap X^e) = \partial \overline{\Omega} \cap X^e$.}
Since \( \int_D g^* \omega = \deg (g, D, q(p)) \int \omega \) it follows that
\[
\deg (g, D, q(p)) = \deg (f^\varepsilon, \Omega \cap \hat{X}^\varepsilon, p),
\]
which proves that the degree is invariant under coordinate changes. By restricting to the subspace \( \varepsilon \) we obtain \( \deg (g_n, D \cap \mathbb{R}^n, q(p)) = \deg (f^\varepsilon, \Omega \cap X^\varepsilon, p) \). The proof follows now from Step (i).

The final step in showing that Definition 22.1 proposes a well-defined notion of degree, is to prove that for \( \varepsilon < \delta / 2 \) the definition is independent of the chosen approximation \( k^\varepsilon \).

\begin{lemma}
 Let \( k^\varepsilon \) and \( \tilde{k}^\varepsilon \) both be finite rank approximations for \( k \) with \( \|k - k^\varepsilon\|_{C^0} \leq \varepsilon \), \( \|k - \tilde{k}^\varepsilon\|_{C^0} < \varepsilon \) and \( \varepsilon < \delta / 2 \). Then
\[
\deg (\text{Id} - k^\varepsilon, \Omega \cap X^\varepsilon, p) = \deg (\text{Id} - \tilde{k}^\varepsilon, \Omega \cap \hat{X}^\varepsilon, p),
\]
for any subspaces \( X^\varepsilon \) and \( \hat{X}^\varepsilon \) containing both \( p \) and the ranges of \( k^\varepsilon \) and \( \tilde{k}^\varepsilon \) respectively.

\end{lemma}

\begin{proof}
 Let \( Z^\varepsilon \subset X \) be a finite dimensional linear subspace containing both \( X^\varepsilon \) and \( \hat{X}^\varepsilon \). From Lemma 22.2 it follows that
\[
\deg (\text{Id} - k^\varepsilon, \Omega \cap X^\varepsilon, p) = \deg (\text{Id} - k^\varepsilon, \Omega \cap Z^\varepsilon, p),
\]
\[
\deg (\text{Id} - \tilde{k}^\varepsilon, \Omega \cap \hat{X}^\varepsilon, p) = \deg (\text{Id} - \tilde{k}^\varepsilon, \Omega \cap Z^\varepsilon, p).
\]
Consider the compact homotopy \( k^\varepsilon_t = (1 - t)k^\varepsilon + t\tilde{k}^\varepsilon \), which yields a homotopy \( f^\varepsilon_t = \text{Id} - k^\varepsilon_t \) and is a proper homotopy. By Property (ii) of Section 7 then \( \deg (\text{Id} - k^\varepsilon, \Omega \cap Z^\varepsilon, p) = \deg (\text{Id} - \tilde{k}^\varepsilon, \Omega \cap Z^\varepsilon, p) \) which then proves that
\[
\deg (\text{Id} - k^\varepsilon, \Omega \cap X^\varepsilon, p) = \deg (\text{Id} - \tilde{k}^\varepsilon, \Omega \cap \hat{X}^\varepsilon, p).
\]
The Leray-Schauder degree is well-defined.

\end{proof}

23. Properties of the Leray-Schauder degree

The properties of the (Brouwer) degree listed in Section 7 hold equally well for the Leray-Schauder degree. For this list we refer to Section 7. For proving these properties for the Leray-Schauder degree one has to make sure that approximations are constructed such that the conditions for the Brouwer degree are met. An important property is the validity property. If \( p \notin f( \Omega ) \), then \( \deg_{LS}(f, \Omega, p) = 0 \). This is proved as follows. Due to the form of \( f \) the set \( f( \Omega ) \) is closed and since \( p \notin f(\overline{\Omega}) \) we have that \( \inf_{y \in f(\overline{\Omega})} \|p - y\| \geq \delta > 0 \). Let \( f^\varepsilon \) be an approximation for \( f \) as described in the definition of the Leray-Schauder degree (Definition 22.1) and with \( X^\varepsilon \) such that \( f^\varepsilon(\overline{\Omega}) \subset X^\varepsilon \) and \( p \in X^\varepsilon \). If we choose \( \varepsilon > 0 \) small enough, i.e. \( \varepsilon \leq \delta / 2 \), then for all \( \|f - f^\varepsilon\|_{C^0} < \varepsilon \) it holds that \( \inf_{y \in f(\overline{\Omega} \cap X^\varepsilon)} \|p - y\| \geq \frac{\delta}{2} \geq \frac{\delta}{2} > 0 \) and \( p \notin f^\varepsilon(\overline{\Omega} \cap X^\varepsilon) \). From the properties of the Brouwer degree we now have that \( \deg (f^\varepsilon, \Omega \cap X^\varepsilon, p) = 0 \). As an immediate consequence it now holds that
\[
\deg_{LS}(f, \Omega, p) \neq 0,
\]
implies that $f^{-1}(p) \neq \emptyset$. Indeed, if $f^{-1}(p) = \emptyset$, then $d_{LS}(f, \Omega, p) = 0$, a contradiction.

Another way to treat the Leray-Schauder degree is to show that the axioms of a degree theory are satisfied and derive the properties from that. We start with the Leray-Schauder degree theory and explain the axioms in a more general context later on.

Consider triples $(f, \Omega, p)$ with $\Omega \subset X$ a bounded and open set in a Banach space $(X, \| \cdot \|)$, $f \in C^0(\overline{\Omega})$ and $p \in X \setminus f(\partial \Omega)$. For such triples we assign the Leray-Schauder degree

$$(f, \Omega, p) \mapsto \deg_{LS}(f, \Omega, p).$$

\section*{23.1 Theorem.} For Leray-Schauder degree we have the following properties:

(A1) if $p \in \Omega$, then $\deg_{LS}(\text{Id}, \Omega, p) = 1$;
(A2) for $\Omega^1, \Omega^2 \subset \Omega$, disjoint open subsets of $\Omega$, and $p \notin f(\overline{\Omega \setminus (\Omega^1 \cup \Omega^2)})$, it
holds that $\deg_{LS}(f, \Omega, p) = \deg_{LS}(f, \Omega^1, p) + \deg_{LS}(f, \Omega^2, p)$;
(A3) for any continuous paths $t \mapsto f_t = \text{Id} - k_t$, $k_t \in K(\overline{\Omega})$ and $t \mapsto p_t$, with $p_t \notin f_t(\partial \Omega)$, it holds that $\deg_{LS}(f_t, \Omega, p_t)$ is independent of $t \in [0, 1]$;

and $\deg_{LS}$ is called a degree theory.

\textbf{Proof:} Under construction.

As in the case of the Brouwer degree the essential properties of the Leray-Schauder degree follow from (A1)-(A3). In Section 7 we choose to prove these properties of the degree using only the axioms. Therefore most properties hold also for the Leray-Schauder degree with the same proofs. There are some differences though. Let us go through the list in Section 7 and point out the differences.

\section*{23.2 Property.} (Validity of the degree, Property 7.4) If $p \notin f(\overline{\Omega})$, then $\deg_{LS}(f, \Omega, p) = 0$. Conversely, if $\deg_{LS}(f, \Omega, p) \neq 0$, then there exists a $x \in \Omega$, such that $f(x) = p$.

\section*{23.3 Property.} (Continuity of the degree, Property 7.5) The degree $\deg_{LS}(f, \Omega, p)$ is continuous in $f = \text{Id} - k$, i.e. there exists a $\delta = \delta(p, f) > 0$, such that for all $g = \text{Id} - \tilde{k}$ satisfying $\|k - \tilde{k}\|_{C^0} < \delta$, it holds that $p \notin g(\partial \Omega)$ and $\deg(g, \Omega, p) = \deg(f, \Omega, p)$.

\section*{23.4 Property.} (Dependence on path components, Property 7.6) The degree only depends on the path components $D \subset X \setminus f(\partial \Omega)$, i.e. for any two points $p, q \in D \subset X \setminus f(\partial \Omega)$ it holds that $\deg_{LS}(f, \Omega, p) = \deg_{LS}(f, \Omega, q)$. For any path component $D \subset X \setminus f(\partial \Omega)$ this justifies the notation $\deg_{LS}(f, \Omega, D)$.

\section*{23.5 Property.} (Translation invariance, Property 7.7) The degree is invariant under translation, i.e. for any $q \in X$ it holds that $\deg_{LS}(f - q, \Omega, p - q) = \deg_{LS}(f, \Omega, p)$.
23.6 Property. (Excision, Property 7.8) Let \( A \subset \Omega \) be a closed subset in \( \Omega \) and \( p \notin f(A) \). Then, \( \deg_{LS}(f,\Omega,p) = \deg_{LS}(f,\Omega \setminus A,p) \).

23.7 Property. (Additivity, Property 7.9) Suppose that \( \Omega_i \subset \Omega \), \( i = 1, \cdots , k \), are disjoint open subsets of \( \Omega \), and \( p \notin f(\overline{\bigcup_i (\cup_i \Omega_i)}) \), then \( \deg_{LS}(f,\Omega,p) = \sum_i \deg_{LS}(f,\Omega_i,p) \).

As for the Brouwer degree the Leray-Schauder degree can also be defined in the \( C^1 \)-case. Let \( p \in X \setminus f(\partial\Omega) \) be a regular value then by the Inverse Function Theorem the set \( f^{-1}(p) \) consists of isolated points. Let \( x_n \in f^{-1}(p) \), then \( x_n = p + k(x_n) \) which has a convergent subsequence by the compactness of \( k \) and therefore \( f^{-1}(p) \) is compact. Combined with isolation this yields that \( f^{-1}(p) \) is a finite set. Using the excision and additivity Properties 23.6 and 23.7 we derive that \( \deg_{LS}(f,\Omega,p) = \sum_i \deg_{LS}(f,\Omega_i,p) \). It holds that

\[
\deg_{LS}(f,\Omega_i(x^i),p) = \deg_{LS}(f,B_{\varepsilon'}(x^i),p) = i(f,x^i),
\]

for all for any \( 0 < \varepsilon' \) sufficiently small. The integer \( i(f,x^i) \) is called the index of an isolated zero. For the Brouwer degree the index is given by the sign of the Jacobian at \( x^i \). Since for each \( x \in f^{-1}(p) \) the operator \( f'(x) = I - k'(x) \in B(X) \) is invertible by the compactness of \( k \), then \( \lambda > 1 \) be an eigenvalue of \( A = I - k'(x) \), i.e. \( A\xi = \lambda\xi, X \ni \xi \neq 0 \), then

\[
n_{\lambda} = \dim \left( \bigcup_{k=1}^{\infty} \ker(\lambda I - A)^k \right) < \infty,
\]

by the compactness of \( k'(x) \).

23.8 Lemma. Let \( x \in \Omega \) be a regular point of \( f = \xi_0^{\beta} \), then

\[
i(f,x) = (-1)^{\beta},
\]

where \( \beta = \sum_{k > 1} n_{\lambda} \).

Proof: Under construction.

23.9 Remark. The formula for the index can also be given for the Brouwer degree because \( (-1)^{\beta} = \text{sign}(J_f(x)) \) is the finite dimensional case. The above consideration also show how the Leray-Schauder degree is defined axiomatically and leads to a similar expression as a sum of indices in the \( C^1 \)-case. The latter can also be used as a first definition.

24. Compact homotopies

Corollary 15.2 relates the existence of a zero of any extension \( f : \overline{B}_1(0) \to \mathbb{R}^k \) of \( f|_{\partial B_1} \) to the homotopy type of the \( \psi = f/|f| : \partial B_1 = S^{n-1} \to S^{k-1} \), i.e. a non-trivial zero exists if and only if \( [\psi] \in \pi_{n-1}(S^{k-1}) \) is non-trivial. The example in the introduction showed that the infinite dimensional version of this result does not hold true without an appropriate notion of homotopy type. A mapping \( f : \partial B_1 \to \mathbb{R}^k \).}

\[\text{The identity operator } x \mapsto x \text{ is denoted by } \text{Id} \text{ and it linearization by } I.\]
$$\mathbb{R}^k \setminus \{0\}$$ that admits an extension $$f : \mathbb{B}_1 \to \mathbb{R}^k$$ for which $$f \neq 0$$ is called inessential with respect to $$B_1$$. Otherwise $$f : \partial B_1(0) \to \mathbb{R}^k \setminus \{0\}$$ is called essential with respect to $$B_1$$, i.e. every continuous extension $$f : \mathbb{B}_1 \to \mathbb{R}^k$$ has the property that $$f^{-1}(0) \neq \emptyset$$.

Let $$\overline{\Omega} \subset X$$ be a closed and bounded subset and $$F : X \to Y$$ a continuous mapping.

\section*{24.1 Definition} Two mappings $$f, g : \overline{\Omega} \subset X \to Y$$ are said to be compactly homotopic relative to $$F$$ if there exists a family of compact mappings $$k(t, \cdot) : \overline{\Omega} \subset X \to Y$$, $$t \in [0, 1]$$, such that

(i) $$h(t,x) = F(x) + k(t,x)$$;
(ii) $$h(0,x) = F(x) + k(0,x) = f(x)$$;
(iii) $$h(1,x) = F(x) + k(1,x) = g(x)$$.

The associated compact homotopy classes are denoted by $$[f]_c$$.

Using the Leray-Schauder degree we have the following analogue of Theorem 14.8 and Corollary 14.13 in the case that $$\Omega \subset X$$ is a bounded and convex domain and $$Y = X$$. As before the Leray-Schauder degree only depends on the boundary behavior of a mapping and for a mapping $$\phi \in C_{0}^0(\partial \Omega; X \setminus \{0\})$$ we define

$$\deg_{LS}(\phi) := \deg_{LS}(f, \Omega, 0),$$

for any continuous extension $$f : \overline{\Omega} \subset X \to X$$.

\section*{24.2 Theorem} Let $$\Omega \subset X$$ be a bounded and convex domain.

(i) Two mappings $$\phi_0, \phi_1 \in C_{0}^0(\partial \Omega; X \setminus \{0\})$$ are compactly homotopy if and only if $$\deg_{LS}(\phi_0) = \deg_{LS}(\phi_1)$$.  
(ii) A mapping $$\phi \in C_{0}^0(\partial \Omega; X \setminus \{0\})$$ is essential with respect to $$\Omega$$ if and only if $$\deg_{LS}(\phi) \neq 0$$.

In the latter case $$f(x) = 0$$ has a non-empty solution set for any continuous extension $$f : \overline{\Omega} \subset X \to X$$ of $$\phi$$.

\textbf{Proof:} Under construction.

The Leray-Schauder degree classifies the homotopy classes and gives necessary and sufficient conditions for the extension, or solvability problem (compare Section 14). The restriction here is that $$\Omega$$ is convex. To extend the Leray-Schauder theory and the results in Section 15 we start with mappings $$f : \Omega \subset X \to Y$$, where $$Y \subset X$$ is a closed linear subspace of finite codimension. The function class with compact perturbations of a fixed linear Fredholm operator $$A : X \to Y$$ is denoted by $$C_{0}^0_A$$. Let $$\phi \in C_{0}^0_A(\partial \Omega; \{0\}),$$ where $$\text{range}(\phi) \subset \bar{Y} \subset Y$$, then considered as map into $$Y \setminus \{0\}$$ if is homotopically trivial. Indeed, $$\deg_{LS}(\phi) = \deg_{LS}(f, \Omega, 0) = 0$$. The latter follows since $$Y \setminus f(\partial \Omega) = Y \setminus \phi(\partial \Omega)$$ is path connected. In the case that $$\Omega = B_1(0)$$ the following result by Srvarc generalizes the theory in Section 15.

In order to explain this extension we first discuss stable homotopy types. We start with the Freudenthal suspension for maps between unit spheres $$S^n \subset \mathbb{R}^{n+1}$$. Let $$\psi : S^n \to S^k$$ be a continuous then the suspension operator defines an operator $$S\psi : S^{n+1} \to S^{k+1}$$. Let $$f : D^{n+1} \subset \mathbb{R}^{n+1} \to \mathbb{R}^{k+1}$$ be any continuous extension of $$\phi$$ then define $$\tilde{f} : D^{n+2} \subset \mathbb{R}^{n+2} \to \mathbb{R}^{k+2}$$ by $$\tilde{f}(x, \overline{x}_{n+2}) := (f(x), \overline{x}_{n+2})$$. Now
set \( S\psi = \tilde{f}/|\tilde{f}| \) by restricting to \( S^{n+1} \) and \( S\psi : S^{n+1} \to S^{k+1} \). It follows from the construction that \([S\psi]\) only depends on \(|\psi|\) and \( S : \pi_n(S^k) \to \pi_{n+1}(S^{k+1}) \) defines an isomorphism. This construction allows to define the notion of stable homotopy type.

\[\textbf{24.3 Theorem.}\] Let \( \Omega = B_1(0) \) and let \( A \in B(X,Y) \) be a Fredholm operator of index \( \ell \geq 0 \). Then the homotopy classes of mappings \( \varphi \in C^0(\partial\Omega;Y\setminus\{0\}) \) are given by the stable homotopy groups \( \pi_{n+\ell}(S^n), n > \ell + 1 \).

\[\textit{Proof:}\] Under construction.

In the case that \( Y = X \) and \( A = \text{Id} \) the stable homotopy is given by the Leray-Schauder degree.

\[\textbf{24.4 Theorem.}\] Let \( \Omega = B_1(0) \) and let \( A \in B(X,Y) \) be a Fredholm operator of index \( \ell \geq 0 \). Then the mapping \( \varphi \in C^0(\partial\Omega;Y\setminus\{0\}) \) is essential with respect to \( \Omega \) if and only if \([\varphi]_c\) corresponds to a non-trivial stable homotopy class in \( \pi_{n+\ell}(S^n) \), \( n > \ell + 1 \).

\[\textit{Proof:}\] Under construction.

\[\textbf{25. Stable cohomotopy}\]

Describe the stable cohomotopy theory and the infinite dimensional framed cobordism theory. Explain some of the theory in Nirenberg, Berger and the papers by Elworthy and Tromba.

\[\textbf{26. Semi-linear elliptic equations and a priori estimates}\]

In this section we will give a application of the Leray-Schauder degree in the context of nonlinear elliptic equations. We follow the notes by L. Nirenberg. The methods that we discuss apply in general for elliptic differential operator of any order. In order to simplify matter here we will restrict ourselves to the Laplace operator with Dirichlet boundary data. Let \( D \subset \mathbb{R}^n \) be a bounded domain with smooth boundary \( \partial D \). Consider the problem

\[-\Delta u = g(x,u,\nabla u), \quad u = 0, \quad x \in \partial D.\]

For the nonlinearity \( g \) we assume that \( C^\infty \)-function of arguments, i.e. \( g \in C^\infty(\overline{D} \times \mathbb{R} \times \mathbb{R}^n) \), and

\[|g(x,u,\nabla u)| \leq C + C|\nabla u|^\gamma, \quad \gamma < 1,\]

uniformly in \( x \in \overline{D} \), and \( u \in \mathbb{R} \). Under these conditions we can prove the following result.

**Theorem 26.1.** Under the assumptions on \( g \) the above elliptic equation has a solution \( u \in C^\infty(\overline{D}) \). Moreover, if \( g(x,0,0) \neq 0 \), then the solution \( u \) is not identically zero.
Proof: The idea behind the proof is the formulate the above elliptic equation as a problem of finding zeroes of an appropriate function \( f \) on a (infinite dimensional) Banach space. Let us start with choosing an appropriate space in which to work. Define \( X = H^2 \cap H^1_0(D) \) to be the intersection of two Sobolev spaces. For details on Sobolev space we refer to the next chapter. We will use the implications of this choice with respect to the well-defined of the elliptic equation, and postpone to proofs to the next chapter. The space \( H^2 \cap H^1_0(D) \) is a Hilbert space with norm \( \|u\|_X = \int_D |\Delta u|^2 \, dx \). Due to the Dirichlet boundary conditions the Laplace operator \( -\Delta : H^2 \cap H^1_0(D) \subset L^2(D) \to L^2(D) \) has a compact inverse \( (-\Delta)^{-1} : L^2(D) \to L^2(D) \). We rewrite the elliptic equation as

\[
(26.1) \quad u - (-\Delta)^{-1} g(x,u,\nabla u) = 0.
\]

The above equation can be regarded as a seeking zeroes of the (Nemytskii) mapping \( f(u) = u - (-\Delta)^{-1} g(x,u,\nabla u) \) on \( H^2 \cap H^1_0(D) \). By the estimate on \( g \) we have that

\[
\int_D |g(x,u(x),\nabla u(x)|^2 \, dx \leq C \int_D \left[ 1 + |\nabla u(x)|^2 \right]^\gamma \, dx
\]

\[
\leq C \left( \int_D \left[ 1 + |\nabla u(x)|^2 \right] \, dx \right)^\gamma
\]

\[
\leq C \left( 1 + \|u\|^2_{H^2_0} \right)^\gamma,
\]

which proves that for \( u \in X \), \( g(x,u,\nabla u) \) is an \( L^2 \)-function. Consequently, the composition \( (-\Delta)^{-1} [g(x,u,\nabla u)] \in X \), proving that \( f : X \to X \) is well-defined. The latter follows from the fact that \( R((-\Delta)^{-1}) = H^2 \cap H^1_0(D) \). As a map from \( L^2 \) to \( H^2 \cap H^1_0 \), the inverse Laplacian is an isometry. Concerning the continuity of this substitution map we refer to the next section. If we define \( Y = H^2_0(\Omega) \) then \( f \) is a map from \( Y \) to \( Y \), and \( f = \text{id} - k \), where \( k : Y \to Y \) is a compact map. Indeed, \( k \) is a composition of the Nemytskii map \( u \mapsto g(x,u,\nabla u) \) (from \( Y \) to \( L^2 \)), the inverse Laplacian \( (-\Delta)^{-1} \) (from \( L^2 \) to \( X \)), and the compact embedding \( X \hookrightarrow Y \), which proves the compactness of \( k \). This brings us into the realm of the Leray-Schauder degree.

Suppose \( u \in X \) is a solution of the equation (26.1), then the estimate on \( g(x,u,\nabla u) \) can be used now to obtain an a priori estimate on the solutions.

\[
\|u\|^2_Y \leq C\|u\|^2_X = C\|g(x,u,\nabla u)\|^2_{L^2}
\]

\[
\leq C \left( 1 + \|u\|^2_Y \right)^\gamma,
\]

which, since \( \gamma < 1 \), implies that \( \|x\|_Y \leq R \).

\section{Exercise. Prove the inequalities \( \|u\|_{L^2} \leq C\|u\|_X \), and \( \|u\|_{H^2_0} \leq C\|u\|_X \), for all \( u \in X \).}

Define the domain \( \Omega = B_{2R}(0) \subset X \). Clearly, \( f \) is a continuous map from \( \overline{\Omega} \) into \( X \), which is of the form identity minus compact. Due to the above a priori estimate
$f^{-1}(0) \subset B_R(0)$, and $0 \notin f(\partial \Omega)$, and therefore the Leray-Schauder degree
$$d_{LS}(f, \Omega, 0),$$
is well-defined.

In order to compute this degree we consider the following homotopy:
$$f_t(u) = u - t(-\Delta)^{-1}[g(x, u, \nabla u)], \quad t \in [0, 1].$$
Notice, for $t \in [0, 1]$ we have via the same a priori estimates, that $f_t^{-1}(0) \subset B_R(0)$, and therefore $0 \notin f_t(\partial \Omega)$ for all $t \in [0, 1]$. Homotopy invariance of the Leray-Schauder degree then yields
$$d(f, \Omega, 0) = d(id, \Omega, 0) = 1,$$
which implies, by validity property of the Leray-Schauder degree, that $f^{-1}(0) \neq \emptyset$. Equation (26.1) thus has a solution $u \in Y$. The equation yields $u = (-\Delta)^{-1}[g(x, u, \nabla u)] \in X$, which that the solution also lies in $X$.

To prove regularity we use a bootstrapping argument. The integral estimates on $g$ can be adjusted to $L^p$-estimates. This gives, by the Sobolev embeddings that:
$$u \in H^{1,p} \implies g(x, u, \nabla u) \in L^p \implies u \in H^{2,p} \implies u \in H^{1,q},$$
where $\frac{1}{p} = \frac{1}{q} - \frac{1}{n}$, provided $n > p$. This yields the recurrence relation
$$\frac{1}{p_{k+1}} = \frac{1}{p_k} - \frac{1}{n}.$$
We can repeat these recurrent steps until $k$ times until $2(k+1) > n > 2k$, and then $u \in H^{2,p_k}$, where $p_k = \frac{2n}{n - 2k}$. Again by the Sobolev embeddings, we have that
$$H^{2,p_k} (\Omega) \hookrightarrow C^{1,\alpha} (\overline{\Omega}),$$
where $\alpha = 1 - \frac{n}{p_k}$, since $\frac{2}{p_k} = \frac{2}{n} - k$, and $k + 1 > \frac{n}{2} > k$, it holds that $0 < \alpha < 1$. We now repeat the bootstrapping in the Hölder space:
$$u \in C^{1,\alpha} \implies g(x, u, \nabla u) \in C^{0,\alpha'} \implies u \in C^{2,\alpha'},$$
where $\alpha' = \gamma \alpha$. The idea now is the use the elliptic regularity theory for the Laplacian be differentiation the equation. Let $v_i = \frac{\partial u}{\partial x_i}$, then
$$-\Delta v_i = \partial_i g + (\partial_u g)v_i + \sum_j \partial_j g \frac{\partial v_j}{\partial x_i}.$$
Since $g$ is a $C^\infty$-function of its arguments, and $u \in C^{2,\alpha'}$, the right hand side is in $C^{0,\alpha'}$, implying that $v_i \in C^{2,\alpha'}$, and thus $u \in C^{3,\alpha'}$. We can repeat this process indefinitely, which proves that $u \in C^\infty (\overline{\Omega})$.

If $g(x, 0, 0) \neq 0$, then $u = 0$ cannot be a solution, and thus $u \neq 0$. $\blacksquare$
VI. Minimax methods

In this Chapter we discuss a first example of a class of methods, also referred to variational methods, that are used to find critical points of functions on finite and infinite dimensional spaces. The main characteristic is to link topological properties of the space to the set of critical points of the function in question.

27. Palais-Smale functions and compactness

The following example shows that critical point theory can break down on various aspects of non-compactness. Consider the function $f(x) = \arctan(x)$. Clearly, $f$ has no critical values, and thus no critical points on $\mathbb{R}$. The values $c = \pm \pi/2$ are special however. One can for example take sequences $\{x^n\}, x^n = \pm n$, such that $f(x^n) \to c$ and $f'(x^n) \to 0$. Regardless of the fact that $f'$ goes to zero along such sequences there are no critical points. One could argue that there exist critical points at ‘infinity’. This would require compactifying our setting.

This simple example already goes to show that due to the non-compactness of $\mathbb{R}$, the domain of definition of $f$, the notion of critical value and critical point lacks uniform estimates. For this very reason Palais and Smale, in their work on Morse theory in infinite dimensions, introduced the following compactness condition.

\begin{definition}
Let $X$ be a (reflexive and separable) a Banach space. A function $f \in C^1(X; \mathbb{R})$ is said to satisfy the Palais-Smale condition at $c$ — (PS) for short —, if any sequence $\{x^n\} \subset X$ for which
\[
f(x^n) \to c, \quad f'(x^n) \to 0,
\]
has a convergent subsequence.
\end{definition}

This condition was referred to as ‘Condition (C)’ in the original work of Palais and Smale. One can also require a function to satisfy (PS) for an interval of values $c$, i.e. a function satisfies (PS) on an interval $I$ if it satisfies (PS) for every $c \in I$. The same holds for the Palais-Smale condition on $X$. In that case we do not specify $c$ beforehand.

The (PS) condition has various consequences for Palais-Smale functions. Denote by $C_f(I)$ the set of critical points of $f$ with critical values restricted to the interval $I$.

\begin{lemma}
Let $f \in C^1(X)$ satisfy (PS) for some $c$, then the set $C_f(c)$ is compact. Moreover, $C_f(I)$ is compact whenever $I$ is compact.
\end{lemma}

\begin{proof}
Compactness is established by pointing out that compactness for a metric space is equivalent to sequential compactness. The space $(C_f(c), d)$ with the induced metric is a metric space itself. For any sequence $\{x^n\}$ we have that $f(x^n) = c$, and $f'(x^n) = 0$. The (PS)-condition then implies that $x^n \to x \in X$. Consequently,
Theorem 28. The deformation lemma

In this section we will start with a standard deformation result for sub-level sets. In the previous section we explained that the Palais-Smale condition yields various properties for the sub-level sets \( f^a := f^{-1}(-\infty, a] \). For sake of simplicity we assume that \( X \) is a Hilbert space here with inner product \((\cdot, \cdot)_X\) or \((\cdot, \cdot)\) when there is no ambiguity about the space involved. In a Hilbert space one can define the gradient of \( f \) via the relation

\[
(\nabla f(x), y) = f'(x)y, \quad \forall y \in X.
\]

The main tool in this is the following differential equation on \( X \):

\[
\frac{dx}{dt} = -\frac{\nabla f(x)}{\|\nabla f(x)\|^2}.
\]

The vector field on the right hand side is well-defined on regular level sets. Let \([a, b]\) be an interval of regular values of \( f \), then by Lemma 27.3 we have that \( \|\nabla f(x)\| \geq \delta > 0 \), which make the above differential equation well-defined on the strip \( f^a_d := f^{-1}[a, b] \). Here we used the identity \( \|\nabla f(x)\|_X = \|f'(x)\|_{X^*} \).

Lemma 28.2. Let \([a, b]\) be an interval of regular values of \( f \). Assume that \( f \in C^2(X) \), and satisfies (PS) on the strip \( f^a_d \). Then \( f^b \) is homotopically equivalent to \( f^a \), i.e. there exists a homotopy \( \eta : [0, 1] \times f^b \to f^b \), such that

\[
\eta_1(f^b) = f^a,
\]

and \( \eta_0 = \text{id on } f^a \).

Proof: Consider Equation 28.1 with initial value \( x(0) = x \in f^a_d \), and denote the solution by \( \xi(x) \). From the theory of ordinary differential equation we know, since \( f \in C^2(X) \), that the solution \( \xi(x) \) is well-defined for all \( x \in f^a_d \) by the above considerations.

Another important consequence of the (PS)-condition is uniformity on lower bounds for \( f' \) regular values.

\begin{lemma}
Let \( c \) be a regular value for \( f \). Then there exists an \( \epsilon > 0 \), such that \( \|f'(x)\|_{X^*} \geq \delta > 0 \) for all \( x \in f^{-1}[c-\epsilon, c+\epsilon] \).
\end{lemma}

Proof: The fact that \( f \) is regular implies that a neighborhood \([c-\epsilon, c+\epsilon]\), for some \( \epsilon > 0 \), consists of regular values. If not, one can choose \( c^n \to c \), and \( x^n \), with \( f(x^n) = c^n \), and \( f'(x^n) = 0 \). By (PS) we have that \( x^n \to x \), with \( f(x) = c \) and \( f'(x) = 0 \), a contradiction.

For any \( c_\epsilon \in [c-\epsilon, c+\epsilon] \) one can find a \( \delta_{c_\epsilon} > 0 \) such that \( \|f'(x)\|_{X^*} \geq \delta_{c_\epsilon} > 0 \) for all \( x \in f^{-1}(c_\epsilon) \). Indeed, otherwise one can find sequences \( \{x^n\} \) such that \( f'(x^n) \to 0 \), which, by (PS), have convergent subsequences converging to a critical point at level \( c_\epsilon \), a contradiction.

Finally, \( \delta_{c_\epsilon} \geq \delta > 0 \) for all \( c_\epsilon \in [c-\epsilon, c+\epsilon] \), by repeating the above argument.
28.3 Exercise. \(\dagger\dagger\) Prove the existence and uniqueness of \(\xi_t\) in the above defined initial value problem.

Consider the composition \(f \circ \xi_t\). Differentiating \(f \circ \xi_t\) yields
\[
\frac{d}{dt} f \circ \xi_t = (\nabla f(\xi(t)), \xi'(t)) = -1.
\]
If we integrate this equation we obtain the following identity
\[
f(\xi(t)) - f(x) = - \int_0^t dt = -t,
\]
and thus \((f \circ \xi_t)(t) = f(x) - t\), which explains that \(f\) decreases along \(\xi_t(t)\). Using this identity we can now define a candidate for a homotopy that is well-defined for all \(x \in f_b\).

\(\eta_t(x) = \begin{cases} 
\xi_{f(x) - a}^t(x) & \text{for } x \in f_b, \\
x & \text{for } x \in f_a.
\end{cases}\)

28.4 Exercise. Verify that \(\eta_t\) is a homotopy as indicated in Lemma 28.2.

From Exercise 28.4 we have that \(\eta_t\) is a proper homotopy that deforms \(f_b\) into \(f_a\), which completes the proof.

28.5 Remark. In Lemma 28.2 we have chosen \(f\) to be of class \(C^2\) to ensure the local Lipschitz continuity of the right hand side of Equation 28.1. In the literature Lemma 28.2 is proved for \(C^1\)-functionals by constructing so-called pseudo gradient vector fields to play the role of \(\nabla f\). This way all the necessary properties for the construction are preserved and the vector field is Lipschitz continuous. Here we choose to leave out this technicality for the sake of focussing on the main ideas involved.

28.6 Remark. In the case \(f\) is invariant under a compact group action, i.e. if \(G\) is a compact group and \(f(g(x)) = f(x)\), for all \(g \in G\), we get additional properties for the deformations in Lemma 28.2. We have \(g(\eta_t(x)) = \eta_t(g(x))\), for all \(t\), and all \(g \in G\).

28.7 Exercise. Prove the above property for \(\eta_t\).

This will be used later on to obtain multiplicity of critical points.

We will now state a general version of the deformation lemma that is also used in Morse theory in next chapter.

Theorem 28.8. Assume that \(f \in C^2(X)\) satisfying (PS) on \(f_b\) for some regular value \(b \leq \infty\). Let \(c < b\), and let \(N\) be a neighborhood of \(C_f(c)\). Then there exists an \(\varepsilon > 0\), and a 1-parameter semi-group of homeomorphisms \(h_t : f_b \to f_b\) such that

1. \(h_0(x) = x\),
2. \(h_t = \text{id}, \text{ if } x \in C_f, \text{ or if } |f(x) - c| \geq \varepsilon\),
3. \(f(h_t(x))\) is non-increasing in \(t\) for all \(x \in f_b\),
4. \(h_1(f_b^c + \varepsilon) \cap N \subset f_b^{c - \varepsilon}\),
5. \(h_1(f_b^{c + \varepsilon}) \subset f_b^{c - \varepsilon} \cup N\).
In addition, if \( f \) is \( G \)-invariant with respect to some compact group \( G \), then the homeomorphisms \( h_t \) are also \( G \)-invariant.

Note that if \( C_f(c) = \emptyset \), then \( N \) can also be chosen to be the empty set.

29. The linking theorem and minimax characterizations

Our next step is explain how the classical minimax characterization of eigenvalues can be generalized to finding critical points of functionals \( f \in C^2(X) \). Let us consider a simple example. Given the function \( f(x,y) = \frac{1}{2}(x^2 + y^2) - \frac{1}{4}x^4 - y^4 \). Upon compute \( f' \) on find nine critical points, of which \((\pm 1, 0)\) and \((0, \pm 2)\) are saddle points. For instance \( f(1, 0) = \frac{1}{2} \). This critical value has on more critical point, namely \((-1, 0)\), which is also a saddle point. The critical value can be thought off as a ‘mountain pass’, i.e. we start walking from the middle at \((0, 0)\) towards the ‘outside’. Let us think of the surface given by \( f(x,y) \) as a mountain landscape. Standing at \((0, 0)\) is as being surrounded by a mountain range. We seek the best way out, i.e. out of the valley, over the mountains, to the hinterland. The way to do this is to seek a mountain passageway. Starting at valley bottom at \((0, 0)\) we consider paths \( \{\gamma(t)\}_{t \in [0,1]} \) to a point where \( f(\gamma(1)) < 0 \). Of all such paths we seek the most economical one, i.e. the one for which \( f(\gamma(t)) \) stays lowest. In mathematical terms:

\[
c = \inf_{\gamma} \max_{t \in [0,1]} f(\gamma(t)).
\]

By construction \( c > 0 \). We will explain now that \( c = \frac{1}{2} \), and that in general such characterizations lead to critical values and critical points.

Let us review these ingredients again. Pick a point \((x^*, y^*)\) such that \( f(x^*, y^*) < 0 \). Instead of considering arbitrary paths we restrict ourselves to paths connecting \((0, 0)\) and \((x^*, y^*)\). Such a path then becomes a continuous map \( h \) from

\[
Q = \{(x,y) = t \cdot (x^*, y^*), \ t \in [0,1]\},
\]
to \( \mathbb{R}^2 \). The minimax described above is therefore

\[
c = \inf_{h \ (x,y) \in Q} \max_{h(x,y)} f(h(x,y)).
\]

The conclusion that \( c > 0 \) can be achieved by finding a second set \( S \) in \( \mathbb{R}^2 \) with the property that \( f|_S \geq \delta > 0 \), \( S \cap Q \neq \emptyset \), and \( \partial Q \cap S = \emptyset \). Here we can take \( S = \{(x,y) \mid x^2 + y^2 = 1\} \). Having these properties we know that \( c \leq \max_{(x,y) \in Q} f(x,y) < \infty \). Since \( S \cap Q \neq \emptyset \), also \( S \cap h(Q) \neq \emptyset \) for any \( h \). Moreover, since \( \partial Q \cap S = \emptyset \), it holds that \( \max_{(x,y) \in Q} f(h(x,y)) \geq \delta > 0 \). Summarizing this yields that \( c > 0 \). The deformation lemma can now be used to prove that \( c \) is indeed a critical value. We will prove this idea in a more general setting.

We explain the general linking of set in \( X \) as introduced by Benci and Rabinowitz.
29.1 Definition. Let $S \subset X$ be a closed subset, and $Q$ is a finite dimensional (compact) submanifold in $X$ with relative boundary $\partial Q$. The sets $S$ and $\partial Q$ link, or form a non-trivial link, in $X$ if:

(i) $S \cap \partial Q = \emptyset$, 
(ii) $h(Q) \cap S \neq \emptyset$, for all maps $h$ in the set 

$$\mathcal{H} = \{ h \in C^0(Q;X) : h|_{\partial Q} = \text{id} \}.$$ 

The sets used in the mountain pass described above are of course examples of linking sets but one can also choose $S$ to be a single point and $Q$ a disc in $X$. Then the minimax describes a local minimum for $f$. Let us now prove the main result of this section, and then given a few more examples of linking sets.

Theorem 29.2. Let $f \in C^2(X)$, and let $S$ and $Q$ be as in Definition 29.1 — linking subsets in $X$. If $\inf_{x \in S} f(x) > \max_{x \in \partial Q} f(x)$, then the minimax

$$c = \inf_{h \in \mathcal{H}} \max_{x \in Q} f(h(x)),$$

well-defined. Moreover, if $f$ satisfies (PS) on $[c - \varepsilon, c + \varepsilon]$, for some $\varepsilon > 0$, then $c$ is a critical value for $f$.

Proof: First, since $S$ and $\partial Q$ are linking sets, $S$ and $h(Q)$ intersect, and therefore $c \geq \inf_{x \in S} f(x) > \max_{x \in \partial Q} f(x) > -\infty$. On the other hand $c \leq \sup_{x \in Q} f(x) < \infty$, which proves the well-definedness of $c$.

As for the critical value we argue by contradiction. Suppose $c$ is a regular value for $f$. Since $f$ satisfies (PS) for interval $[c - \varepsilon, c + \varepsilon]$, for some $\varepsilon > 0$, one can choose an $0 < \varepsilon_1 < \varepsilon$ such that

$$c - \varepsilon_1 > \max_{x \in \partial Q} f(x),$$

the interval $[c - \varepsilon_1, c + \varepsilon_1]$ consists of regular values only. By Lemma 28.2 we have a homotopy $\eta_\varepsilon$ such that

$$\eta_1(f^{c + \varepsilon_1}) = f^{c - \varepsilon_1}, \text{ and } \eta_\varepsilon = \text{id on } f^{c - \varepsilon_1}.$$ 

By the choice of $\varepsilon_1$ the map $\eta_1$ restricted to $Q$ has the property that $\eta_1|_{\partial Q} = \text{id}$. Indeed, since $\max_{x \in \partial Q} f(x) < c - \varepsilon_1$, we have that $\partial Q \subset f^{c - \varepsilon_1}$, and thus $\eta_1 = \text{id on } \partial Q$. Consequently, $\eta_1 \in \mathcal{H}$. The latter implies that

$$\inf_{h \in \mathcal{H}} \max_{x \in Q} f(h(x)) \leq c - \varepsilon_1,$$

which is clearly a contradiction.
29.3 Remark. As pointed out before this general version of the linking can also be be proved for $C^1$-functions. In their work on minimax theory Benci and Rabinowitz introduced the linking theorem in a broader context. Namely, in the general version of Benci and Rabinowitz the set $Q$ is allowed to be a submanifold modeled over an infinite dimensional set. This is very important for applications to strongly indefinite problems such as Hamiltonian systems. To prove the general version of Benci and Rabinowitz more groundwork is needed. Due to the infinite dimensional nature of $Q$ linking becomes a more delicate notion and the Leray-Schauder degree theory is required. This then means that the set $\mathcal{H}$ of deformations also has to be altered. Without going into details the Benci and Rabinowitz theorem is for functions of the form $f(x) = \frac{1}{2}(Lx, x) + b(x)$, where $b$ is a function whose derivative is a compact mapping, and $L$ is an isometry on $X$. This specific form is needed to accommodate the Leray-Schauder degree theory. There are also versions of Theorem 29.2 allowing infinite dimensional sets $Q$, and without $f$ being of the specific form as given above. Such results can be derived using Galerkin type arguments and require a slightly stricter version of the Palais-Smale condition. In practice this stricter version of (PS) is not a problem.

Let us now give a couple of examples of linking sets. Our first example is a direct generalization of the mountain pass construction as sketched above.

Consider a decomposition $X = X_1 \oplus X_2$, with $\dim X_2 < \infty$. Define the sets

\[
S = \{ x \in X_1 : \| x \|_X = r \},
\]

\[
Q = \{ x = te + x_2 : e \in X_1, 0 \leq t \leq R_1, x_2 \in X_2, \| x_2 \|_X \leq R_2 \}.
\]

Under the assumption that $0 < r < R_1$, and $0 < R_2$, the sets $S$ and $\partial Q$ link.

29.4 Exercise. Formulate $S$ and $Q$ in the case of the Mountain Pass Theorem, and show that $S$ and $\partial Q$ link.

29.5 Exercise. Prove in the general case, using degree arguments, that $S$ and $\partial Q$ link.

Our next example displays another class of linking sets. As before consider the decomposition $X = X_1 \oplus X_2$, with $\dim X_2 < \infty$. Define the sets

\[
S = X_1, \quad \text{and} \quad Q = \{ x \in X_2 : \| x \|_X \leq R \}.
\]

Under the assumption that $R > 0$ the sets $S$ and $\partial Q$ link. Clearly $\partial Q = \{ x \in X_2 : \| x \|_X = R \}$, and thus $S \cap \partial Q = \emptyset$ In order to complete the proof that $S$ and $\partial Q$ link we have to show that $S \cap h(Q) \neq \emptyset$ for any $h \in \mathcal{H}$. Let $\pi$ be the projection from $X$ onto $X_2$, then the intersection is non-empty if and only if $0 \in \pi h(Q)$. In terms of the degree this translates into $d(\pi h, Q, 0) \neq 0$. Consider the equation $\pi h(x) = 0$, with $x \in Q$. Define the homotopy

\[
h_t(x) = t\pi h(x) + (1-t)x,
\]

which connects id with $\pi h$. By construction $h_t|_{\partial Q} = \text{id}$, and thus $d(h_t, Q, 0)$ is well-defined for all $t \in [0, 1]$. The homotopy invariance of the degree we then have that

\[
d(\pi h, Q, 0) = d(\text{id}, Q, 0) = 1,
\]
which completes our proof.

The minimax principle explained in the linking theorem is based on a more minimax principle. A minimax theory consists of a function \( f \in C^2(X) \), and collection \( \mathcal{A} \) of subsets of \( X \), and class of maps \( \mathcal{H} \), which have the property that for all \( A \in \mathcal{A} \), and \( h \in \mathcal{H} \) it holds that \( h(A) \in \mathcal{A} \). Define

\[
c = \inf_{A \in \mathcal{A}} \sup_{x \in A} f(x).
\]

The link \( f \) is given as follows: If \( c \in \mathbb{R} \), then there exists an \( h \in \mathcal{H} \), such that for some sufficiently small \( \varepsilon > 0 \) it holds that \( h(f^c+\varepsilon) \subset f^c-\varepsilon \).

**Theorem 29.6.** If \( f \) satisfies (PS) in the interval \( [c-\varepsilon, c+\varepsilon] \) for some \( \varepsilon > 0 \), then \( c \) is a critical value of \( f \).

\[\text{thm:mm1}\]

\[\text{exer:mm}\]

\[\text{sec:category}\]

\[\text{defn:cat1}\]

\[\text{exer:cat2}\]

\[\text{exer:cat3}\]

\[\text{exer:mm}\]

\[\text{sec:category}\]

\[\text{defn:cat1}\]

\[\text{exer:cat2}\]

\[\text{exer:cat3}\]

\[\text{exer:mm}\]

\[\text{sec:category}\]

\[\text{defn:cat1}\]

\[\text{exer:cat2}\]

\[\text{exer:cat3}\]

\[\text{exer:mm}\]

\[\text{sec:category}\]

\[\text{defn:cat1}\]

\[\text{exer:cat2}\]

\[\text{exer:cat3}\]

\[\text{exer:mm}\]

\[\text{sec:category}\]

\[\text{defn:cat1}\]

\[\text{exer:cat2}\]

\[\text{exer:cat3}\]

\[\text{exer:mm}\]

\[\text{sec:category}\]

\[\text{defn:cat1}\]

\[\text{exer:cat2}\]

\[\text{exer:cat3}\]

\[\text{exer:mm}\]

\[\text{sec:category}\]

\[\text{defn:cat1}\]

\[\text{exer:cat2}\]

\[\text{exer:cat3}\]

\[\text{exer:mm}\]

\[\text{sec:category}\]

\[\text{defn:cat1}\]

\[\text{exer:cat2}\]

\[\text{exer:cat3}\]

\[\text{exer:mm}\]
Denote the class of $G$-invariant closed subsets by $\mathcal{A}$, and define $\mathcal{H}$ to be a closed subset of the set of continuous maps $h$ on $X$ that commute with $G$, i.e. $h \circ g = g \circ h$.

For $G \neq \{\text{id}\}$, we define $\text{Fix}(G) = \{x \in X : g(x) = x, \forall g \in G\}$.

**3.34 Definition.** An index theory for $(G, \mathcal{A}, \mathcal{H})$ is a map $i : \mathcal{A} \to \mathbb{N} \cup \{\infty\}$ satisfying the following properties:

1. $i(A) = 0 \iff A = \emptyset$,
2. $A \subset B \Rightarrow i(A) \leq i(B)$,
3. $i(A \cup B) \leq i(A) + i(B)$,
4. $i(A) \leq i(h(A))$, for all $h \in \mathcal{H}$,
5. For $A \in \mathcal{A}$ compact and $A \cap \text{Fix}(G) = \emptyset$, then $i(A) < \infty$, and there exists a neighborhood $N \in A$ of $A$, such that $i(N) = i(A)$,
6. For $x \notin \text{Fix}(G)$ it holds that $i(\cup_{g \in G} g(x)) = 1$.

**3.35 Exercise.** Show that the Ljusternik-Schirelmann theory is an index theory with $\mathcal{H}$ the class of all closed subsets of $X$, and $\mathcal{H} = C^0(X;X)$.

Another example of an index theory is the Krasnoselskii genus. Consider the set $\mathcal{A}$ to be closed subsets of a Hilbert space $X$, satisfying $A = -A$. Choose $G = \{\text{id}, -\text{id}\}$, and $\mathcal{H}$ the set of odd maps. Define the genus as

$$\gamma(A) = \inf \{m \mid \exists h \in C^0(A, \mathbb{R}^m \setminus \{0\}), h(-x) = -h(x)\},$$

and $\gamma(A) = \infty$ if no $h$ can be found for any $m$.

**3.36 Exercise.** Show that $\gamma$ is an index theory for the triple $(G, \mathcal{A}, \mathcal{H})$ defined above.

With the existence of an index theory one can prove the following general multiplicity theorem in minimax theory. This theorem is for the case that $f$ is bounded from below. One can also think about removing this restriction, which requires an extension of the notion of index theory. In the next theorem we assume that $X$ is Hilbert space. In the next theorem let $\mathcal{A}$ be the closed $G$-invariant subsets of $f^b$ for some regular value $b \leq \infty$, and $\mathcal{H}$ the class of $G$-invariant homeomorphisms of $f^b$.

**Theorem 30.7.** Let $f \in C^2(X)$ satisfying (PS) on $f^b$ for some regular value $b \leq \infty$, and let $f$ be invariant with respect to some compact group $G$, i.e. $f(g(x)) = f(x)$ for all $g \in G$. Assume $X$ is $G$-invariant, and $\text{Fix}(G) = \emptyset$. Let $i$ be an index theory for $(G, \mathcal{A}, \mathcal{H})$, and define

$$\hat{i}(f^b) = \sup \{i(K) \mid K \in \mathcal{A}, \text{compact}\}.$$ 

If $f \geq -c$ on $X$, for some $c \geq 0$, then $f$ admits at least $\hat{i}(f^b) \leq \infty$ geometrically distinct critical points modulo $G$.

The above theorem also holds for complete Hilbert manifolds $M$ embedded into $X$. In this case we define to gradient vector field as the component in $TM$.

We start with defining minimax values. Define the subsets $\mathcal{A}_k$ of $\mathcal{A}$ as follows:

$$\mathcal{A}_k = \{A \in \mathcal{A} : i(A) \geq k\}.$$
For any \( k \leq i(X) \) the sets \( A_k \) are non-empty, invariant under homeomorphisms \( h \in \mathcal{H} \). Namely, by Property (iv) we have that \( k \leq i(A) \leq i(h(A)) \). Clearly, \( X \subset A_k \), which proves that \( A_k \neq \emptyset \), for all \( k \leq i(X) \). Now define the minimax values

\[
c_k = \inf_{A \in A_k} \sup_{x \in A} f(x).
\]

\[\blacktriangleright\] **30.8 Lemma.** Let \( f \in C^2(X) \) satisfying (PS) on \( X \), and let \( f \) be invariant with respect to some compact group \( G \). If

\[
-\infty < c_k < \infty,
\]

then \( c_k \) is a critical value for \( f \).

**Proof:** This proof follows along the same lines as the proof of Theorem 29.2. Suppose that \( c_k \) is not a critical value of \( f \), then there exists an \( \varepsilon > 0 \) such that \([c - \varepsilon, c + \varepsilon]\) is an interval of regular values. Let \( A \in \mathcal{A}_k \) such that \( \sup_{x \in A} f(x) \leq c + \varepsilon' \), \( \varepsilon' < \varepsilon \) sufficiently small. Then by Theorem 28.8 there exists an homomorphism \( h_1 \in \mathcal{H} \), such that \( k \leq i(A) \leq i(h_1(A)) \), and \( h_1(A) \subset f^{-\varepsilon'} \). This then gives

\[
c_k \leq c - \varepsilon' < c,
\]

a contradiction.

Clearly the values \( c_k \) are critical for all \( k \leq i(X) \). If they are all distinct, then they all correspond to different critical points. The question now is what happens if some of them are the same.

\[\blacktriangleright\] **30.9 Lemma.** Suppose

\[
-\infty < c_k = c_{k+1} = \cdots = c_{k+l-1} = c < \infty,
\]

then \( i(C_f(c)) \geq l \).

**Proof:** We know from Lemma 30.8 that \( c \) is a critical value. The Palais-Smale condition implies that the set \( C_f(c) \) is compact. By Property (v) (of index) there exists a neighborhood \( N \in \mathcal{A} \) such that \( i(N) = i(A) \). As in the proof of the previous lemma we can choose a set \( A \in \mathcal{A}_{k+l-1} \), such that \( f(x) \leq c + \varepsilon' \), for all \( x \in A \). Clearly, by Theorem 28.8 \( h_1(A) \in \mathcal{A}_{k+l-1} \), and

\[
h_1(A) \subset f^{-\varepsilon'} \cup N.
\]

Since, for all \( A \in \mathcal{A}_k \) it holds that \( f(A) \geq c \), and \( f(f^{-\varepsilon}) \leq c - \varepsilon \), we have that \( f^{-\varepsilon} \notin \mathcal{A}_k \), and thus

\[
i(f^{-\varepsilon}) < k,
\]

From the properties (ii) - (iv) we obtain: \( i(N) + i(f^{-\varepsilon}) \geq i(f^{-\varepsilon} \cup N) \), \( i(f^{-\varepsilon} \cup N) \geq i(h_1(A)) \). Combining these estimates gives:

\[
i(N) \geq i(f^{-\varepsilon} \cup N) - i(f^{-\varepsilon})
\]

\[
> i(h_1(A)) - k \geq i(A) - k
\]

\[
\geq k + l - 1 - k = l - 1.
\]

Consequently, \( i(N) = i(C_f(c)) \geq l \), which proves the lemma.
Proof of Theorem 30.7: For any compact set $K \in \mathcal{A}$ it holds that $i(K) \leq i(f^b)$. Thus for all finite $k \leq i(K) \leq \sup_K i(K) = \hat{i}(f^b)$ the minimax values are bounded from above. The boundedness from below follows from the fact that $f$ is bounded from below.

If all $c_k$ are different we obtain all distinct critical points. In the case of some $c_k = \cdots = c_{k+l-1} = c$’s being the same Lemma 30.9 yields that $i(C_f(c)) \leq l > 1$. Then there are at least $l$ critical points. As a matter of fact by Property (vi) the number is infinite.

\section*{Remark}
The statement in Theorem 30.7 can be improved in the case of the Lusternik-Schnirelmann category. Assume $\text{cat}_{f^b}(f^b) < \infty$, then $f^b \subset \cup_{i} A_i$ (finite covering). Then the set $K = \{x_i\}$, $x_i \in A_i$, and $x_i \notin A_j$, $i \neq j$, is compact and $\text{cat}_{f^b}(K) = \text{cat}_{f^b}(f^b)$. Consequently, $\text{cat}_{f^b}(f^b) = \text{cat}_{f^b}(f^b)$. When $\text{cat}_{f^b}(f^b) = \infty$, then easily follows that $\sup_K \text{cat}_{f^b}(K) = \infty$, proving that $\text{cat}_{f^b}(f^b) = \text{cat}_{f^b}(f^b)$ in general. Therefore,

$$|C_f(f^b)| \geq \text{cat}_{f^b}(f^b),$$

for functionals that are bounded from below.

In the case that a functional $f$ is a bounded functional, i.e. for any bounded set $K$, it holds that $f(K)$ is a bounded interval, we can replace $\hat{i}(f^b)$ by $i(f^b)$ in Theorem 30.7, so that $|C_f(f^b)| \geq i(f^b)$.

\section*{Remark}
For functionals that are not bounded from above most of the minimax theory applies, except for lower bounded in terms of the index. In that case we need to resort to relative index theories. For instance for the category we can use the relative category. We then get

$$|C_f(f^b)| \geq \text{cat}_{(f^b, f^b)}(f^b),$$

which is explained in detail in [ ].

In this chapter the minimax and Morse theory will be applied to a model class of nonlinear elliptic differential equations. For our purposes here we are concerned with elliptic problems of the form

$$-\Delta u = g(x,u), \quad x \in \Omega,$$

$$u = 0, \quad x \in \partial \Omega,$$

where $\Omega \subset \mathbb{R}^n$ is a bounded domain with smooth boundary $\partial \Omega$, and $g(x,u)$ is a $C^\infty$-nonlinearity that satisfies the growth estimate

$$|g_a(x,u)| \leq C + C|u|^{p-1}, \quad p > 1,$$

uniformly in $x \in \overline{\Omega}$, for all $u \in \mathbb{R}$. We will study solutions of this problem using minimax and Morse theory. Standard regularity theory for this equation reveals that solutions are $C^\infty(\Omega)$. Regularity issues will be postponed till later.
31. Variational principles and critical points

As opposed to applying degree theory and fixed point arguments the equation above possesses a alternative formulation for finding solutions; variational principle. Consider the integral

$$\int_{\Omega} \left[ \frac{1}{2} |\nabla u(x)|^2 - G(x,u(x)) \right] dx,$$

where $G(x,u) = \int_0^u g(x,s) ds$. Clearly, the integral is well-defined for all $u \in C^\infty(\Omega) \cap C^1(\overline{\Omega})$. Denote the integral as functional on functions $u(x)$ by $f$. Let us consider the first variation of the integral with respect to test functions $\varphi \in C^\infty(\Omega)$. This yields

$$f(u + \varphi) - f(u) = \int_{\Omega} \left[ \nabla u \cdot \nabla \varphi - g(x,u) \varphi \right] dx + \int_{\Omega} \left[ \frac{1}{2} |\nabla \varphi|^2 - g(x,u) \varphi \right] dx.$$

As explained before, under the assumption that $p < \frac{n+2}{n-2}$, when $n \geq 3$, the function $f$ extends to the Sobolev space $H^1_0(\Omega) = \text{clos}_{H^1}(C^\infty_0(\Omega))$, with equivalent norm

$$\|u\|_{H^1_0} := \sqrt{\int_{\Omega} |\nabla u(x)|^2 dx}.$$

This uses the compact embeddings

$$H^1_0(\Omega) \hookrightarrow \begin{cases} C^0(\overline{\Omega}), & n = 1, \\ L^{p+1}(\Omega), & n = 2, 0 < p < \infty, \\ L^{p+1}(\Omega), & n \geq 3, 0 \leq p < \frac{n+2}{n-2}. \end{cases}$$

If we use the above variation formula we obtain that for fixed $u \in H^1_0(\Omega)$ it holds that

$$|f(u + \varphi) - f(u) - \int_{\Omega} \left[ \nabla u \cdot \nabla \varphi - g(x,u) \varphi \right] dx| = o(\|\varphi\|_{H^1_0}),$$

which proves that $f$ is differentiable on $H^1_0(\Omega)$. Notation:

$$f'(u) \varphi = \int_{\Omega} \left[ \nabla u \cdot \nabla \varphi - g(x,u) \varphi \right] dx.$$

\section*{31.1 Exercise.} Prove the above identity for the Fréchet derivative in the case $g(x,u) = \lambda u + |u|^{p-1}u$, using the first variation and the Sobolev embeddings.

\section*{31.2 Exercise.} In Section 28 we introduced the notion of gradient. Compute the gradient $\nabla f(u)$ in $H^1_0(\Omega)$.

Similarly, the second variation yields

$$|f'(u + \psi) \varphi - f'(u) \varphi - \int_{\Omega} \left[ \nabla \psi \cdot \nabla \varphi - g_u(x,u) \psi \varphi \right] dx| = o(\|\psi\|_{H^1_0}) \|\varphi\|_{H^1_0}. $$
which proves that $f$ twice continuously differentiable on $H^1_0(\Omega)$. Notation: Notation:

$$f''(u)\psi \varphi = \int_{\Omega} \left[ \nabla \psi \cdot \nabla \varphi - g_u(x,u)\psi \varphi \right] dx.$$

**31.3 Exercise.** Establish the expression for the second derivative in the case $g(x,u) = \lambda u + |u|^{p-1} u$, by proving the identity for the second Fréchet derivative. 

The expression for the first derivative explains that the elliptic equation is satisfied in a ‘weak’ sense, i.e. weak solution $u \in H^1_0(\Omega)$. If additional regularity is known then a simple integration by part provides the identity

$$\int_{\Omega} \left[ \left(-\Delta u - g(x,u)\right) \varphi \right] dx = 0,$$

for all $\varphi \in H^1_0(\Omega)$, which reveals the equation again and $u$ is a ‘strong’ solution. This identity, without a priori regularity, can also be interpreted in distributional sense, i.e. $-\Delta$ is regarded as a map from $H^1_0(\Omega)$ to its dual Sobolev space $H^{-1}(\Omega)$.

**31.4 Exercise.** Interpret the above identity in the dual space $H^{-1}(\Omega)$. 

Having established all these preliminary differentiability properties we conclude that solutions of the elliptic equation can be regarded as critical points of the function $f$. This variational principle allows us to attack the elliptic problem via critical point theory.

Before going to the actual application in the next section we first prove a result concerning the Palais-Smale condition.

**31.5 Lemma.** Let $g$ and $\Omega$ be as above and let $1 < p < \infty$, for $n \leq 2$, and $1 < p < \frac{n+2}{n-2}$, for $n \geq 3$. In addition assume that for some $\gamma > 2$,

$$0 < \gamma G(x,u) \leq u g(x,u), \text{ for } |u| \geq r > 0.$$

Then, the function $f$ satisfies the Palais-Smale condition on $H^1_0(\Omega)$.

**Proof:** The requirements on $p$ are needed in order for $f$ to be well-defined and differentiable. Let $\{u^n\}$ be a sequence satisfying

$$f(u^n) \to c \in \mathbb{R}, \text{ and } f'(u^n) \to 0,$$

as $n \to \infty$ — a Palais-Smale sequence. In terms of the above integrals this reads:

$$\int_{\Omega} \left[ \frac{1}{2} |\nabla u^n|^2 - G(x,u^n) \right] dx \to c, \text{ and }$$

$$\left| \int_{\Omega} \left[ \nabla u^n \cdot \nabla \varphi - g(x,u^n) \varphi \right] dx \right| \leq \varepsilon_n \|\varphi\|_{H^1_0}, \varepsilon_n \to 0, \forall \varphi \in H^1_0(\Omega).$$
exer:PS2  ▶ 31.6 Exercise. Derive the above inequalities from the definitions of $f$ and $f'$.

The first step is to show that a Palais-Smale sequence $\{u^n\}^\infty_n=1$ is uniformly bounded in $H^1_0(\Omega)$, with the bound only depending on $c$. In the expression for the derivative we choose $\varphi = \gamma^{-1}u^n$. This gives, upon substitution, that

$$-\varepsilon_n\|u^n\|_{H^1_0}^2 \int_\Omega \frac{1}{2} \nabla u^n \cdot \nabla u^n - g(x,u^n)u^n \, dx \leq \varepsilon_n\|u^n\|_{H^1_0}.$$

Combining this inequality with the expression for $f(u^n)$ we obtain:

$$\left(\frac{1}{2} - \frac{1}{\gamma}\right) \int_\Omega \nabla u^n \cdot \nabla u^n = \int_\Omega \left[ G(x,u^n) - \gamma^{-1} u^n g(x,u^n) \right] \, dx$$

$$\quad \quad \quad \quad \quad \quad \quad \quad + \ c + \varepsilon_n + \varepsilon_n \gamma^{-1} \|u^n\|_{H^1_0} \leq C + \gamma^{-1} \varepsilon_n \|u^n\|_{H^1_0}.$$

This inequality yields the estimate $\|u^n\|_{H^1_0} \leq C$.

Since $H^1_0(\Omega)$ is a Hilbert space the boundedness of $\{u^n\}$ implies that $u^n \rightharpoonup u$ in $H^1_0(\Omega)$. Since the embeddings of $H^1_0(\Omega)$ into $L^{p+1}(\Omega)$ are all compact, provided $p < \frac{n+2}{n-2}$, $n \geq 3$, it holds that $u^n \rightarrow u$ in $L^{p+1}$. Consequently, $\int_\Omega G(x,u^n) \, dx \rightarrow \int_\Omega G(x,u) \, dx$. If we combine this with the convergence of $f$ we obtain:

$$\frac{1}{2} \int_\Omega \nabla u^n \cdot \nabla u^n = f(u^n) + \int_\Omega G(x,u^n) \, dx \rightarrow c + \int_\Omega G(x,u) \, dx,$$

which proves that $\|u^n\|_{H^1_0} \rightarrow \|u\|_{H^1_0}$, and convergence of $\{u^n\}$ in $H^1_0(\Omega)$, completing the proof.

Having established the Palais-Smale condition for $f$ allows us now to apply critical point methods.

32. Existence of solutions

We start with applying the Linking Theorem 29.2 to establish non-trivial solutions to the elliptic equation in Section 31. In order to simplify matters we assumed in the next theorem $u = 0$ is a solution with given local behavior. Let us summarize the hypotheses on $g \in C^0(\overline{\Omega} \times \mathbb{R})$:

- $(g1)$ $|g(x,u)| \leq C + C|u|^{p-1}$, uniformly in $x \in \overline{\Omega}$, with $1 < p < \infty$ when $n \leq 2$, and $1 < p < \frac{n+2}{n-2}$ when $n \geq 3$,
- $(g2)$ $0 < \gamma G(x,u) \leq ug(x,u)$, for $|u| \geq r > 0$,
- $(g3)$ $|g(x,u) - \lambda u| = o(|u|)$, for $|u| \leq \delta$.

Under these assumption we can prove the following existence result.

**Theorem 32.1.** Let $\Omega$ be a smooth bounded domain in $\mathbb{R}^n$, and let $g$ satisfy Hypotheses $(g1)$-$(g3)$. Then $f$ has a non-trivial critical point $u \in H^1_0(\Omega)$, with $f(u) > 0$.

**Proof:** By Hypotheses $(g1)$-$(g2)$ the function $f$ satisfies (PS) on all of $H^1_0(\Omega)$ (Lemma 31.5).
By (g3) we have that \( u = 0 \) is a solution. Upon computing \( f''(0) \) we obtain:

\[
f''(0)\varphi\psi = \int_{\Omega} \left[ \nabla \varphi \nabla \psi - \lambda \varphi \psi \right] dx.
\]

Depending \( \lambda \), we can determine is local behavior using the quadratic form. In order to apply the Linking Theorem we need to construct sets \( S \) and \( Q \). To do this we start with investigating the local behavior at \( u = 0 \). Consider the eigenfunctions of \(-\Delta\) as a map from \( L^2(\Omega) \) to \( L^2(\Omega) \) with Dirichlet boundary conditions; say \( \phi_k \), with \(-\Delta \phi_k = \lambda_k \phi_k \), and \( \lambda_k > 0 \), \( \lambda_k \to \infty \) as \( k \to \infty \). By the spectral theorem for self-adjoint operator we then have that function \( u \in H^1_0(\Omega) \) can be decomposed as

\[
u = \sum_{k=1}^{\infty} u_k \phi_k,
\]

where \( u_k \) — the ‘Fourier coefficients’ — are given by \( u_k = (u, \phi_k)_{H^1_0} \). Clearly, \( \|u\|_{H^1_0}^2 = \sum_k u_k^2 \) — Parseval’s identity.

\( \blacktriangleleft \) 3.2. Exercise. \( \blacktriangleright \) Formulate the spectral theory for \(-\Delta\) on \( L^2(\Omega) \) with Dirichlet boundary conditions.

For the second derivative we observe that \( f''(0) \) is negative definite on \( X^- = \text{span}\{\phi_k : \lambda_k - \lambda < 0\} \), positive definite on \( X^+ = \text{span}\{\phi_k : \lambda_k - \lambda > 0\} \), and zero on \( X^0 = \text{span}\{\phi_k : \lambda_k - \lambda = 0\} \). Clearly, \( X^+ \) infinite dimensional and \( X^0 \), and \( X^- \) are finite dimensional. Write \( H^1_0(\Omega) = X^1 \oplus X^2 \), where \( X^1 = X^+ \), and \( X^2 = X^0 \oplus X^- \). Define

\[
S = \{u \in X^1 : \|u\|_{H^1_0} = r\},
\]
\[
Q = \{u = te + u_2 : e \in X^1, 0 \leq t \leq R_1, u_2 \in X^2, \|u_2\|_{H^1_0} \leq R_2\}.
\]

We proved in the previous section that \( S \) and \( \partial Q \) linking whenever \( 0 < r < R_1 \), and \( 0 < R_2 \).

Let us start with \( S \). From the integral form of \( f \) we have, for \( \|u\|_{H^1_0} \) sufficiently small, that

\[
f|_S = \frac{1}{2} \sum_{\lambda_k - \lambda > 0} (\lambda_k - \lambda) u_k^2 - \int_{\Omega} \left[ G(x,u) - \frac{\lambda}{2} u^2 \right] dx,
\]

\[
\geq \frac{1}{2} \left(1 - \frac{\lambda}{\lambda_{k_0}}\right) \|u\|_{H^1_0}^2 - \int_{\Omega} \left[ G(x,u) - \frac{\lambda}{2} u^2 \right] dx,
\]

\[
\geq \frac{1}{2} \left(1 - \frac{\lambda}{\lambda_{k_0}}\right) \|u\|_{H^1_0}^2 - \epsilon \int_{\Omega} u^2 dx,
\]

\[
\geq c_0 \|u\|_{H^1_0}^2 = c_0 r > 0,
\]

where \( \lambda_{k_0} \) is the smallest eigenvalue for which \( \lambda_{k_0} - \lambda > 0 \).
\[ \int_{\Omega} \left[ G(x,u) - \frac{\lambda}{2} u^2 \right] dx = o(\|u\|_{H^1_0}^3), \text{ as } \|u\|_{H^1_0} \to 0. \]

The next step is to choose the parameters in $Q$ such that $f_{|Q} < 0$. That way the main assumption of the Linking Theorem is satisfied and we obtain non-trivial critical points. In order to simplify exposition here let us assume that $\lambda \neq \lambda_k$ for any $k$. Then $X^0 = \{0\}$, which simplifies the proof. We have

\[ f_{|Q} = f(te + u_2) = \frac{t^2}{2} \sum_{\lambda_k - \lambda > 0} (\lambda_k - \lambda) e_k^2 - \sum_{\lambda_k - \lambda < 0} (\lambda - \lambda_k) u_{2,k}^2 \]

\[ \leq \frac{t^2}{2} \|e\|_{H^1_0}^2 - c_1 \|u_2\|_{H^1_0}^2 \]

\[ \leq \int_{\Omega} \left[ G(x,te + u_2) - \frac{\lambda}{2} |te + u_2|^2 \right] dx, \]

where $te + u_2$ lies on one of the three pieces of the boundary $\partial Q$, i.e. (i) The cylinder: $0 \leq t \leq R_1$, and $\|u_2\|_{H^1_0} = R_2$, (ii) The bottom: $t = 0$, and $\|u_2\|_{H^1_0} \leq R_2$, and (iii) The lid (top): $t = R_1$, and $\|u_2\|_{H^1_0} \leq R_2$.

\[ \lambda \leq \left( 1 - \frac{\lambda}{2} \right) \leq \frac{\lambda}{2} \]

\[ f_{|Q} = f(te + u_2) \leq \frac{t^2}{2} \|e\|_{H^1_0}^2 - c_1 \|u_2\|_{H^1_0}^2 - c_3 \int_{\Omega} |te + u_2|^2 - c_3 |\Omega|, \]

\[ \leq c_4 t^2 - c_1 \|u_2\|_{H^1_0}^2 - c_5 \left( \int_{\Omega} |te + u_2|^2 \right)^{\gamma/2} - c_6, \]

\[ \leq c_4 t^2 - c_1 \|u_2\|_{H^1_0}^2 - c_5 \left( t^2 \int_{\Omega} |e|^2 + \int_{\Omega} |u_2|^2 \right)^{\gamma/2} - c_6, \]

\[ \leq c_4 t^2 - c_7 \gamma - c_1 \|u_2\|_{H^1_0}^2 - c_6. \]

Let us now use this estimates on the three parts of $\partial Q$. On the bottom we have $t = 0$, and thus $f_{|Q} \leq -c_1 \|u_2\|_{H^1_0}^2 - c_6 \leq 0$. On the lid we have $t = R_1$, and so $f_{|Q} \leq c_4 R_1^2 - c_7 \gamma - c_1 \|u_2\|_{H^1_0}^2 - c_6 \leq 0$, provided $R_1 > r$ is chosen large enough. Finally, on the cylinder we have $f_{|Q} \leq c_4 t^2 - c_7 \gamma - c_1 R_1^2 - c_6$. Let $M = \max_{x \in [0,R_1]} (c_4 t^2 - c_7 \gamma) > 0$. Choose $R_2$ large enough such that $M - c_1 R_2^2 - c_6 < 0$. This completes the choices of $r$, $R_1$, and $R_2$, and proof of theorem.

\[ \lambda \leq \left( 1 - \frac{\lambda}{2} \right) \leq \frac{\lambda}{2} \]

\[ f_{|Q} = f(te + u_2) \leq \frac{t^2}{2} \|e\|_{H^1_0}^2 - c_1 \|u_2\|_{H^1_0}^2 - c_3 \int_{\Omega} |te + u_2|^2 - c_3 |\Omega|, \]

\[ \leq c_4 t^2 - c_1 \|u_2\|_{H^1_0}^2 - c_5 \left( \int_{\Omega} |te + u_2|^2 \right)^{\gamma/2} - c_6, \]

\[ \leq c_4 t^2 - c_1 \|u_2\|_{H^1_0}^2 - c_5 \left( t^2 \int_{\Omega} |e|^2 + \int_{\Omega} |u_2|^2 \right)^{\gamma/2} - c_6, \]

\[ \leq c_4 t^2 - c_7 \gamma - c_1 \|u_2\|_{H^1_0}^2 - c_6. \]

Let us now use this estimates on the three parts of $\partial Q$. On the bottom we have $t = 0$, and thus $f_{|Q} \leq -c_1 \|u_2\|_{H^1_0}^2 - c_6 \leq 0$. On the lid we have $t = R_1$, and so $f_{|Q} \leq c_4 R_1^2 - c_7 \gamma - c_1 \|u_2\|_{H^1_0}^2 - c_6 \leq 0$, provided $R_1 > r$ is chosen large enough. Finally, on the cylinder we have $f_{|Q} \leq c_4 t^2 - c_7 \gamma - c_1 R_1^2 - c_6$. Let $M = \max_{x \in [0,R_1]} (c_4 t^2 - c_7 \gamma) > 0$. Choose $R_2$ large enough such that $M - c_1 R_2^2 - c_6 < 0$. This completes the choices of $r$, $R_1$, and $R_2$, and proof of theorem.
\[\text{32.6 Remark. Hypothesis (g3) is somewhat questionable in the sense if necessary or not. For example if } g(x,u) = g(u) \text{ — autonomous case — one does not need (g3). In this case a first solution is readily found by taking a zero of } g. \text{ For this point on the proof is the same as the proof of Theorem 32.1.}\]

\[\text{32.7 Exercise. } \uparrow\uparrow \text{ Consider the differential equation } \Delta u + |u|^{p-1} u = 0, \text{ with } u = 0 \text{ on } \partial\Omega. \text{ Describe a method for finding positive solutions via the Mountain Pass Theorem (Hint: use the Maximum Principle for the Laplacian } \Delta, \text{ and consider } (u^+)^p \text{ instead of } |u|^{p-1} u.\]

\[\text{32.8 Exercise. } \uparrow\uparrow \text{ Consider the differential equation } \Delta^2 u - \lambda_1^2 u - |u|^{p-1} u = 0, \text{ with } u = \Delta u = 0 \text{ on } \partial\Omega, \text{ and where } \lambda_1 \text{ is the first eigenvalue of } -\Delta \text{ with Dirichlet boundary conditions. Use the Linking Theorem to prove the existence of a non-trivial solution (Hint: Choose } S \text{ and } Q \text{ as in the above example.}.\]